Relativistic magnetohydrodynamics

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Abstract

The lowest-order description of a magnetized plasma is given by magnetohydrodynamics (MHD). However, many astrophysical plasmas (as well as some laboratory plasmas) are relativistic, in that either the thermal speed or the fluid flow speed approaches the speed of light, and conventional MHD is not consistent with special relativity. Beginning with exact laws of motion, this work derives a generalization of MHD that is Lorentz covariant and therefore applicable to relativistic plasma. The resulting closed set of fluid equations is then seen to reduce to conventional MHD in the nonrelativistic limit.

keywords: stars: atmospheres, galaxies: jets, methods: analytical, physical data and processes, relativistic plasmas

I. INTRODUCTION

Relativistic plasma

A relativistic plasma is one in which either the thermal speed—the *rms* speed of individual particles—measured in the fluid rest frame, or the local bulk flow measured in some convenient frame, can approach the speed of light. Various astrophysical and cosmic plasmas (galactic and extra-galactic plasma jets [1], electron-positron streams in pulsar atmospheres and in the accretion disks of active galactic nuclei [2], the Mev era of the early universe) [3] as well as hot electrons in some laboratory experiments (especially fusion experiments), are relativistic in this sense.

Many relativistic plasmas of interest are magnetized—that is, their dynamics is dominated by the magnetic field. (A general definition of "magnetized" is given below.) However, the conventional description of magnetized plasma dynamics, magnetohydrodynamics (MHD), is not consistent with special relativity.

The object of this work is to derive, beginning with exact laws of motion, a generalization of MHD that is Lorentz covariant and therefore applicable to relativistic plasmas. We do not begin with the existing theory and then somehow "relativize" it, but rather derive *ab initio* a closed fluid description of a magnetized system. This closure is then seen to reduce to conventional MHD in the nonrelativistic limit.

The manifestly covariant version of MHD appears to be new, and we find it illuminating. Moreover, when the equations are expressed in a more conventional, 3-vector form, relativistic corrections appear that seem potentially important.

Finally we find the covariant derivation of MHD to be simpler and more transparent than the nonrelativistic version found in textbooks. In particular the relativistic derivation suggests natural avenues for the generalization of MHD, including pressure anisotropy and finite gyroradius physics. Such generalization will be the subject of future work.

Electromagnetic coupling

A plasma is distinguished from other physical systems by its strong coupling to the electromagnetic field. This coupling enters a fluid description through the second moment equation, the conservation law for energy momentum [4]

$$\frac{\partial T^{\mu\nu}}{\partial x^{\nu}} - F^{\mu\nu} J_{\nu} = 0 \tag{1}$$

where $T^{\mu\nu}$ is the (total) plasma energy-momentum tensor, $F^{\mu\nu}$ is the electromagnetic field strength tensor, or Faraday tensor, and J_{ν} is the four-vector current.

All fluid descriptions of *magnetized* plasma evolution use this second moment to compute the four-vector current in terms of the fields, and thus to provide closure relations for Maxwell's equations:

$$\frac{\partial F^{\mu\nu}}{\partial x^{\nu}} = J^{\mu} \tag{2}$$

$$\frac{\partial F_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial F_{\beta\gamma}}{\partial x^{\alpha}} + \frac{\partial F_{\gamma\alpha}}{\partial x^{\beta}} = 0. \tag{3}$$

In other words the current is computed in terms of the plasma energy-momentum tensor. Therefore Maxwell's equations are closed by expressing that tensor in terms of the fields.

Magnetohydrodynamics (MHD) is the simplest such closure. Our objective is to derive a fully covariant version of MHD. First however it is necessary to define the word "magnetized" in an Lorentz-invariant way.

Magnetized system

A plasma system is magnetized if two criteria are satisfied:

1. The two electromagnetic field invariants, $W \equiv B^2 - E^2$ and

$$\lambda \equiv \frac{\boldsymbol{E} \cdot \boldsymbol{B}}{W} \tag{4}$$

must satisfy

$$W > 0 \tag{5}$$

$$\lambda \ll 1. \tag{6}$$

Here of course $\boldsymbol{E}$ and $\boldsymbol{B}$ are respectively the electric and magnetic fields.

2. The thermal gyroradius must be small compared to any gradient scale length:

$$\delta \ll 1 \tag{7}$$

where δ is the ratio of the thermal gyroradius of any plasma species to any gradient scale length.

Note that a "thermal gyroradius" exists only for plasmas in the vicinity of thermal equilibrium, with roughly Maxwellian distributions. A convenient ordering turns out to be $\lambda \sim \delta$.

II. FLUID CLOSURE

Moment equations

To avoid trivial complications involving mass ratios, we assume a two-species plasma of ions and electrons, with disparate masses. Then the total mass density $\rho \approx \rho_i$ and center of mass motion can be approximated by that of the ions. In particular the relativistic γ-factor is computed from the ion velocity.

Consider the two lowest moments of the kinetic equation for the plasma ions:

$$\frac{\partial \Gamma^\nu}{\partial x^\nu} = 0 \tag{8}$$

$$\frac{\partial T_i^{\mu\nu}}{\partial x^\nu} - qF^{\mu\nu}\Gamma_\nu = 0 \tag{9}$$

where the flow vector Γ^μ and the energy-momentum tensor $T_i^{\mu\nu}$ are the first two moments of the ion distribution function. The tensor appearing in (1) is of course the sum of T_i and the corresponding electron tensor. Equation (9) omits collisional effects (friction force), as is appropriate when the gyrofrequency exceeds the collision frequency.

Note also that

$$\Gamma^\mu = n_R U^\mu = n_R \gamma V^\mu \tag{10}$$

where n_R is the (scalar) density measured in the ion rest-frame, $U^\mu = (\gamma, \gamma \boldsymbol{V})$ is the four-velocity of the fluid, and $\boldsymbol{V}$ is the ordinary three-vector flow velocity. It follows that

$$\gamma^2 = 1 + \mathbf{\Gamma} \cdot \mathbf{\Gamma} / n_{R^2}. \tag{11}$$

Significance of vanishing λ

If λ were finite, we could turn (1) into an equation for J^μ using the identity

$$\mathcal{F}_{\mu\kappa} \mathcal{F}^{\kappa\nu} = \eta_\mu{}^\nu \lambda \mathcal{W} \tag{12}$$

where $\eta_\mu{}^\nu$ is the Minkowski metric tensor and $\mathcal{F}_{\mu\kappa}$ is the dual Faraday tensor. Thus the Faraday tensor and its dual are matrix inverses. However in the magnetized, $\lambda \to 0$, case the relation (12) plays a very different role. It no longer provides a useful inverse, because of the small denominator that would occur. Indeed, in the magnetized case $\mathcal{F}$ is evidently an *annihilator* for F, rather than its inverse. The fact that both the Faraday tensor and its inverse have null spaces (which turn out to be two-dimensional) in a magnetized plasma plays an important role in MHD closure.

Energy-momentum tensor

We begin with the thermodynamic form,

$$T^{\mu\nu} = p\eta^{\mu\nu} + \frac{h}{n_R^2} \Gamma^\mu \Gamma^\nu \tag{13}$$

where p is the total plasma pressure and h the enthalpy density.

We recall the thermodynamic relation, characterizing a Maxwellian distribution,

$$h = 4p + \rho \tag{14}$$

where ρ, the mass scalar density, is given by

$$\rho = mn_R \frac{K_1(z)}{K_2(z)}. \tag{15}$$

Here K_1 and K_2 are Macdonald functions, and

$$z \equiv m/T = mn_R/p$$

is the ratio of rest energy to thermal energy.

The use of the thermodynamic equilibrium form for $T^{\mu\nu}$ is the most questionable and limiting assumption of MHD. It can be justified approximately when the collision frequency is large, and even for small collisionality if the plasma is in some sense confined.

In view of (14), (15) and the fact that

$$\Gamma^0 = \gamma n_R$$

the expression for $T^{\mu\nu}$ contains five independent variables: h, p, and three components of Γ^μ. Hence we need five equations to close the system. One of the five is particle conservation, (8). The other four are approximate, and use the fact that the system is magnetized.

First we exploit magnetization of the *field*: $\lambda \to 0$. After multiplying (1) by $\mathcal{F}$ and using (12) we find that

$$\mathcal{F}_{\mu\kappa} \frac{\partial \mathcal{T}^{\kappa\nu}}{\partial \S^\nu} = \prime \tag{16}$$

which constitutes two independent equations; these will be seen to determine the pressure and the parallel component of Γ. Second, we use the fact that the *plasma* is magnetized to conclude from (9), in lowest $\delta-$order, that

$$F_{\mu\nu}\Gamma^\nu = 0. \tag{17}$$

Note that the ion flow has been identified with the mass flow appearing in (13), as noted previously. Again there are two independent equations, specifying essentially the components of Γ perpendicular to the magnetic field.

Thus we have five equations for the five unknowns in the energy-momentum tensor.

III. MANIFESTLY COVARIANT MHD

Plasma current

Here we show explicitly how the energy-momentum tensor determines the plasma current, thus closing the system.

First we introduce the quantity

$$e^{\mu\nu} \equiv -\frac{F^{\mu\kappa}F_{\kappa}^{\ \nu}}{W}.$$

It is straightforward to show that

$$F^{\mu\kappa}e_{\kappa}^{\ \nu} = F^{\mu\nu} - \lambda\mathcal{F}^{\mu\nu} \tag{18}$$

and therefore that $e^{\mu\nu}$ becomes a projection operator in the magnetized limit:

$$e^{\mu\kappa}e_{\kappa\nu} = e^{\mu}_{\ \nu} + \mathcal{O}(\lambda^{\epsilon}).$$

Indeed $e^{\mu\nu}$ projects onto the subspace perpendicular to the magnetic field.

We use the projector to derive from (1) an expression for the perpendicular current density:

$$e^{\mu\nu}J_{\nu} = -\frac{F^{\mu}_{\ \kappa}}{W}\frac{\partial T^{\kappa\nu}}{\partial x^{\nu}}. \tag{19}$$

In the magnetized limit this relation provides two independent equations, specifying the components of the current density in the plane transverse to the magnetic field. The remaining two equations needed to fix the four-vector current are charge conservation

$$\frac{\partial J^{\nu}}{\partial x^{\nu}} = 0, \tag{20}$$

and quasineutrality, which has the covariant form

$$J^\nu \Gamma_\nu = 0. \tag{21}$$

Here we briefly digress to comment on the quasineutrality condition. Recall that quasineutrality is fundamentally a statement about scale lengths: it pertains when the scale of interest L is much larger than the Debye length λ_D, and instructs us to use the MHD equation of motion, rather than Poisson's equation, to advance the electric field. Poisson's equation is not useful because it involves the product of a very small charge density with a very large parameter $(L/\lambda_D)^2$. In relativistic MHD the charge density can be presumed to vanish only in the local rest-frame, but otherwise the argument is the same. In particular, the obvious consequence of (21) and (2),

$$\Gamma_\mu \frac{\partial F^{\mu\nu}}{\partial x^\nu} = 0,$$

the relativistic Poisson equation, is not used in determining the field. Instead the field is advanced using its relation to the fluid flow, (17), the fluid evolution law (16), and the other Maxwell equations. In other words only three of the four components of (2) are used in a quasineutral plasma.

We have found that closure is achieved if the energy-momentum tensor of the plasma can be expressed in terms of the electromagnetic field. MHD is the simplest means for finding this expression in a magnetized plasma.

Covariant MHD summary

Beginning with the thermodynamic *ansatz*,

$$T^{\mu\nu} = p\eta^{\mu\nu} + \frac{h}{n_R^2}\Gamma^\mu\Gamma^\nu. \tag{22}$$

MHD uses the following six equations to determine the six variables in $T^{\mu\nu}$ (h, p, n_R, and the three independent components of Γ):

$$\mathcal{F}_{\mu\kappa}\frac{\partial\mathcal{T}^{\kappa\nu}}{\partial\S^{\nu}} = 0 \tag{23}$$

$$F_{\mu\nu}\Gamma^{\nu} = 0 \tag{24}$$

$$h = 4p + mn_R K_1(z)/K_2(z) \tag{25}$$

$$\frac{\partial\Gamma^{\nu}}{\partial x^{\nu}} = 0. \tag{26}$$

IV. THREE-VECTOR FORM

Plasma flow

It is often convenient to sacrifice manifest covariance and work with such three-vectors as $\boldsymbol{E}$, $\boldsymbol{B}$ and the three-vector fluid velocity

$$\boldsymbol{V}_k \equiv \boldsymbol{\Gamma}_k/\gamma n_R, \quad k = 1, 2, 3. \tag{27}$$

Hence this section derives, from (19) – (26), the three-vector form of relativistic MHD.

Our starting point is the identity

$$F_{\mu\kappa}K^{\kappa} = (-\boldsymbol{E}\cdot\boldsymbol{K}, \boldsymbol{E}K^0 + \boldsymbol{K}\times\boldsymbol{B}) \tag{28}$$

where K^{κ} is an arbitrary four-vector. This relation follows immediately from the definition of the Faraday tensor; applied to (24) it quickly provides the familiar relation

$$\boldsymbol{V}_{\perp} = \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2} + \boldsymbol{V}_{\parallel} \tag{29}$$

where

$$\boldsymbol{V}_{\parallel} = \boldsymbol{B}\boldsymbol{B}\cdot\boldsymbol{V}/B^2$$

is the (at this point arbitrary) parallel flow.

Energy-momentum evolution

Consider next the four-divergence of the energy-momentum tensor. From (13) we find after using (8) that

$$\frac{\partial T^{0\nu}}{\partial x^\nu} = -\frac{\partial p}{\partial t} + \gamma n_R \frac{d}{dt}\left(\frac{\gamma h}{n_R}\right) \tag{30}$$

$$\frac{\partial T^{i\nu}}{\partial x^\nu} = \frac{\partial p}{\partial x^i} + \gamma n_R \frac{d}{dt}\left(\frac{\gamma h V_i}{n_R}\right). \tag{31}$$

Here we have introduced the conventional notation

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{V}\cdot\nabla = (\gamma n_R)^{-1}\Gamma^\nu \frac{\partial}{\partial x^\nu}.$$

Since the dual tensor $\mathcal{F}$ is obtained from the Faraday tensor by the replacements

$$E \to B$$

$$B \to -E$$

we find that (23) has the temporal component

$$\boldsymbol{B}\cdot\nabla p + \gamma n_R \boldsymbol{B}\cdot\frac{d}{dt}\left(\frac{\gamma h \boldsymbol{V}}{n_R}\right) = 0. \tag{32}$$

The spatial components of (23) require that

$$\boldsymbol{B}\left[\frac{\partial p}{\partial t} - \gamma n_R \frac{d}{dt}\left(\frac{\gamma h}{n_R}\right)\right] - \boldsymbol{E}\times\left[\nabla p + \gamma n_R \frac{d}{dt}\left(\frac{\gamma h \boldsymbol{V}}{n_R}\right)\right] = 0$$

or, after multiplication by $\boldsymbol{B}$ and use of (32),

$$\frac{dp}{dt} + \gamma n_R\left[-\frac{d}{dt}\left(\frac{\gamma h}{n_R}\right) + \boldsymbol{V}\cdot\frac{d}{dt}\left(\frac{\gamma h \boldsymbol{V}}{n_R}\right)\right] = 0.$$

This result is simplified, using the identity

$$\frac{d\gamma}{dt} = \gamma^3 \boldsymbol{V}\cdot\frac{d\boldsymbol{V}}{dt},$$

to yield

$$\frac{dp}{dt} - n_R \frac{d}{dt}\left(\frac{h}{n_R}\right) = 0. \tag{33}$$

Using $p = n_R T$ and the definition (14) of h, it is straightforward to integrate (33) and thus obtain the adiabatic equation of state [5]

$$\frac{z n_R}{K_2(z)} \exp[-z K_3(z)/K_2(z)] = \text{constant}. \tag{34}$$

This well-known relation yields familiar limits for both nonrelativistic and super-relativistic temperatures.

We have now expressed (23) and (24) in three-vector form. Since the three-vector form of (26),

$$\frac{\partial \gamma n_R}{\partial t} + \nabla \cdot \gamma n_R \boldsymbol{V} = 0 \tag{35}$$

is obvious, we turn our attention to the current.

Current density

Straightforward manipulation shows that

$$e_0{}^\nu J_\nu = -W^{-1}(E^2 J_0 + \boldsymbol{E} \times \boldsymbol{B} \cdot \boldsymbol{J}), \tag{36}$$

$$e_i{}^\nu J_\nu = W^{-1}[(\boldsymbol{E} \times \boldsymbol{B})_i J_0 - E_i(\boldsymbol{J} \cdot \boldsymbol{E}) + B^2 J_{\perp i} \tag{37}$$

where the $\perp$ subscript refers to the components perpendicular to $\boldsymbol{B}$. Similarly we compute

$$F_{0\kappa}\frac{\partial T^{\kappa\nu}}{\partial x^{nu}} = -\boldsymbol{E} \cdot \left[\nabla p + \gamma n_R \frac{d}{dt}\left(\frac{\gamma h \boldsymbol{V}}{n_R}\right)\right] \tag{38}$$

$$\begin{aligned} F_{i\kappa}\frac{\partial T^{\kappa\nu}}{\partial x^{nu}} &= -E_i\left[\frac{\partial p}{\partial t} - \gamma n_R \frac{d}{dt}\left(\frac{\gamma h}{n_R}\right)\right] \\ &\quad - \left(\boldsymbol{B} \times \left[\nabla p + \gamma n_R \frac{d}{dt}\left(\frac{\gamma h \boldsymbol{V}}{n_R}\right)\right]\right)_i . \end{aligned} \tag{39}$$

We first substitute (37) and (39) into (19). After multiplying the result by $\boldsymbol{E}$ we obtain the energy evolution law

$$\frac{\partial p}{\partial t} - \gamma n_R \frac{d}{dt}\left(\frac{\gamma h}{n_R}\right) + \boldsymbol{J} \cdot \boldsymbol{E} = 0, \tag{40}$$

an alternative version of (33), as well as an expression for the perpendicular current:

$$\boldsymbol{J}_\perp + \boldsymbol{V}_\perp J_0 = B^{-2}\boldsymbol{B} \times \left[\nabla p + \gamma n_R \frac{d}{dt}\left(\frac{\gamma h \boldsymbol{V}}{n_R}\right)\right]. \tag{41}$$

Notice that substitution of (41) into (40) reproduces (33).

Charge conservation, (20), provides the charge density in a relativistic quasineutral plasma:

$$J_0 = -\boldsymbol{V} \cdot \boldsymbol{J}. \tag{42}$$

By combining this expression with the temporal component of (19), we find that

$$J_0 = -\boldsymbol{V} \cdot \boldsymbol{J} = \frac{\boldsymbol{E}}{W} \cdot \left[\nabla p + \gamma n_R \frac{d}{dt} \left(\frac{\gamma h \boldsymbol{V}}{n_R} \right) \right] - \frac{B^2}{W} J_\parallel V_\parallel. \tag{43}$$

This expression completes our result for the perpendicular current density, (41). The relativistic current differs significantly from the conventional result, mainly because of the charge density observed in a moving quasineutral system.

Summary

Recall that MHD closure consists in expressing the current density in terms of the electromagnetic field, and combining this "constitutive relation" with Maxwell's equations. Recall also that

1. The enthalpy density h is known from (25) in terms of p and n_R.
2. The plasma flow is known from (29) in terms of the fields and its parallel component, $V_\parallel$

Thus we see that (41) and (43), combined with charge conservation,

$$\frac{\partial \boldsymbol{V} \cdot \boldsymbol{J}}{\partial t} + \nabla \cdot \boldsymbol{J} = 0 \tag{44}$$

provides such as expression for $\boldsymbol{J}$ provided p, n_R, and $V_\parallel$ can be found. These three quantities are determined by the three equations (32), (33) and (35).

V. CONCLUSION

Starting from the fundamental orderings that characterize a magnetized plasma—small gyroradius and small parallel electric field—we have derived a closed fluid description that is valid for arbitrary thermal speed and arbitrary flow speed. The manifestly covariant version of this closure is given by (22)–(26); its three-vector form is summarized in Sec. IV.

When all speeds are small compared to the light speed, our equations reduce to those ordinary MHD (with the retention of the Maxwell term in Ampere's law). However the relativistic corrections uncovered by our derivation seem potentially important, and the covariant expression of relativistic MHD seems instructive.

Like MHD, our system is valid only to lowest order in the small gyroradius parameter, and depends upon a form for the pressure tensor that is strictly valid only near thermal equilibrium. Allowing for higher-order corrections as well as a more generally realistic pressure tensor will be the objects of future work.

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