Landau Damping

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The damping of a longitudinal plasma wave of finite amplitude is considered. It is shown that the Landau result is the first term in a systematic expansion in a small parameter, and the corrections for finite wave amplitude are shown to be 5^{th} order in the small parameter. The contributions to the damping from particles with different velocities near the phase velocity are explicitly calculated and this leads to a simple physical picture of the damping process.

I. INTRODUCTION

Landau [1] predicted the damping of small amplitude longitudinal plasma oscillations in a collisionless plasma. There is little doubt that this is correct in the limit of small amplitude waves, however the Landau derivation does not tell us just how small these waves must be. The purpose of the present paper is to give an alternate derivation of the Landau results that provides a simple picture of the damping process, together with a bound on the wave amplitude.

The starting point is the Vlasov-Poisson equation set:

$$\frac{\partial F}{\partial t} + \mathrm{v}\frac{\partial F}{\partial \mathrm{x}} + \frac{e}{m}E\frac{\partial F}{\partial \mathrm{v}} = 0 \tag{1}$$

$$\frac{\partial E}{\partial \mathrm{x}} = 4\pi e \int F(\mathrm{x}, \mathrm{v}, t) d\mathrm{v} - 4\pi n_0 e \tag{2}$$

Landau set $F(\mathrm{x}, \mathrm{v}, t) = f_0(\mathrm{v}) + f_k(\mathrm{x}, \mathrm{v}, t)$ where $f_0(\mathrm{v})$ is the unperturbed distribution function which he took to be a Maxwellian, $f_0(\mathrm{v}) = n_0/\sqrt{2\pi}\overline{\mathrm{v}}\ \exp\{-(1/2)(\mathrm{v}/\overline{\mathrm{v}})^2\}$, and $f_k(\mathrm{x}, \mathrm{v}, t)$ is a perturbation. The Vlasov equation thus becomes

$$\frac{\partial f_k}{\partial t} + \mathrm{v}\frac{\partial f_k}{\partial \mathrm{x}} + \frac{e}{m}E\frac{\partial f_0}{\partial \mathrm{v}} = -\frac{e}{m}E\frac{\partial f_k}{\partial \mathrm{v}}, \tag{3}$$

and he linearized this equation by dropping the term $(e/m)E\partial f_k/\partial \mathrm{v}$, which is bilinear in the small quantities. He then considered the initial value problem: For $t < 0$, $f_k(\mathrm{x}, \mathrm{v}, \mathrm{t})$ and E are zero, and at $t = 0$ there is a small perturbation, $f_k^0(\mathrm{v}) \cos(\mathrm{kx})$ which provides a perturbed electric field. Using Fourier transforms in space and Laplace transforms in time, Landau then integrated the linearized equation to find a solution of the form

$$E = E_0 \exp(-\gamma t) \sin(k\mathrm{x} - \omega t)\ \ + \text{transients which decay rapidly} \tag{4}$$

where:

$$\omega^2 = \omega_p^2 \left(1 + 3\left(\frac{k\overline{\mathrm{v}}}{\omega_p}\right)^2\right); \tag{5}$$

$$\frac{\gamma}{\omega} = -\frac{\pi}{2}\left(\frac{\omega_p}{k}\right)^2 \frac{1}{n_0}\frac{\partial f_0}{\partial \mathrm{v}}\bigg|_{\mathrm{v}=\frac{\omega}{\mathrm{k}}};$$

$$\frac{\gamma}{\omega} \ll \left(\frac{k\overline{\mathrm{v}}}{\omega_p}\right)^2 \ll 1. \tag{6}$$

Since the Landau result was derived from the linearized Vlasov equation it provides no information on the effect of finite amplitude on the damping. The Landau derivation also provides little insight into the wave particle interactions that produce the damping. Plainly, from the form of the damping coefficient, it is the particles with velocities near the phase velocity which lead to the damping, but how the particles with different velocities in this range contribute to the damping is not clear.

Our objective is to rederive the Landau result in a way that addresses these questions.

II. PHASE SPACE PICTURE

In his treatment of the linearized theory, Landau divided $F(\mathrm{x},\mathrm{v},t)$ into two terms, $f_0(\mathrm{v})$, the unperturbed distribution function, and $f_k(\mathrm{x},\mathrm{v},t)$, the perturbation. The following considerations suggest a somewhat different division of $F(\mathrm{x},\mathrm{v},t)$ for the nonlinear initial value problem. At $t=0$ we have the unperturbed distribution function, $f_0(\mathrm{v})$, and the initial perturbation, $f_k^0(\mathrm{v})\cos kx$, which provides an initial electric field. This field causes the unperturbed distribution function, $f_0(\mathrm{v})$, which represents the bulk of the particles, to change, leading to an electric field from these particles. It is this motion of the bulk of the particles which then evolves into the plasma wave. It thus seems natural to break up $F(\mathrm{x},\mathrm{v},t)$ into $f(\mathrm{x},\mathrm{v},t)+\widehat{f}(\mathrm{x},\mathrm{v},t)$ where $f(\mathrm{x},\mathrm{v},t)$ is the distribution which evolves from $f_0(\mathrm{v})$, and $\widehat{f}(\mathrm{x},\mathrm{v},t)$ is the distribution which evolves from $f_k^0(\mathrm{v})\cos k\mathrm{x}$, i.e. $f(\mathrm{x},\mathrm{v},0)=f_0(\mathrm{v})$, and $\widehat{f}(\mathrm{x},\mathrm{v},0)=f_k^0(\mathrm{v})\cos kx$. Both $f(\mathrm{x},\mathrm{v},t)$ and $\widehat{f}(\mathrm{x},\mathrm{v},t)$ obey the Vlasov equation, and thus, in particular, $f(\mathrm{x},\mathrm{v},t)=f(\mathrm{x}_0,\mathrm{v}_0,0)=f_0(\mathrm{v}_0)$, where x and v are the phase space coordinates of a particle which was at x_0 and v_0 at $t=0$. If we know $\mathrm{x}=\mathrm{x}(\mathrm{x}_0,\mathrm{v}_0,t)$ and $\mathrm{v}=\mathrm{v}(\mathrm{x}_0,\mathrm{v}_0,t)$ and if we can invert these to find $\mathrm{x}_0=\mathrm{x}_0(\mathrm{x},\mathrm{v},t)$ and $\mathrm{v}_0=\mathrm{v}_0(\mathrm{v},\mathrm{x},t)$ we can solve for $f(\mathrm{x},\mathrm{v},t)=f_0[\mathrm{v}_0(\mathrm{x},\mathrm{v},t)]$.

If we write $\mathrm{v}_0=\mathrm{v}-(\mathrm{v}-\mathrm{v}_0)$ and expand $f_0(\mathrm{v}_0)$ we have

$$f(\mathrm{x},\mathrm{v},t)\cong f_0(\mathrm{v})-(\mathrm{v}-\mathrm{v}_0)\frac{\partial \mathrm{f}_0}{\partial \mathrm{v}}=\mathrm{f}_0(\mathrm{v})-\delta\mathrm{v}\ \frac{\partial \mathrm{f}_0}{\partial \mathrm{v}} \tag{7}$$

where $\delta\mathrm{v}=\mathrm{v}-\mathrm{v}_0$.

This approximation requires only $(\delta\mathrm{v})/\mathrm{v}\ll(\overline{\mathrm{v}}/\mathrm{v})^2$. As discussed in Sec. VI, this is easily satisfied even when nonlinear effects are important for particles moving with velocities near the phase velocity.

Similarly, we can write $\widehat{f}(\mathrm{x},\mathrm{v},t) = f_k^0[v_0(\mathrm{x},\mathrm{v},t)]\cos[kx_0(\mathrm{x},\mathrm{v},t)]$. However, as shown in Sec. III, this term phase mixes and does not contribute to the damping.

To make use of this we need to find the orbit of a particle moving in an electric field, $E = E_0\exp(-\gamma t)\sin(k\mathrm{x}-\omega t)$. Since it is the particles with velocities near the phase velocity, ω/k, that produce the damping, we move to the wave frame, $v = \mathrm{v} - \omega/k$, $x = \mathrm{x} - (\omega/\mathrm{k})\mathrm{t}$, $E(x,t) = E_0\exp(-\gamma t)\sin kx$. In the wave frame we have

$$\dot{v} \;=\; \frac{e}{m}\,E_0\exp(-\gamma t)\sin(kx) \tag{8}$$

$$\dot{x} \;=\; v. \tag{9}$$

Introducing the dimensionless variables

$$\boldsymbol{v} = \frac{kv}{\omega_0},\;\; \boldsymbol{x} = kx\;,\;\; \boldsymbol{t} = \omega_0\,t\;,\;\; \Gamma = \frac{\gamma}{\omega_0}$$

where $\omega_0 = \sqrt{\frac{keE_0}{m}}$ we have

$$\dot{\boldsymbol{v}} \;=\; \exp(-\Gamma\boldsymbol{t})\sin\boldsymbol{x} \tag{10}$$

$$\dot{\boldsymbol{x}} \;=\; \boldsymbol{v}. \tag{11}$$

For large amplitude waves in the sense that $\Gamma \ll 1$, O'Neil [2] has addressed this problem and finds that after an initial transient, the damping stops. In the Landau limit $E_0 \to 0$, and thus $\Gamma \to \infty$. We seek the solution for small but finite amplitude waves for which $\Gamma \gg 1$, and proceed by expanding in powers of $(\frac{1}{\Gamma})$ keeping terms up through the fifth order in $\frac{1}{\Gamma}$.

Integrating Eq. (10) we note that $(\boldsymbol{v}-\boldsymbol{v}_0) = \int_0^{\boldsymbol{t}} d\boldsymbol{t}'\exp(-\Gamma\boldsymbol{t}')\sin\boldsymbol{x}$ is bounded by $(\frac{1}{\Gamma})$. We are interested in times, t, for which $\gamma t \sim \mathcal{O}(1)$. Thus, since $\Gamma\boldsymbol{t} = \gamma t$, and $\Gamma \gg 1$, $\boldsymbol{t}$ is of order $(\frac{1}{\Gamma})$. Integrating Eq. (11) this gives

$$\boldsymbol{x} - \boldsymbol{x}_0 \;=\; \int_0^{\boldsymbol{t}} d\boldsymbol{t}'\left[\boldsymbol{v}_0 + \mathcal{O}\left(\frac{1}{\Gamma}\right)\right] = \boldsymbol{v}_0\boldsymbol{t} + \mathcal{O}\left(\frac{1}{\Gamma}\right)^2$$

and

$$(\boldsymbol{v}-\boldsymbol{v}_0) \;=\; \int_0^{\boldsymbol{t}} d\boldsymbol{t}'\exp(-\Gamma\boldsymbol{t}')\sin(\boldsymbol{x}_0+\boldsymbol{v}_0\boldsymbol{t}') + \mathcal{O}\left(\frac{1}{\Gamma}\right)^3$$

From the above $\boldsymbol{x}_0+\boldsymbol{v}_0t' = \boldsymbol{x}-\boldsymbol{v}_0(\boldsymbol{t}-\boldsymbol{t}')+\mathcal{O}(\frac{1}{\Gamma})^2 = \boldsymbol{x}-(\boldsymbol{v}+\mathcal{O}(\frac{1}{\Gamma})(\boldsymbol{t}-\boldsymbol{t}') = \boldsymbol{x}-\boldsymbol{v}(\boldsymbol{t}-\boldsymbol{t}')+\mathcal{O}(\frac{1}{\Gamma})^2$. Inserting this in the integral for $\boldsymbol{v}-\boldsymbol{v}_0$ we have

$$\delta\boldsymbol{v} = \boldsymbol{v}-\boldsymbol{v}_0 = \int_0^{\boldsymbol{t}} d\boldsymbol{t}'\exp(-\gamma\boldsymbol{t}')\sin\left(\boldsymbol{x}-\boldsymbol{v}(\boldsymbol{t}-\boldsymbol{t}')\right) + \mathcal{O}\left(\frac{1}{\Gamma}\right)^3 \tag{12}$$

which is just the linearized limit. The corrections to this up through terms of order $(\frac{1}{\Gamma})^5$ are calculated in the appendix. As will be discussed, the first correction to the Landau damping result is of order $(\frac{1}{\Gamma})^5$.

Integrating Eq. (12) and expressing the result in terms of dimensional variables in the wave frame gives

$$\delta v = v - v_0 = -\frac{eE_0}{m}\frac{1}{(kv)^2+\gamma^2}\left\{\exp(-\gamma t)\left[kv\cos kx + \gamma\sin kx\right] - \left[kv\cos(kx-kvt)+\gamma\sin(kx-kvt)\right]\right\}.$$

Letting $t \to \infty$ yields the time asymptotic change in the velocity

$$\delta v_\infty = \frac{e}{m}E_0\left\{\frac{kv\cos(kx-kvt)+\gamma\sin(kx-kvt)}{(kv)^2+\gamma^2}\right\}$$

which is sometimes called the ballistic term since it represents particles traveling with constant velocity. It is δv_∞ which leads to plasma wave echos.

III. THE ENERGY TRANSFER

At a point, x, v, in phase space in the laboratory frame, the rate of energy transfer between the field and the particles is given by $P(\mathrm{x},\mathrm{v}) = eE(\mathrm{x},t)\mathrm{v}f(\mathrm{x},\mathrm{v},t)$ and since $f(\mathrm{x},\mathrm{v},t) = f_0 - \delta\mathrm{v}\frac{\partial f_0}{\partial \mathrm{v}}$, the spatially-averaged power transfer is

$$\overline{P}(\mathrm{v}) = -eE_0\exp(-\gamma t)\mathrm{v}\frac{\partial f_0}{\partial \mathrm{v}}\;\frac{k}{2\pi}\int_0^{2\pi/k} d\mathrm{x}\;\sin(k\mathrm{x}-\omega t)\delta\mathrm{v}.$$

Expressing this in wave frame coordinates and carrying out the integration gives

$$\overline{P}(v) = \frac{e^2E_0^2}{2m}\exp(-\gamma t)\left\{(\omega/k+v)\frac{\partial f_0}{\partial v}\left[\frac{\gamma\exp(-\gamma t) - kv\sin(kvt) - \gamma\cos(kvt)}{(kv)^2+\gamma^2}\right]\right\}. \quad (13)$$

Consider the terms in curly brackets. The term proportional to $\gamma\exp(-\gamma t)$ is negative (since $\partial f_0/\partial v < 0$) and leads to energy transfer *from* particles of all velocities *to* the electric field. The velocity dependence has a resonance, $\gamma/((kv)^2+\gamma^2)$, at $v=0$ with a half-width of γ/k. Since the width of the resonance is small compared to the range over which the term $(\omega/k+v)\partial f_0/\partial v < 0$ varies, this term will be evaluated at the peak of the resonance and treated as a constant in the resonance region. Integrating v over the resonance yields $\pi/k\exp(-\gamma t)(\omega/k)\partial f_0/\partial v|_{v=0}$. For the main part of the velocity spectrum, away from the resonance, this term is given by

$$\exp(-\gamma t)\frac{\gamma}{(kv)^2+(\gamma^2)}\left(\frac{\omega}{k}+v\right)\frac{\partial f_0}{\partial v} \cong \exp(-\gamma t)\frac{\gamma}{(\omega-k\mathrm{v})^2}\;\mathrm{v}\partial f_0/\partial \mathrm{v},$$

which, as shown in Sec. V, simply leads to the loss of particle "sloshing energy" as the wave damps.

The terms, $-kv\sin(kvt)$, and $-\gamma\cos(kvt)$, after integrating over the resonance, each give $(\pi/k)\exp(-\gamma t)\partial f_0/\partial v|_{v=0}$. The resonant contribution from these two terms thus leads to an energy transfer *from* the field *to* the particles which is twice the energy transfer in the opposite direction from the $\gamma\exp(-\gamma t)$ term. If we integrate these terms over the broad spectrum of the particle velocities, excluding the resonance, phase mixing leads to a rapid decay in time, and this nonresonant contribution can be neglected.

The net energy transfer from all of the terms is thus *to* the particles in the resonance region and *from* the particles in the bulk of the plasma.

The total power transfer is thus given by:

$$-\frac{\partial}{\partial t}\frac{k}{2\pi}\int dx\frac{E^2}{8\pi} = \int dv\overline{P}(\mathrm{v}) = \frac{e^2E_0^2\exp(-2\gamma t)}{2m}\left\{-\frac{\pi\omega}{k^2}\frac{\partial f_0}{\partial \mathrm{v}}\bigg|_{\mathrm{v}=\frac{\omega}{k}} + \gamma\,\mathcal{P}\int\frac{dv\,\mathrm{v}}{(kv-\omega)^2}\frac{\partial f_0}{\partial \mathrm{v}}\right\} \tag{14}$$

where $\mathcal{P}$ means principal part.

Carrying out the integral, the "sloshing energy" term on the right-hand side of Eq. (14) has the same size but opposite sign as the field energy term on the left-hand side. This yields the Landau result

$$\gamma = -\frac{\pi}{2}\left(\frac{\omega_p}{k}\right)^2\frac{1}{n}\frac{\partial f_0}{\partial \mathrm{v}}\bigg|_{v=\frac{\omega}{k}}.$$

Returning to Eq. (13), we note that since $\omega/k + \mathrm{v}$ and $\partial f_0/\partial\mathrm{v}$ are essentially constant over the narrow resonance, and since $\partial f/\partial\mathrm{v}$ is negative, the velocity dependence of $P(\mathrm{v})$ is proportional to

$$\overline{P}(\mathrm{v}) \propto -\left\{\frac{\gamma\exp(-\gamma t) - kv\sin(kvt) - \gamma\cos(kvt)}{(kv)^2+\gamma^2}\right\}. \tag{15}$$

We thus have a simple picture of the energy transfer as a function of the particle velocity. The velocity dependence of Eq. (15) is shown in Fig. 1 for various values of γt. From this we conclude.

1. The bulk of the energy transfer comes from particles for which $|v| \lesssim \gamma/k$, and almost all of these gain energy.

2. The energy transfer is an even function of v.

3. $|\delta v|/(\gamma/k) \lesssim 1/\Gamma^2$ so the mixing of velocities is very local.

4. The distance these particles move in a time, t, is $(vt) \lesssim \frac{\gamma t}{k}$, so they travel only a fraction of a wavelength in one damping time.

Qualitatively, the contribution of the resonant particles to the damping arises from the increase in energy of almost all of the particles without significant mixing: the particles moving more slowly than the wave contribute exactly the same both in size and sign as the particles moving faster than the wave; and during the damping, these particles move less than a wavelength.

The decay of the contribution from $\widehat{f}(\mathrm{x}, \mathrm{v}, t)$ follows from similar considerations.

$$\widehat{f}(\mathrm{x}, \mathrm{v}, t) \cong \left[f_k^0(\mathrm{v}) - (\mathrm{v} - \mathrm{v}_0) \frac{\partial f_k^0(\mathrm{v})}{\partial \mathrm{v}} \right] \cos[k(\mathrm{x} - \mathrm{v}t)].$$

The first term phase mixes in time when integrated over v unless there is a sharp resonance in $f_k^0(\mathrm{v})$ and the second term is out of phase with $E_0 \sin(kx - \omega t)$ when integrated over a wavelength.

IV. THE DISPERSION RELATION

The charge density in the wave frame is given by

$$e \int dv \delta f = -e \int dv \; \delta v \; \frac{\partial f_0}{\partial v}.$$

Writing δv as $\mathcal{I}m(eE_0/m)\exp(-\gamma t) e^{ikx} \{ \frac{e^{-i(kv+i\gamma)t} - 1}{-i(kv+i\gamma)} \}$ where $\mathcal{I}m$ means imaginary part, and then moving back to the laboratory frame we have

$$\delta f(\mathrm{x}, \mathrm{v}, \mathrm{t}) = -\frac{\mathrm{e}}{\mathrm{m}} \mathrm{E}_0 \exp(-\gamma \mathrm{t}) \mathcal{I}\mathrm{m} \, \mathrm{e}^{\mathrm{i}(\mathrm{kx} - \omega \mathrm{t})} \left\{ \frac{\mathrm{e}^{-\mathrm{i}(\mathrm{kv} - \omega + \mathrm{i}\gamma)\mathrm{t}} - 1}{\mathrm{i}[(\mathrm{kv} - \omega) + \mathrm{i}\gamma]} \right\} \frac{\partial \mathrm{f}_0}{\partial \mathrm{v}}.$$

The charge density is thus

$$e \int \delta f(\mathrm{x}, \mathrm{v}, \mathrm{t}) \mathrm{dv} = \frac{\mathrm{e}^2}{\mathrm{m}} \mathrm{E}_0 \exp(-\gamma \mathrm{t}) \; \mathcal{I}\mathrm{m} \; \mathrm{e}^{\mathrm{i}(\mathrm{kx} - \omega \mathrm{t})} \int \mathrm{dv} \left\{ \frac{1 - \mathrm{e}^{-\mathrm{i}(\mathrm{kv} - \omega + \mathrm{i}\gamma)\mathrm{t}}}{\mathrm{i}[(\mathrm{kv} - \omega) + \mathrm{i}\gamma]} \right\} \frac{\partial \mathrm{f}_0}{\partial \mathrm{v}}. \tag{16}$$

The integrand has a pole in the lower half plane at $\mathrm{v} = \omega/k - i(\gamma/k)$, and for the first term in the bracket in the integrand we choose the contour shown in Fig. 2(a), which gives

$$\frac{e^2}{m} E_0 \exp(-\gamma t) \left\{ \mathcal{P} \int dv \frac{\partial f_0}{\partial \mathrm{v}} \left(\frac{(k\mathrm{v} - \omega) \cos kx + \gamma \sin kx}{(k\mathrm{v} - \omega)^2} \right) - \frac{\pi}{k} \frac{\partial f_0}{\partial \mathrm{v}} \bigg|_{\mathrm{v} = \omega/k} \sin kx \right\}, \tag{17}$$

where $\mathcal{P}$ means principal part. For the second term we choose the contour shown in Fig. 2(b). For this contour the phase-mixing term, $e^{-i(kv-\omega)t}$, $\sim \exp(-Mt)$ along the lower leg of the contour, and we choose M sufficiently large so that this part can be neglected. The remaining part is

$$+\frac{e^2}{m}E_0\exp(-\gamma t)\frac{2\pi}{k}\frac{\partial f_0}{\partial \mathrm{v}}\Big|_{\mathrm{v}=\frac{\omega}{k}}\sin kx. \tag{18}$$

Combining the contributions the result is:

$$\begin{aligned}\nabla\cdot E &= kE_0\exp(-\gamma t)\cos kx = 4\pi e\int f(x,\mathrm{v},t)d\mathrm{v} - 4\pi e n_0\\ &= \omega_p^2 E_0\exp(-\gamma t)\left\{\mathcal{P}\int d\mathrm{v}\frac{\partial f_0}{\partial \mathrm{v}}\frac{(k\mathrm{v}-\omega)\cos kx+\gamma\sin kx)}{(k\mathrm{v}-\omega)^2+\gamma^2}+\frac{\pi}{k}\frac{\partial f_0}{\partial \mathrm{v}}\Big|_{v=\omega/k}\sin kx\right\}.\end{aligned} \tag{19}$$

Equating the coefficients of $\cos kx$, and $\sin kx$ separately to zero respectively gives the Landau dispersion relation and damping:

$$\omega^2=\omega_p^2\left(1+3\left(\frac{k\overline{\mathrm{v}}}{\omega_p}\right)^2\right),\qquad \frac{\gamma}{\omega}=-\frac{\pi}{2}\left(\frac{\omega_p}{k}\right)^2\frac{1}{n_0}\frac{\partial f_0}{\partial \mathrm{v}}\Big)_{\mathrm{v}=\omega/k}.$$

V. THE AVERAGE DISTRIBUTION FUNCTION

As discussed above the energy exchange between the electric field and the particles in the resonance region is a function of position and time. However, the spatial average of $f(x,\mathrm{v},t)$ can be used to provide a simple description of the energy exchange.

Integrating $df/dt=0$ over x gives

$$\frac{k}{2\pi}\int_0^{\frac{2\pi}{k}}dx\frac{df}{dt}(x,v,t)=\frac{k}{2\pi}\int_0^{\frac{2\pi}{k}}dx\frac{\partial f}{\partial t}+\frac{k}{2\pi}\int_0^{\frac{2\pi}{k}}dx\frac{e}{m}E\exp(-\gamma t)\sin kx\frac{\partial}{\partial \mathrm{v}}\left\{f_0-\delta v\frac{\partial f_0}{\partial \mathrm{v}}\right\}. \tag{20}$$

Denoting the first term on the right-hand side of Eq. (28) as $d\overline{f}_0/dt$ this gives

$$\frac{d\overline{f}_0}{dt}=-\frac{1}{2}(eE_0)^2/m\ \exp(-\gamma t)\frac{\partial}{\partial \mathrm{v}}\left\{\left[\frac{\gamma\exp(-\gamma t)-(k\mathrm{v}-\omega)\sin[(k\mathrm{v}-\omega)t]-\gamma\cos[(k\mathrm{v}-\omega)t]}{(\omega-k\mathrm{v})^2+\gamma^2}\right]\frac{\partial f_0}{\partial \mathrm{v}}\right\}, \tag{21}$$

Let $\mathcal{E}$ be the plasma kinetic energy, then

$$\begin{aligned}\frac{d\mathcal{E}}{dt}&=\int d\mathrm{v}\frac{1}{2}m\mathrm{v}^2\frac{d\overline{f}_0}{dt}=-\frac{1}{2}\left(\frac{e}{m}E_0\right)^2\exp(-\gamma t)\int d\mathrm{v}\frac{1}{2}m\mathrm{v}^2\frac{\partial}{\partial \mathrm{v}}\\ &\times\left\{\left[-\frac{\gamma\exp(-i\gamma t)-(k\mathrm{v}-\omega)\sin[(k\mathrm{v}-\omega)t]+\gamma\cos[(k\mathrm{v}-\omega)t]}{(\omega-k\mathrm{v})^2+\gamma^2}\right]\frac{\partial f_0}{\partial \mathrm{v}}\right\}.\end{aligned} \tag{22}$$

The integral over the bulk of the particles give $-\gamma(E_0^2/8\pi)\exp(-2\gamma t)$ and the integral over the resonance gives $2\gamma(E_0^2/8\pi)\exp(-2\gamma t)$. The net of these two terms gives

$$\frac{d\mathcal{E}}{dt}=-\frac{\partial}{\partial t}\frac{E_0^2}{16\pi}\exp(-2\gamma t) \tag{23}$$

which just balances the rate of decrease of the field energy, $(E_0^2/16\pi)\exp(-2\gamma t)$.

As is well known, the "wave energy" of a plasma wave is

$$\omega\left(\frac{\partial\epsilon(\omega)}{\partial\omega}\right)\left(\frac{|E_0|^2}{16\pi}\right)\cong\frac{2|E_0|^2}{16\pi}$$

where $\epsilon(\omega)$ is the dielectric function. This is shared equally by the electric field and the "sloshing energy" of the particles. Equation (21) describes the flow of this energy to the resonant particles as the wave decays.

VI. FINITE E_0 CONSIDERATIONS

The effect of finite values of E_0 can be determined by continuing the expansion which led to the order $(1/\Gamma)$ results above. This is done in the appendix. The result is that there are contributions to δv from terms of order $(1/\Gamma)^3$ and $(1/\Gamma)^5$. The terms of order $(\frac{1}{\Gamma})^3$ are spatial harmonics, e.g. $\sin(2kx)$, $\sin kx\ \cos(kx-kvt)$, etc., and are thus out of phase with $E_0\sin(kx)$, and do not lead to net energy transfer. The terms of order $(1/\Gamma)^5$, give a contribution to the energy transfer, which is equal to the contribution from the terms of order $1/\Gamma$ times the factor

$$\frac{-183+498\gamma t-510\gamma^2t^2+428\gamma^3t^3}{768\Gamma^4}+\frac{\exp(-2\gamma t)(237+12\gamma t+96\gamma^2t^2+128\gamma^3t^3)}{768\Gamma^4}.$$

For $\Gamma^4\gg 1$, these nonlinear corrections to δv thus produce a negligible change in the damping for times of physical interest.

Similarly, the diffusion of $\overline{f}_0(v,t)$ is equally negligible. The flattening of $f_0(v)$ can be estimated from Eq. (20). Writing this as

$$\frac{\partial\overline{f}_0}{\partial t}=\frac{\partial}{\partial \mathrm{v}}D\frac{\partial f_0}{\partial v} \tag{24}$$

and approximating D as

$$D\sim\left(\frac{eE}{m}\right)^2\frac{1}{\gamma}\sim\frac{(\Delta v)^2}{t}$$

where $(\Delta v)^2$ is the range of the flattening in a time, t, yields

$$\frac{(\Delta v)^2}{(\gamma/k)^2}=\left(k\frac{eE}{m}\right)^2\frac{1}{\Gamma^4}(\gamma t)=\frac{1}{\Gamma^4}(\gamma t).$$

Thus for $\gamma t\sim\mathcal{O}(1)$, the flattening is negligible.

Here we note that the approximation leading to Eq. (7) can be written as:

$$\frac{\delta v}{v} \cong \frac{eE}{m\gamma}\frac{k}{\omega} = \left(\frac{\omega_0}{\gamma}\right)^2 \frac{\gamma}{\omega} \ll \left(\frac{\overline{v}}{v}\right)^2 = \left(\frac{k\overline{v}}{\omega}\right)^2$$

which gives $\frac{1}{\Gamma^2} \ll \frac{\omega}{\gamma}(\frac{k\overline{v}}{\omega})^2$, and this is easily satisfied.

VII. EARLIER WORK

Landau damping is discussed in many papers and in most plasma physics textbooks, however, since it makes use of the linearized Vlasov equation, Landau's paper does not provide any information about the effect of finite wave amplitude. With respect to the nonlinear effects several authors have suggested that the nonlinear corrections should depend on the smallness of $1/\Gamma = \omega_0/\gamma$. However these works have been largely qualitative in nature and there appears to be no systematic calculation which leads to a specific dependence on this parameter.

Similarly, the Landau paper does not provide a physical picture of the contribution of particles of different velocities to the damping. Aside from a detailed calculation by Dawson [3], the earlier discussions of the energy transfer to or from particles of different velocities are mostly of a qualitative nature and lead to similarly qualitative descriptions of the origin of the damping.

One such description, sometimes called the "surfer picture" [3–6] is that in a damping wave, particles moving slightly slower than the wave tend to be speeded up, and particles moving slightly faster than the wave tend to be slowed down. The damping then results from the fact that since $\partial f_0/\partial \mathrm{v} < 0$ there are more particles gaining energy from the wave than there are particles losing energy to the wave. This is in direct contradiction to the results given in Sec. III where it is shown that both the particles with velocity less than the wave velocity and the particles with velocity greater than the wave velocity gain energy.

Something akin to the "surfer picture" is, however meaningful for large amplitude plasma waves. As discussed in Sec. II, O'Neil [2] has considered the damping of large amplitude plasma waves, i.e., the limit, $\gamma/\omega_0 \ll 1$. In this limit particles with $|v| < \omega_0/k$ are trapped. The physical picture developed by O'Neil is that the bulk of the energy loss of the wave is due to the net increase in energy of these particles as their phase space orbits in the wave frame change from straight-line orbits to rotational orbits, and then phase mix. In this process particles initially going slower than the wave are speeded up and particles going faster than the wave are slowed, which is consistent with the "surfer picture." By contrast, in the present paper we are dealing with the limit $\gamma/\omega_0 \rightarrow \infty$, i.e., the limit of $E \rightarrow 0$, and there are no trapped particles.

A similar picture [7] envisions a mixing of the slower and faster particles. This is in direct contradiction to the results in Sec. III where it is shown that there is virtually no mixing of the slower and faster particles.

Another picture [8, 9] is that the damping results from the oscillation of trapped particles in a damping wave. This is in direct contradiction to the results of Sec. III where it is shown that the particles which contribute to the damping do not oscillate and move much less that a wavelength during a damping time $1/\gamma$.

Dawson's calculation involves an integration along the linearized particle orbit, however he does not include the damping in these orbits. In effect, he finds for the in-phase terms $\delta v \sim \frac{kv \sin kvt}{(kv)^2}$, rather than $\delta v \sim \frac{-\gamma \exp(-\gamma t)+kv \sin(kvt)+\gamma \cos(kvt)}{(kv)^2+\gamma^2}$. The result is that he finds a resonance of the form $\frac{\sin(kvt)}{kv}$ which has a half-width of π/kt, rather than a resonance of the form $\frac{\gamma}{(kv)^2+\gamma^2}$ with a half-width of γ/k. The integral over velocity of this gives $\frac{\pi eE}{m}$ rather than $\frac{\pi eE}{m}\exp(-\gamma t)$.

It was shown in Sec. II that for $(1/\Gamma)^4 \ll 1$, the linearized theory is correct and the orbits are essentially straight lines in phase space. But for $\Gamma \rightarrow 0$, the orbits of particles with small velocity in the wave frame are oscillatory, i.e. there are trapped particles, and the linearized theory is not applicable. Thus Dawson's use of the linearized theory with $\Gamma \rightarrow 0$ is inconsistent with the equations of motion for the particles with velocities near the wave velocity.

VIII. GROWING WAVES

In the foregoing only the case of damping was considered. However, it is well known that if the distribution function has a gentle bump on the tail, i.e., a range of velocities in which $\partial f_0/\partial \mathrm{v} > 0$, plasma waves will grow. Although the growth rate is given by the same formula there is a significant difference in the physical interactions. In particular, in Eq. (13), if the sign of γ is negative, the $\gamma \exp(-\gamma t)$ term will have a growing exponential coefficient, whereas the $kv\sin kvt$ and $\gamma\cos kvt$ terms cancel each other when integrated over the resonance. This is reflected in the calculation of the dispersion relation. For $\gamma < 0$ the pole is above the axis and the contour of integration of the first term [Fig. 2(a)] must be a semi-circle going *under* the pole. For the second term [Fig. 2(b)], $\gamma < 0$ leads to the entire contour being along the line $k\mathrm{v} = -iM$, and this entire term drops out. This removes the oscillatory time dependence from $f(x, \mathrm{v}, t)$ and from the averaged distribution function, $\overline{f}_0$. The result is that Eq. (20) becomes a simple diffusion equation as in the quasilinear theory [10] leading to a flattening of the velocity distribution for a growing wave.

$$\frac{\partial\overline{f}}{\partial t} = \frac{\partial}{\partial \mathrm{v}} D(\mathrm{v}) \frac{\partial\overline{f}}{\partial \mathrm{v}} \qquad \text{where} \quad D(\mathrm{v}) = \frac{1}{2}\left(\frac{eE_0}{m}\exp(-\gamma t)\right)^2 \frac{(-\gamma)}{(\omega - k\mathrm{v})^2 + \gamma^2}. \tag{25}$$

IX. EXPERIMENTAL RESULTS

Predictions based on the Landau theory for both damping and growth of plasma waves have been experimentally tested by Malmberg *et al.* [11] and by Roberson *et al.* [12] The results show good agreement over a wide range of parameters. The amplitude bound, $(\frac{1}{\Gamma})^4 \ll 1$, is not very restrictive but should be measurable. Both of these experiments were done with plasma temperatures of about 10eV and waves for which $\omega/k\overline{\mathrm{v}}$ was in the range of 3.5 to 4. From $(\frac{1}{\Gamma})^4 \ll 1$, which can be written as

$$\left(\frac{eV_0}{\kappa T}\right)^2 \ll \left[\frac{\pi}{8}\left(\frac{\omega}{kv}\right)^8 e^{-(\frac{\omega}{k\overline{\mathrm{v}}})^2}\right]^2$$

where T is the plasma temperature and κ is the Boltzman constant, this would require that 2 millivolt $< V_0 <$ 200 millivolt. If the criterion $1/\Gamma \ll 1$, suggested in the literature, had been used, these bounds would be reduced by an order of magnitude. Unfortunately neither of these experimental papers report on the size of V_0, although it was estimated in Ref. [8] to be in the 100 MeV range.

X. SUMMARY

The two principal results are the bound on E_0, given by $(\frac{1}{\Gamma})^4 \ll 1$, and the energy exchange picture developed in Sec. III. It is worth noting that the pedagogical steps leading to these results are both simple and few. To obtain the energy bound requires three steps: 1) recognize that the well-known result, $f(\mathrm{x}, \mathrm{v}, t) = f_0(\mathrm{x}_0, \mathrm{v}_0)$, when applied to the initial value problem at hand, leads to the nonlinear result: $\delta f = -(\mathrm{v}-\mathrm{v}_0)\frac{\delta f_0}{d\mathrm{v}}$; 2) recognize that $\mathrm{v}-\mathrm{v}_0$ depends on the constants, $k, eE_0/m$, and γ, only through the ratio, $\gamma/\omega_0 = \Gamma$, and note that $\Gamma \gg 1$ for the Landau problem; and 3) show that the corrections to $\mathrm{v}-\mathrm{v}_0$ are of fifth order in $\frac{1}{\Gamma}$ insofar as they affect the damping, a simple matter if this term is not explicity evaluated.

Since this shows that the linearized theory is correct for $(1/\Gamma)^4 \ll 1$, the energy exchange picture is obtained by simply integrating the linearized equations and examining the energy transfer. The result, discussed in detail in Sect. III, is that essentially all the particles with velocities near the phase velocity gain energy during the damping without significant mixing of the particles in phase space. This energy comes equally from the electric field and the "sloshing" energy of the bulk of the particles moving in the field.

The remaining sections simply fill out this physical picture, and attempt to put this work in perspective.

Acknowledgment

The author would like to thank the students in his plasma seminar course whose persistent questions led to the present work.

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Appendix — Evaluation of the Higher-Order Terms

We wish to solve Eqs. (a-1) and (a-1) by expanding in powers of $1/\Gamma$, where $\Gamma \gg 1$

$$\dot{\boldsymbol{v}} = \exp(-\Gamma \boldsymbol{t}) \sin \boldsymbol{x} \tag{a-1}$$

$$\dot{\boldsymbol{x}} = \boldsymbol{v}.$$

Let $\boldsymbol{v} = \boldsymbol{v}_0 + \delta \boldsymbol{v}, \quad \boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{v}_0 \boldsymbol{t} + \delta \boldsymbol{x}$. Then

$$\delta \dot{\boldsymbol{v}} = \exp(-\Gamma \boldsymbol{t}) \sin(\boldsymbol{x}_0 + \boldsymbol{v}_0 \boldsymbol{t} + \delta \boldsymbol{x}) \tag{a-2}$$

$$\delta \dot{\boldsymbol{x}} = \delta \boldsymbol{v}.$$

We anticipate that $\delta \overline{\boldsymbol{v}}$ will have terms of order $(1/\Gamma), (1/\Gamma)^3, (1/\Gamma)^5 + ...$ and set $\delta \boldsymbol{v} = \delta \boldsymbol{v}_1 + \delta \boldsymbol{v}_3 + \delta \boldsymbol{v}_5 + ...$ Similarly we anticipate that $\delta \boldsymbol{x} = \delta \boldsymbol{x}_2 + \delta \boldsymbol{x}_4 + ...$ From (a-2)

$$\delta \boldsymbol{x} = \int_0^{\boldsymbol{t}} d\boldsymbol{t}' \delta \boldsymbol{v} = \int_0^{\boldsymbol{t}} d\boldsymbol{t}' \left(\delta \boldsymbol{v}_1 + \delta \boldsymbol{v}_3 + ... \right) = \delta \boldsymbol{x}_2 + \delta \boldsymbol{x}_4 ... \ .$$

where $\delta \boldsymbol{x}_2 = \int_0^{\boldsymbol{t}} d\boldsymbol{t}' \delta \boldsymbol{v}_1$ and $\delta \boldsymbol{x}_4 = \int_0^{t} d\boldsymbol{t}' \delta \boldsymbol{v}_3$. From (a-2)

$$\begin{aligned}
\delta v &= \int_0^{\boldsymbol{t}} d\boldsymbol{t}' \exp(-\Gamma \boldsymbol{t}') \sin\left[\boldsymbol{x}_0 + \boldsymbol{v}_0 \boldsymbol{t}' + \delta \boldsymbol{x}_2 + \delta \boldsymbol{x}_4\right] \\
&= \int_0^{\boldsymbol{t}} d\boldsymbol{t}' \exp(-\Gamma \boldsymbol{t}') \sin(\boldsymbol{x}_0 + \boldsymbol{v}_0 \boldsymbol{t}') + \int^{\boldsymbol{t}} d\boldsymbol{t}' \exp(-\Gamma \boldsymbol{t}') (\delta \boldsymbol{x}_2 + \delta \boldsymbol{x}_4) \cos(\boldsymbol{x}_0 + \boldsymbol{v}_0 \boldsymbol{t}') \\
&\quad - \frac{1}{2} \int_0^{\boldsymbol{t}'} d\boldsymbol{t}' \exp(-\Gamma \boldsymbol{t}') (\delta \boldsymbol{x}_2 + \delta \boldsymbol{x}_4)^2 \cos(\boldsymbol{x}_0 + \boldsymbol{v}_0 \boldsymbol{t}'), \quad \text{which gives}
\end{aligned}$$

$$\delta \boldsymbol{v}_1 = \int_0^{\boldsymbol{t}} d\boldsymbol{t}' \exp(-\Gamma \boldsymbol{t}') \sin(\boldsymbol{x}_0 + \boldsymbol{v}_0 \boldsymbol{t}'), \tag{a-3}$$

$$\begin{aligned}
\delta \boldsymbol{v}_3 &= \int_0^{\boldsymbol{t}} d\boldsymbol{t}' \exp(-\Gamma \boldsymbol{t}') \delta \boldsymbol{x}_2 \cos(\boldsymbol{x}_0 + \boldsymbol{v}_0 \boldsymbol{t}') \\
\delta \boldsymbol{v}_5 &= \int_0^{\boldsymbol{t}} d\boldsymbol{t}' \exp(-\Gamma \boldsymbol{t}') \left(\delta \boldsymbol{x}_4 \cos(\boldsymbol{x}_0 + \boldsymbol{v}_0 \boldsymbol{t}') - \frac{1}{2} (\delta \boldsymbol{x}_2)^2 \sin(\boldsymbol{x}_0 + \boldsymbol{v}_0 \boldsymbol{t}') \right).
\end{aligned}$$

Using $\delta \boldsymbol{v}_1$ we find

$$\delta \boldsymbol{x}_2 = \int_0^{\boldsymbol{t}} d\boldsymbol{t}' \int_0^{\boldsymbol{t}'} d\boldsymbol{t}'' \exp(-\Gamma \boldsymbol{t}') \sin(\boldsymbol{t}_0 + \boldsymbol{v}_0 \boldsymbol{t}'')$$

which gives

$$\delta \boldsymbol{v}_3 = \int_0^{\boldsymbol{t}} d\boldsymbol{t}' \exp(-\Gamma \boldsymbol{t}') \int_0^{\boldsymbol{t}'} d\boldsymbol{t}'' \int_0^{\boldsymbol{t}''} d\boldsymbol{t}^{(3)} \exp(-\Gamma \boldsymbol{t}^{(3)}) \sin(\boldsymbol{x}_0 + \boldsymbol{v}_0 t^{(3)}) \cos(\boldsymbol{x}_0 + \boldsymbol{v}_0 \boldsymbol{t}')). \tag{a-4}$$

this gives $\delta \boldsymbol{x}_4$

$$\delta\boldsymbol{x}_4 = \int_0^{\boldsymbol{t}} d\boldsymbol{t}' \int_0^{\boldsymbol{t}'} d\boldsymbol{t}'' \exp(-\Gamma\boldsymbol{t}'') \int_0^{\boldsymbol{t}''} d\boldsymbol{t}^{(3)} \int_0^{\boldsymbol{t}^{(3)}} d\boldsymbol{t}^{(4)} \exp(-\Gamma\boldsymbol{t}^{(4)}) \sin(\boldsymbol{x}_0 + v_0\boldsymbol{t}(4)) \cos(\boldsymbol{x}_0 + \boldsymbol{v}_0\boldsymbol{t}'')$$

and

$$\begin{aligned} \delta\boldsymbol{v}_5 = \int_0^{\boldsymbol{t}} d\boldsymbol{t}' \exp(-\Gamma\boldsymbol{t}') \int_0^{\boldsymbol{t}'} d\boldsymbol{t}'' \int_0^{\boldsymbol{t}''} d\boldsymbol{t}^{(3)} \exp(-\Gamma\boldsymbol{t}^{(3)}) \int_0^{\boldsymbol{t}^{(3)}} d\boldsymbol{t}^{(4)} \int_0^{\boldsymbol{t}^{(4)}} d\boldsymbol{t}^{(4)} \\ \times \exp(\Gamma\boldsymbol{t}^{(4)}) \sin(\boldsymbol{x}_0 + \boldsymbol{v}_0\boldsymbol{t}^{(5)} \cos(\boldsymbol{x}_0 + \boldsymbol{v}_0\boldsymbol{t}^{(3)}) \cos(\boldsymbol{x}_0 + \boldsymbol{v}_0\boldsymbol{t}') \\ -\frac{1}{2}\int_0^{\boldsymbol{t}} d\boldsymbol{t}' \exp(-\Gamma\boldsymbol{t}') \left\{ \int_0^{\boldsymbol{t}'} d\boldsymbol{t}'' \int_0^{\boldsymbol{t}''} d\boldsymbol{t}^{(3)} \exp(-\Gamma\boldsymbol{t}^{(3)}) \sin(\boldsymbol{y}_0 + \boldsymbol{v}_0\boldsymbol{t}^{(3)}) \right\}^2 \sin(\boldsymbol{x}_0 + \boldsymbol{v}_0\boldsymbol{t}') \end{aligned} \tag{a-5}$$

We now must express δv in terms of x and v, rather than x_0 and v_0. We have

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{x}_0 + \boldsymbol{v}_0\boldsymbol{t} + \boldsymbol{x}_2 + \boldsymbol{x}_4 \\ \boldsymbol{v} &= \boldsymbol{v}_0 + \boldsymbol{v}_1 + \boldsymbol{v}_3 + \boldsymbol{v}_5. \end{aligned}$$

Thus $(\boldsymbol{x} + \boldsymbol{v}_0\boldsymbol{t}') = \boldsymbol{x} - \boldsymbol{v}(\boldsymbol{t} - \boldsymbol{t}') - \boldsymbol{\eta}$, where

$$\boldsymbol{\eta} = \{\boldsymbol{x}_2 + \boldsymbol{x}_4 - (\boldsymbol{v}_1 + \boldsymbol{v}_2)(\boldsymbol{t} - \boldsymbol{t}')\}\,. \tag{a-6}$$

To get $(\boldsymbol{x} + \boldsymbol{v}_0 t')$ to 4th-order $\boldsymbol{x}_2$ and $\boldsymbol{v}_1$ must be expressed in terms of $\boldsymbol{x}$ and $\boldsymbol{v}$ which gives

$$\begin{aligned} \boldsymbol{x}_2 &= \widetilde{\boldsymbol{x}}_2 - \widetilde{\boldsymbol{x}}_2 \int_0^{\boldsymbol{t}} d\boldsymbol{t}' \int_0^{\boldsymbol{t}'} dt'' \exp(-\Gamma/\boldsymbol{x}\ \boldsymbol{t}'') \cos(\boldsymbol{x} - \boldsymbol{v}(\boldsymbol{t} - \boldsymbol{t}'')) \\ &+ \widetilde{\boldsymbol{v}}_1 \int_0^{\boldsymbol{t}} d\boldsymbol{t}' \int_0^{\boldsymbol{t}'} d\boldsymbol{t}'' \exp(-\Gamma/\boldsymbol{x}\ \boldsymbol{t}''(\boldsymbol{t} - \boldsymbol{t}'') \cos(\boldsymbol{x} - \boldsymbol{v}(\boldsymbol{t} - \boldsymbol{t}''))) \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{v}_1 &= \widetilde{\boldsymbol{v}}_1 - \widetilde{\boldsymbol{x}}_2 \int_0^{\boldsymbol{t}} d\boldsymbol{t}' \exp(-\Gamma/\gamma\ \boldsymbol{t}') \cos(\boldsymbol{x} - \boldsymbol{v}(\boldsymbol{t} - \boldsymbol{t}')) \\ &+ \boldsymbol{v}_1 \int_0^{\boldsymbol{t}} d\boldsymbol{t}' \exp(-\Gamma/\gamma\ \boldsymbol{t}')(\boldsymbol{t} - \boldsymbol{t}') \cos(\boldsymbol{x} - \boldsymbol{v}(\boldsymbol{t} - \boldsymbol{t}')) \end{aligned}$$

where $\widetilde{\boldsymbol{x}}_2$, $\widetilde{\boldsymbol{v}}_1$ denote $\boldsymbol{x}_2$ and $\boldsymbol{v}_2$, expressed in terms of $\boldsymbol{x}$ and $\boldsymbol{v}$, and $\boldsymbol{x}_4 = \widetilde{\boldsymbol{x}}_4$, $\boldsymbol{v}_3 = \widetilde{\boldsymbol{v}}_3$ to the same order.

Proceeding in this way we get about a dozen terms of order $(\frac{1}{\Gamma})^3$ and $(\frac{1}{\Gamma})^5$.

The 3rd-order terms involve the product of two sine or cosine functions equations, e.g.:

$$\widetilde{\boldsymbol{x}}_2 \int_0^{\boldsymbol{t}} d\boldsymbol{t}'' \exp(-\gamma\boldsymbol{t}'') \cos(\boldsymbol{x} - \boldsymbol{v}(\boldsymbol{t} - \boldsymbol{t}')) \qquad \text{is}$$

$$\int_0^{\boldsymbol{t}} d\boldsymbol{t}' \int_0^{\boldsymbol{t}'} d\boldsymbol{t}' \exp(\gamma\boldsymbol{t}') \sin(\boldsymbol{x} - \boldsymbol{v}(\boldsymbol{t} - \boldsymbol{t}')) \int_0^{\boldsymbol{t}} d\boldsymbol{t}'' \exp(-\gamma\boldsymbol{t}'') \cos(\boldsymbol{x} - \boldsymbol{v}(\boldsymbol{t} - \boldsymbol{t}'').$$

We are interested in the energy transfer, $\sim \int_0^{2\pi} E_0 \sin(\boldsymbol{x})\delta\boldsymbol{v}d\boldsymbol{x}$ and for the above terms

$$\int_0^{2\pi} sin\boldsymbol{x}\ \sin(\boldsymbol{x}-\boldsymbol{v}(\boldsymbol{t}-\boldsymbol{t}'))\cos(\boldsymbol{t}-\boldsymbol{v}(\boldsymbol{t}-\boldsymbol{t}'))d\boldsymbol{x}=0.$$

This is true for all $\delta\widetilde{\boldsymbol{v}}$ terms of 3rd order. Only the 1st- and 5th-order terms remain.

The 5th-order terms can be evaluated in a straightforward manner. To see this, consider the term:

$$\delta\widehat{\boldsymbol{v}}_5=\int_0^{\boldsymbol{t}} d\boldsymbol{t}'\exp(-\Gamma\boldsymbol{t}')\int_0^{\boldsymbol{t}'} d\boldsymbol{t}''\int_0^{\boldsymbol{t}''} d\boldsymbol{t}^{(3)}\exp(-\Gamma\boldsymbol{t}^{(3)})\int_0^{\boldsymbol{t}^{(3)}} d\boldsymbol{t}^{(4)}\int_0^{\boldsymbol{t}^{(4)}} d\boldsymbol{t}^{(5)}$$
$$\times\exp(-\Gamma\boldsymbol{t}^{(5)})\sin\left(\boldsymbol{x}-\boldsymbol{v}(\boldsymbol{t}-\boldsymbol{t}^{(5)}\right)\cos\left(\boldsymbol{x}-\boldsymbol{v}(\boldsymbol{t}-\boldsymbol{t}^{(3)}\right)\cos\left(\boldsymbol{x}-\boldsymbol{v}(\boldsymbol{t}-\boldsymbol{t}')\right).$$

Multiplying this by $\sin(\boldsymbol{x})$ and integrating over $\boldsymbol{x}$ the trigonometric terms gives

$$\frac{\pi}{4}\left\{\cos\left[\boldsymbol{v}(\boldsymbol{t}+\boldsymbol{t}'-\boldsymbol{t})^{(3)}-\boldsymbol{t}^{(5)}\right]+\cos\left[\boldsymbol{v}(\boldsymbol{t}-\boldsymbol{t}'+\boldsymbol{t})^{(3)}-\boldsymbol{t}^{(5)}\right]+\cos\left[\boldsymbol{v}(\boldsymbol{t}-\boldsymbol{t}'-\boldsymbol{t}^{(3)}+\boldsymbol{t}^{(5)}\right]\right\}.$$

The remaining integrals are straightforward; however, there are a large numbers of terms. These, however, can be simply handled by the use of Mathematica. A typical term is

$$\frac{e^{itv}\pi\left(2v^5-2iv^4\Gamma-v^3\Gamma^2-3iv^2-3iv^2\Gamma^3+v\Gamma^4+3i\Gamma^5\right)}{16(iv-\Gamma)^3(v-i\Gamma)^4(v-3i\Gamma)\Gamma^2}$$

which has a strong resonance of $v=0$ with a half-width of Γ. These can also be evaluated with the use of Mathematica and the result for $\delta\widehat{\boldsymbol{v}}_5$ is

$$\frac{3\pi^2\exp(-3\gamma t)+\pi^2\exp(-\gamma t)(-3+6\gamma t-6(\gamma t)^2+4(\gamma t)^3)}{96\Gamma^4}.$$

Here we see that when integrated over v, the term, which is of order $(\frac{1}{\Gamma})^5$ is lowered to the order $(\frac{1}{\Gamma})^4$. The numerator contains only powers of $\exp[-\gamma t]$ and of γt. This is true of all such terms. Evaluating and adding all of the 5th-order terms leads to a factor

$$\pi\exp(-\gamma t)\left(\frac{(-183+498\gamma t-510\gamma^2t^2+428\gamma^3t^3}{768\Gamma^4}+\frac{\exp(-2\gamma t)(237+12\gamma t+96\gamma^2t^2+128\gamma^3t^3}{768\Gamma^4}\right)$$

which is to be compared with the 1st-order result, $\pi\exp(-\gamma t)$.

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FIGURE CAPTIONS

FIG. 1. Relative power transfer in the resonant region as a function of kv/γ for $\gamma t = .1, .5, 1$.

FIG. 2. (a)-(b) Contours for evaluation of the integrals leading to Eqs. (16) and (17).

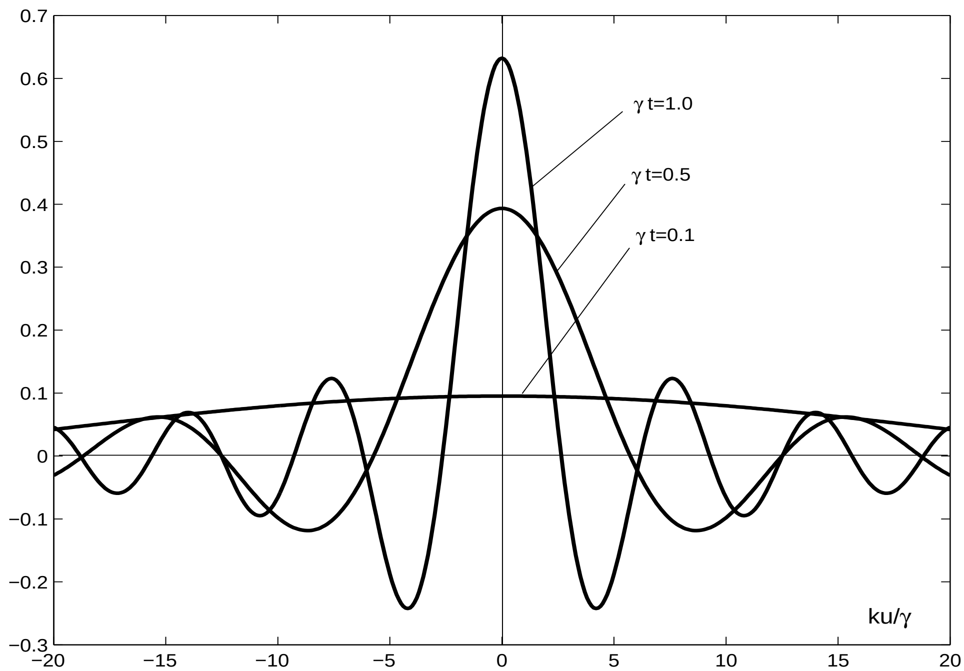
γ t=1.0
γ t=0.5
γ t=0.1
ku/γ
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−20
−15
−10
−5
0
5
10
15
20

Fig. 1

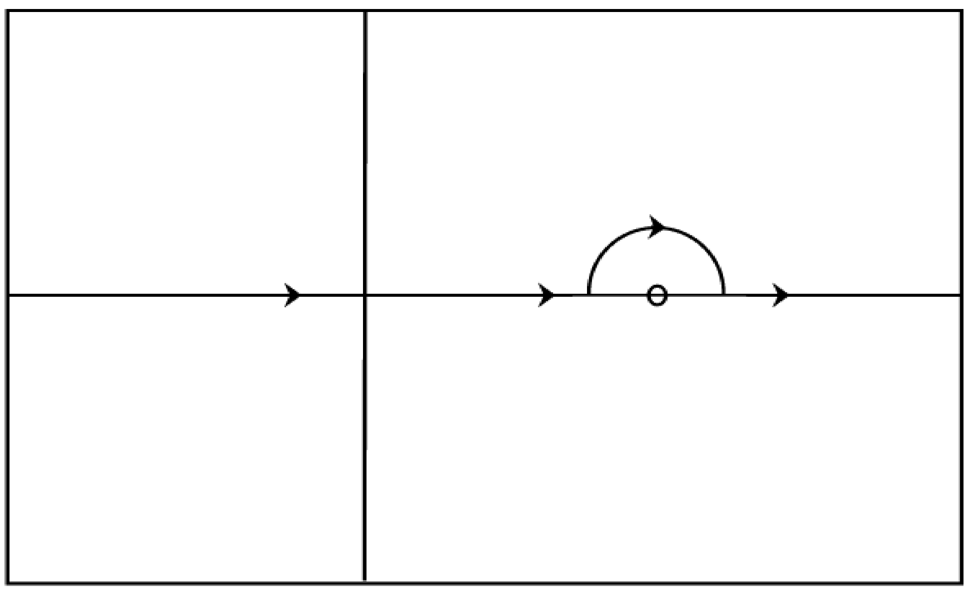

Fig. 2a

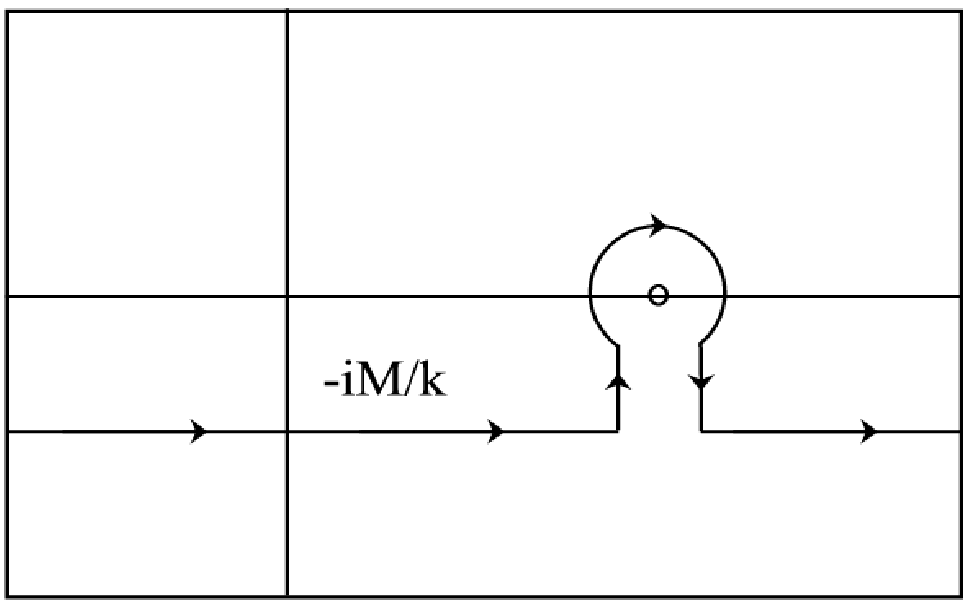

Fig. 2b