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General-Relativistic Plasma Physics in the Early Universe

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Abstract

We apply the "3+1" formalism of Thorne and Macdonald to construct the linearized theory of a general-relativistic electron-positron plasma in the early Universe. Close formal correspondence between the theory of such plasmas and that of their special-relativistic counterparts is demonstrated. The time variation of the plasma modes due to the expansion of the background is determined for the case of a radiation-dominated Universe; it is found that the frequencies of the basic modes redshift like the frequency of a free photon. A simple kinetic argument is used to justify the neglect of creation and annihilation (collisional) effects. The formulation is sufficiently straighforward to be readily amenable to numerical implementation. Our results can be applied to the study of the origin of primordial intergalactic magnetic fields, as well as to the problem of matter fluctuations in the early Universe.

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I. Introduction

Plasmas are the main constituent of the Universe. They are found in stellar interiors, in the corona of the Sun, in H II regions, and anywhere else that temperatures are likely to be high and densities relatively low, such as the inner regions of an accretion disk. In particular, the conditions in the early Universe (from approximately $t=10^{-2}$ second to the time of recombination) are such that the "ordinary" matter content will be in the form of plasma. From approximately $t=10^{-2}$ to $t=1$ second, the plasma is dominated by electrons and positrons, while from $t\approx 1$ to the time of recombination at roughly 10^{14} seconds, the plasma consists mostly of electrons and protons (hydrogen ions), with an admixture of ions of other light elements. These plasmas are believed to be in thermal equilibrium[1], or at least strongly coupled, with photons. Thus the "classical" period prior to recombination is traditionally called the "radiation epoch" of the Universe. However, it could just as well be called the "plasma epoch," as the main dynamics are those of plasmas. Plasma phenomena should thus be expected to play an important role in the early history of the Universe.

Nevertheless, plasma physics has, in the past, generally not intersected with cosmology. The dynamo effect has been invoked to study the origin of the intergalactic magnetic fields[2,3]. It has also been shown[4] that even in thermal equilibrium a cosmic plasma has thermal fluctuations of density and current, which can give rise to magnetic fields; it can be speculated that effects of the earlier epochs of the Universe cause additional (and much larger) magnetic fields due to larger fluctuations ("turbulence") of plasma density and current[3]. Plasmas also find application to cosmology through the quark-gluon plasma that is expected to exist in the very early Universe; this medium can be studied by methods comparable to ours[5], although in this case the pertinent field theory is quantum chromodynamics rather than electrodynamics. However, such topics are only beginning to be explored.

One possible inhibitor to the development of the theory of plasma physics in cosmology is the role of gravity. In the early Universe, when the rate of expansion is relatively high, general relativity will surely have a significant effect on the physics. The most interesting plasma phenomena involve collective effects and are strongly nonlinear even in the Newtonian or special-relativistic case; the addition of gravity can be expected to cause even more highly nonlinear and violent phenomena. Often a numerical approach is the only one feasible. However, existing plasma theories and computational codes are based on Newtonian physics, perhaps accounting for *special* relativity; how can we incorporate general-relativistic effects into our analytic methods and our computational codes, without starting entirely from scratch? The solution to the problem of writing the general-relativistic electromagnetic equations in a form similar to those familiar from Newtonian physics was provided by Thorne and Macdonald[6]. Although this approach does not solve our problems for us, it gives us a firm basis for the understanding of general-relativistic plasmas, without requiring that we discard the knowledge we have already gained from the Newtonian world.

In this article, we formulate the dynamical aproach of general-relativistic plasmas relevant to the early Universe, based on the "3 + 1" formalism of Thorne and Macdonald, in a manner directly amenable to analysis and simulation of nonlinear and collective plasma phenomena. The plasma is assumed to exist in a spacetime described by the standard solution for a radiation-dominated, Friedmann-Robertson-Walker cosmology; we do not account for the self-gravity of the matter. Section II describes the formalism we employ. In §III, we derive the basic properties of plasma waves in this system, including the electromagnetic waves in an electron-positron plasma, plasma oscillations, and phonons in the plasma. Our conclusions are summarized in §IV of the paper.

II. Formalism

Our work is based on the "3 + 1" formulation of general relativity. This approach was developed by Arnowitt, Deser, and Misner[7] [ADM] to study the quantization of the gravitational field, but it has seen its most extensive application in numerical relativity[8]. (A particularly good introduction to the mathematics of the ADM method is given by York[9]; the reader interested in more details than we shall present here is referred to this paper.) In the ADM formalism, spacetime is filled by a foliation of 3-dimensional spacelike hypersurfaces, which form the level surfaces of a family, or congruence, of timelike curves. The choice of these hypersurfaces constitutes a particular *time slicing* of spacetime. The tangent vectors to the timelike congruence specify the universal time coordinate for this slicing; each hypersurface is thus a surface of constant universal time. It is obvious that not every spacetime will admit such a foliation, but we shall consider only those which do.

In this approach, the covariant formulation of general relativity is "unrolled" to show its structure as a Cauchy (initial value) problem; the ten Einstein equations split into four constraint equations, which must be satisfied on each slice, and six dynamical equations, which describe the change in the geometry of the hypersurface slices as universal time passes. Thus we view spacetime as the time history of a particular slice on which we specify our initial data. This is entirely analogous to the Cauchy problem for the Maxwell equations, in which the Poisson equation for the electric field must be satisfied at all times, but gives no information about time evolution, while the time development of the fields is specified by the dynamical Maxwell equations. Just as it can be shown that the constraints are preserved by the Maxwell equations, so can it be demonstrated that the Bianchi identities guarantee that the four constraint equations will be preserved by the dynamical Einstein equations.

The inclusion of electromagnetism requires that the covariant equations of motion for the electromagnetic field and the charged particles be decomposed into their "3+1" forms. Ellis[10] performed such calculations for the Maxwell equations, but the full program was carried out by Thorne and Macdonald[6], who then applied it to the particular case of the Kerr metric, in order to study electromagnetic effects near a rotating black hole[11]. We shall present here only a brief summary of their results. Our notation conforms to that usually employed by numerical relativists; the notation of Thorne and Macdonald differs slightly. Furthermore, we shall use units in which G=1, but $c \neq 1$.

Consider a set of observers at rest with respect to "absolute space," that is, the space defined by the hypersurfaces of constant universal time. Such observers are called fiducial observers, or FIDOs, by Thorne and Macdonald, although they are usually known as Eulerian observers to numerical relativists, in analogy with the Eulerian viewpoint of Newtonian hydrodynamics[12]. The FIDOs measure their proper time τ using clocks that they carry with them, and they can make local measurements of physical quantities such as electric and magnetic fields; whenever we write a quantity such as $\vec{E}$ or $\vec{B}$, it will be understood, unless otherwise specified, to refer to the value as measured by the FIDOs. It is important to point out that

FIDOs are *not* unaccelerated observers, and their motion may become pathological, as seen by observers keeping universal time, in the neighborhood of a singularity; a set of observers, called freely-falling observers (FFOs) by Thorne and Macdonald, can be constructed whose motion is always nonpathological with respect to observers at a great distance from the singularity.

The FIDOs have associated with them a four-velocity vector field $\mathbf{n}$. This vector field is, by construction, orthogonal to the spatial slices. In general, any vector or vector field can be decomposed into components lying in and normal to the slices; any vector whose orthogonal component vanishes identically will be called a spatial vector. This concept can be generalized to tensors as well; spatial vectors and tensors form the foundation of the "$3+1$" split. In particular, a vector $\mathbf{v}$ is spatial if

$$v^{\mu} n_{\mu} = 0, \tag{1}$$

while a tensor $\mathbf{T}$ is spatial if

$$T^{\mu\nu} n_{\nu} = 0. \tag{2}$$

The most important spatial tensors are the three-metric $\gamma_{\mu\nu}$ and the extrinsic curvature tensor $K_{\mu\nu}$; these quantities are known in the language of differential geometry as, respectively, the first and second fundamental forms of a surface. The three-metric describes the intrinsic curvature of the surface, while the extrinsic-curvature tensor specifies the embedding of the surface into the higher-dimensional spacetime. In the "$3+1$" formalism, both $\gamma_{\mu\nu}$ and $K_{\mu\nu}$ will change with universal time as the geometry unfolds under the action of the dynamical Einstein equations.

It can be shown that

$$n_{\alpha;\beta} = \omega_{\alpha\beta} + \sigma_{\alpha\beta} + \frac{1}{3}\theta\gamma_{\alpha\beta} - a_{\alpha}n_{\beta}, \tag{3}$$

where

$$\omega_{\alpha\beta} \equiv \frac{1}{2}(n_{\alpha;\mu}\gamma^{\mu}_{\beta} - n_{\beta;\mu}\gamma^{\mu}_{\alpha}) \tag{4}$$

is the rotation of the fiducial world lines;

$$\sigma_{\alpha\beta} \equiv \frac{1}{2}(n_{\alpha;\mu}\gamma^{\mu}_{\beta} + n_{\beta;mu}\gamma^{\mu}_{\alpha}) - \frac{1}{3}\theta\gamma_{\alpha\beta} \tag{5}$$

is the shear of the fiducial world lines;

$$a^\alpha \equiv n^\alpha_{;\beta} n^\beta \tag{6}$$

is their acceleration, and

$$\theta \equiv n^\mu_\mu \tag{7}$$

is their expansion. The expansion has the property that

$$\theta = -K, \tag{8}$$

where K is the trace of the extrinsic curvature tensor, $K \equiv K^\mu_{;\mu}$. We shall not consider spacetimes with rotation, since such spacetimes do not admit a universal time coordinate; thus always in our case $\omega_{\alpha\beta} = 0$.

The four degrees of freedom of general relativity are accomodated by a function, α, and a spatial vector $\vec{\beta}$. The function α describes the ratio of the rate of fiducial proper time to that of universal time, i.e.

$$\alpha = \frac{d\tau}{dt}, \tag{9}$$

and so is called the *lapse function.* The vector $\vec{\beta}$ indicates how we must "shift" our spatial coordinates as we march from one hypersurface to the next; thus it is clear why this vector is called the *shift vector.* The gravitational acceleration is determined by the lapse function; it can be shown that

$$\vec{g} \equiv -\vec{a} = -c^2 \vec{\nabla} \ln \alpha. \tag{10}$$

Now that we have defined all the quantities we shall need, we can proceed to write down our equations. We shall not give their derivations here; the interested reader is referred to Thorne and Macdonald. The equations will closely resemble their Newtonian counterparts; this is the great power of the "$3+1$" approach to general relativity, as it allows us to make use of our intuition from Newtonian physics. From a numerical point of view, this is especially advantageous; the ability to exploit well-established techiques developed for Newtonian physics is important for code development. For example, this formalism greatly aided the application of relativistic hydrodynamics to astrophysics[13].

In this section, we shall write the fundamental equations of electrodynamics in the "3+1" formalism without derivation; more details may be found in Thorne and Macdonald. In our notation, the Maxwell equations become[14]

$$\nabla_i E^i = 4\pi\rho_e, \tag{11}$$

$$\nabla_i B^i = 0, \tag{12}$$

$$\frac{1}{c}(\partial_t E^i - \alpha K E^i - \pounds_\beta E^i) = \varepsilon^{ijk}\nabla_j(\alpha B_k) - \frac{4\pi\alpha}{c}J^i, \tag{13}$$

$$\frac{1}{c}(\partial_t B^i - \alpha K B^i - \pounds_\beta B^i) = -\varepsilon^{ijk}\nabla_j(\alpha E_k), \tag{14}$$

where ∇ is the spatial covariant derivative operator; i.e. $\nabla\gamma_{ij} = 0$, and $\pounds_\beta$ is the *Lie derivative* along the vector field $\vec{\beta}$; the Lie derivative of a vector field is a generalized directional derivative. The quantity ε^{ijk} is the antisymmetric pseudotensor, through which generalized curl operations are defined.

The equation of charge conservation can be derived from the Maxwell equations, yielding the result

$$\partial_t\rho_e + \nabla_i(\alpha J^i) = \alpha K\rho + \pounds_\beta, \tag{15}$$

where ρ_e is the charge density measured by the FIDOs.

Let $\vec{v}$ be the three-velocity vector of a particle, $\vec{p}$ its three-momentum, and μ its rest mass. Define the usual special-relativistic boost factor Γ by

$$\vec{p} = \mu\Gamma\vec{v}. \tag{16}$$

The particle equation of motion is

$$\frac{D\vec{p}}{D\tau} = -\mu\Gamma\vec{g} - 2\widetilde{\sigma}\cdot\vec{p} + \frac{2}{3}K\vec{p} + \alpha^{-1}\pounds_\beta\vec{p} + q(\vec{E} + \frac{\vec{v}}{c}\times B), \tag{17}$$

where $\widetilde{\sigma}$ is coordinate-free notation for the shear tensor, and $D\vec{p}/D\tau$ is the "convective derivative"

$$\frac{D\vec{p}}{D\tau} = \frac{1}{\alpha}(\partial_t + \vec{v}\cdot\vec{\nabla}). \tag{18}$$

(Unless otherwise specified, $\vec{\nabla}$ in vector notation indicates the ordinary flat-space gradient operator.)

II. Cosmology

As a special case of this formalism, we consider cosmological plasmas. We shall assume that our spacetime is described by a line element of the simple form

$$ds^2 = -c^2 dt^2 + A^2(t)(dx^2 + dy^2 + dz^2). \tag{19}$$

This corresponds to a homogeneous, isotropically-expanding Universe; the spatial coordinates are *comoving* and Cartesian. Since we are interested in the early Universe, which we assume to be dominated by radiation, we shall take for the expansion factor A the standard solution for a $k = 0$, radiation-dominated, Friedmann-Robertson-Walker [FRW] cosmology

$$A(t) = R_i(t/t_i)^{\frac{1}{2}}. \tag{20}$$

If we evaluate the constants of integration by demanding that $R_i = 1$ at $t_i = 1$, then this reduces to $A(t) = \sqrt{t}$. It is obvious that for this line element,

$$\alpha = 1, \tag{21a}$$

$$\vec{\beta} = (0, 0, 0), \tag{21b}$$

and

$$\gamma_{ij} = \mathrm{diag}(A^2, A^2, A^2). \tag{21c}$$

It can be shown that for any FRW spacetime, the trace of the extrinsic-curvature tensor is given by

$$K = -3\left(\frac{\dot{A}}{A}\right), \tag{22}$$

where a dot denotes the time derivative; in this particular case,

$$K = -\frac{3}{2t}. \tag{23}$$

We can now evaluate the covariant derivatives appearing in Eqns (11)–(17) in order to write the Maxwell equations and the particle equation of motion in terms of our coordinates. For this metric, the calculations are short and straightforward, and we shall present only the results.

The Maxwell equations become

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho_e, \tag{24}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{25}$$

$$\frac{\partial \vec{E}}{\partial t} = K\vec{E} + cA^{-1}\vec{\nabla} \times \vec{B} - 4\pi\vec{J}, \tag{26}$$

and

$$\frac{\partial \vec{B}}{\partial t} = K\vec{B} - cA^{-1}\vec{\nabla} \times \vec{E}, \tag{27}$$

where $\vec{\nabla}\cdot$ and $\vec{\nabla}\times$ are the "ordinary" flat-space divergence and curl, in Cartesian coordinates. Equations (24) and (25) are unchanged from their Newtonian form, because our three-metric depends only on time. The equation of charge conservation simplifies to

$$\frac{\partial \rho_e}{\partial t} = K\rho_e + \vec{\nabla}\vec{J}. \tag{28}$$

The particle equation of motion reduces to

$$\frac{d\vec{p}}{dt} = \frac{2}{3}K\vec{p} + q(\vec{E} + A\frac{\vec{v}}{c} \times \vec{B}). \tag{29}$$

Since in this metric, FIDO proper time and universal time coincide, the new positions of the particles may be computed simply from

$$\frac{d\vec{x}}{dt} = \vec{v}, \tag{30}$$

$$\vec{v} = \frac{\vec{p}}{\mu\Gamma}. \tag{31}$$

These are the basic equations needed for computational work in plasma cosmology. For theoretical work, we will often make use of the fluid approximation for the plasma; these equations will be written as they are needed.

III. Linear Theory of Cosmological Plasma Waves

We will now derive the basic properties of general-relativistic plasma waves in an expanding Universe. In this paper, we shall discuss only the *linear* theory of small-amplitude plasma waves. Investigation of nonlinear properties will be deferred to future papers.

We can use our general-relativistic equations to replicate the standard results of plasma theory for a "warm" plasma in an expanding background. However, we cannot make the usual assumption of harmonic time dependence for all quantities. It is clear that the ordinary plane wave with time dependence of $\exp i(-\omega t)$ cannot be a solution for the cosmological wave, because the amplitude of such a wave must decay in time, and its frequency must redshift. Thus we must seek more general solutions.

(a) Free photons

It is simple to derive the appropriate wave equation from Equations (13) and (14); we obtain

$$\frac{1}{c^2}(A^2\partial_t^2\vec{E} + [\dot{A} - 2AK]\partial_t\vec{E} + [K^2A^2 - A\dot{A}K - A^2\dot{K}]\vec{E}) = -\vec{\nabla}\times\vec{\nabla}\times\vec{E} - \frac{4\pi A}{c^2}(\partial_t - K)A\vec{J}. \qquad (32)$$

In order to proceed, we must substitute the appropriate functions for A and K, given in our case by Equations (20) and (23). (We remind the reader that we have chosen $R_i = t_i = 1$, so our time variable is normalized in that sense.) Let us write the electromagnetic wave equation for a free wave in a cosmological background, by setting $\vec{J} = 0$ and requiring that $\nabla\cdot\vec{E} = 0$. The resulting equation for this transverse (divergence-free) wave is

$$\frac{1}{c^2}(t\partial_t^2\vec{E} + \frac{7}{2}\partial_t\vec{E} + \frac{3}{2t}\vec{E}) = \nabla^2\vec{E}. \qquad (33)$$

The words "transverse" and "longitudinal" will refer to directions perpendicular and parallel to the direction of wave propagation, respectively. This equation can be solved by separation of variables. The spatial equation is unchanged from the Newtonian case, yielding the usual solution of $\exp(i\vec{k}_i\cdot\vec{r})$, where k_i is the separation constant (corresponding to the initial time t_i). The time equation is found to be Bessel's equation of order one-half. If we require a propagating solution, then the appropriate functions are the Hankel functions $H^{(1)}_{1/2}$ and $H^{(2)}_{1/2}$; the one represents a wave traveling to the right, the other a wave traveling to the left, so we may choose either one for our solution. Let us select $H^{(2)}_{1/2}$, which corresponds asymptotically to $\exp(-i\omega t)$, and denote it simply by $H_{1/2}$; the full solution is then

$$\vec{E} = E_0\hat{\epsilon}t^{-5/4}H_{1/2}(2k_ict^{1/2})e^{i\vec{k}_i\cdot\vec{r}}, \qquad (34)$$

where $\hat{\epsilon}$ is the polarization vector of the wave, and E_0 is an arbitrary amplitude constant. We should point out that the behavior of the Bessel function at the origin is irrelevant in our case, since in the standard "hot big bang" model, *all* physical properties become singular at the time origin. Thus our choice of Bessel (or Hankel) function is somewhat arbitrary.

If we explicitly combine all the factors of time in this solution, we find that

$$|\vec{E}| \sim t^{-3/2}, \tag{35}$$

a dependence which is expected from the form of the Maxwell equations and the particular form of K in our background model (Equation (23)).

The asymptotic form of Equation (34) is

$$\vec{E} \sim \vec{E}_0(\vec{x},t)\exp[i(\vec{k}_i \cdot \vec{r} - 2k_i c t^{1/2})]. \tag{36}$$

The characteristic is described asymptotically by

$$\vec{k}_i \cdot \vec{r} - 2k_i c t^{1/2} = 0. \tag{37}$$

However, the horizon, defined in our metric by

$$L_t = \int_0^t \frac{c}{t'^{1/2}}\,dt', \tag{38}$$

is found to be

$$L_t = 2ck_i t^{1/2}, \tag{39}$$

so we see that, asymptotically, the characteristic is just

$$k_i(\hat{k} \cdot \vec{x}_{cm} - L_t), \tag{40}$$

where x_{cm} is the comoving coordinate; this expression is entirely analogous to that for the special-relativistic characteristics, which are given by $k(\hat{k} \cdot \vec{x} - ct)$, where ct is the horizon length in Minkowski spacetime.

If we set $\omega_i = k_i c$ in the usual way, where again the subscript refers to the initial time, then the redshift of a photon in our FRW background is[1]

$$\omega = \frac{\omega_i}{t^{1/2}}. \tag{41}$$

(Once again, we are assuming $t_i = 1$.) Thus we see that we can write the time-dependent portion of the wave as

$$t^{-5/4} H_{1/2}(2\omega t), \tag{42}$$

where now ω is the angular frequency of the photon at time t. We can understand the argument of the Hankel function in the following way. When we observe the frequency of a photon at the current epoch, the period of time over which we carry out our measurement is very much shorter than the cosmic time t. In particular, suppose we begin our measurement at cosmic time t_m and continue for an interval δt. The argument of the asymptotic characteristic (37) can then be expanded around t_m, becoming

$$\vec{k}_i \cdot \vec{x}_{cm} - k_i c t_m - k_i c \delta t = const - k_i c \delta t, \tag{43}$$

which justifies our assertion that, *in vacuo*, the angular frequency should be identified as the wavenumber times the speed of light, just as in a Minkowski background.

(b) Photons in the plasma

Electromagnetic disturbances in a plasma can be either propagating or evanescent. Here we shall consider the propagation of small-amplitude electromagnetic waves, i.e. photons, in the plasma. To compute the frequencies of these transverse waves, we start from the equation of motion for a particle in a plasma in an expanding background:

$$\frac{d(\vec{p}_T t)}{dt} = qt(\vec{E}_T + A^{-1}\frac{\vec{v}}{c} \times \vec{B}_T). \tag{44}$$

Linearizing and ignoring the term in $\vec{v} \times \vec{B}$ results in

$$\frac{d(\vec{p}_T t)}{dt} = qt(\vec{E}_T). \tag{45}$$

This equation can be treated in a manner similar to its Newtonian counterpart. Let us assume the time dependence specified by Equation (34) for the transverse part of the electric field. Since the Hankel functions are analogous in our background to the sinusoidal functions of Minkowski spacetime, and since the various Bessel functions, when appropriately weighted, form an orthogonal set, in terms of which we can expand an arbitrary function, this procedure is appropriate. The spherical Bessel (or Hankel) functions are nothing more than combinations of trigonometric functions divided by powers of their arguments, so we can easily carry out the integration of Equation (45) to obtain

$$\vec{p}_T = -\frac{iqt^{1/2}}{\omega_i}\vec{E}_T, \tag{46}$$

where ω_i is the assumed angular frequency of the wave at the fixed initial time. This remarkable result is exactly analogous to the relationship we obtain for the Newtonian plane-wave solution, which supports our view that the Hankel functions play the role in our background that the circular functions play in the familiar flat backgrounds.

The FIDO's will define $\vec{J}$ in terms of the velocity they measure, not the momentum. For a single particle undergoing transverse oscillations due to the influence of an electromagnetic wave, we use Equation (16) to write

$$\frac{d(\vec{p}_T)}{dt} = \mu \frac{d(\Gamma_{th}\vec{v})}{dt}. \tag{47}$$

The boost factor that appears here is obtained from the usual formula

$$\Gamma_{th} = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \tag{48}$$

For a single particle, this boost factor is related to its particular velocity, and hence ultimately to the temperature of the plasma.

In the transverse case, the time derivative of the boost factor is negligible. This allows us to write

$$\vec{v}_T = -\frac{iqt^{1/2}}{\mu\Gamma_{th}\omega_i}\vec{E}_T, \tag{49}$$

where μ is the rest mass of the particle.

We may now compute the dielectric constant and the dispersion relation for transverse waves. For this case, we approximate

$$\vec{J} = \sum \rho_e \vec{v} = \sum \frac{\rho_e}{\mu\Gamma_{th}}\vec{p} = \sum \frac{qn}{\mu\Gamma_{th}}\vec{p}. \tag{50}$$

Inserting this into the Maxwell equation (26), and assuming an $e^+ - e^-$ plasma, gives

$$\frac{t^{1/2}}{c}(\partial_t \vec{E}_T - K\vec{E}_T) + \frac{2\cdot 4\pi n e^2 t^{1/2}}{\mu\Gamma_{th}}\vec{p}_T = (\vec{\nabla}\times\vec{B})_T. \tag{51}$$

It is straightforward to compute $\partial_t \vec{E}_T - K\vec{E}_T$ for our assumed time behavior; when this result and Equation (46) are substituted into (51), we obtain

$$\left(-\frac{i\omega_i}{c} + \frac{8\pi n_0 e^2 t}{\mu\Gamma_{th}\omega_i}\right)\vec{E}_T = (\vec{\nabla}\times\vec{B})_T, \tag{52}$$

where n_0 is the background number density of the particles; there is no perturbation of number density in the transverse direction to linear order, so only the background contributes.

This reduces to

$$-\frac{i\omega_i}{c}\varepsilon(\omega)\vec{E}_T = (\vec{\nabla} \times \vec{B})_T, \tag{53}$$

where the dielectric constant is given by

$$\varepsilon = (1 - \frac{\omega_{pT}^2}{\omega^2}), \tag{54}$$

and

$$\omega_{pT}^2 \equiv \frac{8\pi n_0 e^2}{\mu \Gamma_{th}}. \tag{55}$$

We identify the electron plasma frequency as

$$\omega_{pe}^2 \equiv \frac{4\pi n_0 e^2}{\mu}, \tag{56}$$

which leads to

$$\omega_{pT}^2 \equiv \frac{2\omega_{pe}^2}{\Gamma_{th}}, \tag{57}$$

just as in the special-relativistic case, where the factor of 2 arises from the sum over electrons and positrons.

The dielectric constant of the plasma has the same form as in the Newtonian case, although here both the plasma frequency ω_{pT} and the photon frequency ω vary in time. Let us compute the time behavior of ω_{pT}. We note from Equation (28) that the background number density of the particles diminishes like the inverse volume of the Universe, which goes like $t^{-3/2}$ in our case. Thus

$$\omega_{pe} \sim t^{-3/2}. \tag{58}$$

Equations (35) and (46) tell us that $\vec{p} \sim t^{-1}$, from which we find that

$$\Gamma_{th} \sim t^{-1/2}. \tag{59}$$

We conclude from this that

$$\omega_{pT} \sim t^{-1/2}. \tag{60}$$

Hence $\varepsilon(\omega)$ does not depend explictly on time. The dispersion relation for the photons may be computed from Equation (53) in the usual manner; we use the Fourier transform to write the curl as a cross product, and substitute for curl $\vec{B}$ from Equation (26). The result is

$$[\frac{c^2k_i^2}{\omega_i^2}\varepsilon(\omega)]\vec{E}_T = 0, \tag{61}$$

from which we obtain

$$\omega_T^2 = \omega_{pT}^2 + k^2c^2, \tag{62}$$

exactly as in the Newtonian (special-relativistic) case, except that now ω_T changes with time. We have shown that the dielectric constant is *independent* of time; Equation (62) then leads us to the conclusion that ω and k vary in the same manner. Comparing this to Equation (41), we see that, at least for this metric, the redshift of a photon in a plasma is the *same* as that for a free photon. Thus the photon in the plasma is self-similar to the photon in a vacuum; the three quantities ω_{pT}, k, and ω_T "conspire" to maintain the same redshift factor, whether the photon is free or in a plasma. This result is nontrivial and is specific to the time dependence of the plasma temperature, here indicated by Equation (59), which is a function of the background model employed (but not of the equation of state of the plasma).

(c) Longitudinal oscillations in an unmagnetized plasma

Next we consider small longitudinal oscillations of a warm plasma in an expanding background. In order to describe the longitudinal properties of the plasma correctly, we should start from the relativistic kinetic (Boltzmann) equation and carry out a statistical analysis. As we shall show later, however, it suffices for our present purposes to employ the moment equations, treating collisional effects through the macroscopic pressure. Thus we shall need only the equations of matter conservation and of conservation of momentum. We shall neglect annihilation and creation of particles, an assumption which will be justified below.

Represent the mass energy density by

$$\rho = \rho_0 + \rho_1, \tag{63}$$

where ρ_1 is a small correction to the background value. The number density n obeys a similar equation, since $\rho = \mu n$, and μ, the rest mass of the particle, is an invariant. Consider next the time evolution of the unperturbed background; it is described by

$$\partial_t \rho_0 + \frac{3}{2t}\rho_0 = 0, \tag{64}$$

from which we immediately conclude that

$$\rho_0 = \rho_i t^{-3/2}. \tag{65}$$

The conservation of momentum is given by[6,15]

$$\frac{D(t^{5/2}\vec{S})}{Dt} = t^{5/2} nq\vec{E} - t^{3/2}\vec{\nabla} P_r, \tag{66}$$

where $\frac{D}{Dt}$ denotes a convective derivative, P_r is the pressure observed in the rest frame of the gas, and the relativistic momentum is

$$\vec{S} \equiv \rho h \Gamma \vec{v}, \tag{67}$$

where now Γ is the boost factor between the FIDO's and the *bulk* motion of the plasma. The quantity h, the relativistic specific enthalpy, is defined as

$$h \equiv 1 + \frac{e_r}{\rho c^2} + \frac{P_r}{\rho c^2}, \tag{68}$$

with ρ_r the mass density as seen in the rest frame, and e_r the rest-frame internal energy. The specific enthalpy is important for relativistic gases, since the quantity ρh plays the role of the "inertial mass" of the gas.

For a plasma which obeys an adiabatic equation of state, that is, a plasma for which

$$P_r = \sigma \rho_r^{\gamma}, \tag{69}$$

then the pressure gradient is

$$\vec{\nabla} P_r = h c_s^2 \vec{\nabla} \rho_r, \tag{70}$$

where

$$c_s^2 \equiv \frac{\gamma P_r}{\rho_r h} \tag{71}$$

is the relativistic soundspeed for such a gas.

Acoustic waves are those for which both the electrons and positrons move in the same direction; thus the electric field vanishes, and only the pressure term in Equation (66) contributes. We can find the frequency of acoustic oscillations by computing the divergence of Equation (66), without the electric-field term, and substituting into the time derivative of the linearization of Equation (28). For the case of an ultra-relativistic plasma, for which we have

$$\gamma = \frac{4}{3} \qquad \text{and} \qquad P_r \gg \mu c^2, \tag{72}$$

it follows that

$$h = \frac{4P_r}{\rho c^2} \sim t^{-1/2}. \tag{73}$$

Furthermore,

$$c_s = \frac{c}{\sqrt{3}}, \tag{74}$$

so the soundspeed is time-independent for an ultrarelativistic gas, even in an expanding background.

Using this relationship, we find that the mass density is described by the equation

$$t^2 \partial_t^2 \rho_1 + \frac{7}{2} t \partial_t \rho_1 + [\frac{3}{2} + (\frac{c_s^2 k^2}{\Gamma^3}) t] \rho_1 = 0. \tag{75}$$

(In deriving this equation, we have assumed that $\vec{\nabla}\rho_r \approx \vec{\nabla}\rho_1$, which should be valid in the linear regime.) Equation (75) has exactly the same *form* as the wave equation for a free photon, and thus the analogous solution. Hence we find that the acoustic oscillations obey the dispersion relation

$$\omega_s^2 = (\frac{c_s^2}{\Gamma^3}) k^2. \tag{76}$$

The factor of Γ^3 occurs due to the need for a derivative of $\vec{v}$, not of $\Gamma\vec{v}$. In the general-relativistic case, this factor can be extracted only approximately, in the limit in which $\Gamma \approx 1$. However, this condition must hold in order for our linear theory to be valid. Thus the Γ^3 should be taken to be a lowest-order correction for special-relativistic effects. On the other hand, there are no restrictions on h other than that the perturbations be small. Therefore, our analysis is valid for highly relativistic *internal* motions, i.e. temperatures, but only for weakly relativistic *bulk* motions of the plasma.

The electrostatic longitudinal modes (i.e. plasma oscillations) occur when electrons and positrons move in the opposite direction. To obtain the dispersion relation for these modes, we recall that

$$\vec{\nabla} \cdot \vec{E} = 4\pi q n_1. \tag{77}$$

Using Equations (70) and (77) in Equation (66), taking the divergence, and dropping second-order terms in $\vec{v}$ yields

$$\vec{\nabla} \frac{d}{dt}(t^{1/2}\vec{v}) \approx \frac{4\pi n_i e^2 t^{3/2}}{\mu h_i \Gamma^3} + \frac{c_s^2 t^{3/2}}{n_i \mu \Gamma^3} k^2 n_1. \tag{78}$$

Combining Equation (78) with (28) produces an equation which is again the familiar form of Bessel's equation. A representative solution to this equation is

$$n_1 = t^{-5/4} H_{1/2}(4\omega_L t), \tag{79}$$

where

$$\omega_L^2 \equiv \frac{2\omega_{pe}^2}{h\Gamma^3} + (\frac{c_s^2}{\mu\Gamma^3})k^2, \tag{80}$$

and $H_{1/2}$ is again $H_{1/2}^{(2)}$. Note the factor of h in the denominator of the first term; it is the fluid analogue of the factor of Γ_{th} in the denominator of Equation (57). The factor of Γ^3 again appears in the denominator as a special-relativistic effect in the longitudinal direction. The plasma frequency that makes up the first term of (80) is never greater than, and is typically less than, the the plasma frequency of the transverse wave of Equation (62), which determines the cutoff frequency. We see that both terms in Equation (80) vary in time as $t^{-1/2}$. Thus the frequencies of the longitudinal modes also decay in time like the redshift of a free photon.

A careful treatment of the plasma from a kinetic viewpoint would detemine the Landau damping suffered by the longitudinal modes. Even in the special-relativistic case, however, the momentum (velocity) integration over the distribution function involves a nontrivial integral. For the purposes of the present paper, the information gained from such an analysis would not be particularly illuminating, so we shall not consider this approach here.

(d) Alfvén waves

Lastly, we consider a magnetized plasma. We assume an ambient magnetic field, which may vary in time, but is taken to be uniform in space; we will compute the frequencies of small-amplitude Alfvén waves in this plasma. For this we shall need the equations of relativistic MHD. Special-relativistic MHD equations are nothing new[16,17], but in our case, we must start from the "3+1" general-relativistic MHD equations[6,15,18]. The linearized fluid equations for our metric become

$$\partial_t(t^{5/2}\rho h\Gamma\vec{v}) = \frac{t^3}{c}\vec{J}\times\vec{B} - t^{3/2}hc_s^2\vec{\nabla}\rho_1, \tag{81}$$

$$\partial_t(t^{3/2}\rho_1) = -t^{3/2}\rho_0\vec{\nabla}\cdot\vec{v}. \tag{82}$$

The Maxwell equations we shall need are

$$t\vec{\nabla}\times\vec{B} = \frac{1}{c}\partial_t(t^{3/2}\vec{E}) + \frac{4\pi}{c}t^{3/2}\vec{J}, \tag{83}$$

$$t\vec{\nabla}\times\vec{E} = \frac{1}{c}\partial_t(t^{3/2}\vec{B}). \tag{84}$$

We shall assume that

$$\vec{B} = \vec{B}_0 + \vec{B}_1, \tag{85}$$

where $\vec{B}_1$ is small compared to $\vec{B}_0$, and similarly for $\vec{E}$.

The condition of perfect conductivity is

$$E_i = -\varepsilon_{ijk}\frac{v^j}{c}B^k. \tag{86}$$

Converting to vector notation yields

$$\vec{E} = -\frac{t^{1/2}}{c}\vec{v}\times\vec{B}. \tag{87}$$

When equations (84) and (87) are combined and orders equated, we obtain

$$\partial_t(t^{3/2}\vec{B}_0) = 0. \tag{88}$$

This implies that $\vec{B}_0 \sim t^{-3/2}$, *not* $\vec{B}_0 \sim t^{-1}$, as is commonly "derived" in the literature from a naive extension of the nonrelativistic equations. Our result, however, is fully consistent with flux conservation. The general-relativistic flux is[14]

$$\dot{\Phi} = \frac{d}{dt}\int \gamma^{1/2}B\,d\Sigma^c, \tag{89}$$

where $d\Sigma^c$ is the *coordinate* area element, and γ is the determinant of the three-metric. For our comoving coordinates, $d\Sigma^c$ does not depend on time. Thus

$$\dot{\Phi} = \frac{d}{dt}\int A^3 B\, dS = \frac{d}{dt}\int t^{3/2} B dS = -\mathcal{E}. \tag{90}$$

We see that if $\mathcal{E} = 0$, then the magnetic field must decay in time as $t^{-3/2}$. Since the uniform ambient field should not produce an emf, this confirms our result. In our analysis, we shall neglect the relativistic displacement current in Equation (83). We should point out, however, that in a fully relativistic calculation, this term *must* be included, as it will prevent the Alfvén velocity from exceeding the speed of light.

These equations can be combined with the usual manipulations to obtain an equation for the velocity of the fluid; the only special caution we must take is to consider the time dependence of quantities such as the mass-energy density and the relativistic enthalpy. The result is

$$\Gamma^3 \partial_t (t^{1/2}\partial_t[t^{1/2}\vec{v}]) + \vec{V}_{Ai} \times \vec{\nabla} \times \vec{\nabla} \times \vec{v} \times \vec{V}_{Ai} - c_s^2 \vec{\nabla}(\vec{\nabla} \cdot \vec{v}) = 0, \tag{91}$$

where

$$\vec{V}_{Ai} \equiv \vec{V}_A t^{1/2}, \tag{92a}$$

and

$$\vec{V}_A \equiv \frac{\vec{B}_0}{(4\pi\rho_0 h)^{1/2}} \tag{92b}$$

is the *relativistic* Alfvén velocity. Note that $\vec{V}_{Ai}$ is *time-invariant.*

The operator $\partial_t(t^{1/2}\partial_t[t^{1/2}\vec{v}])$ can be expanded to $t^2\partial_t^2\vec{v} + (3/2)t\partial_t\vec{v}$, and we may Fourier-transform the spatial components in the usual manner. The resulting equation is quite complex, and has, in general, three branches, as in the nonrelativistic case. It can be simplified, however, depending on the relative orientations of the wavevector $\vec{k}$, the fluid velocity $\vec{v}$, and the magnetic field $\vec{B}_0$. We consider three cases.

(i) $\vec{k} \perp \vec{B}_0$. In this case, $\vec{v} \parallel \vec{k}$. Equation (91) reduces for the magnitude of $\vec{v}$ to

$$t^2\partial_t^2 v + (3/2)t\partial_t v + ([\frac{c_s^2 + V_{Ai}^2}{\Gamma^3}]k^2)tv. \tag{93}$$

This is again a form of Bessel's equation of order one-half. The representative solution in this case takes the familiar form

$$v = t^{-1/4} H_{1/2}(2\omega_{MS} t), \tag{94}$$

where

$$\omega_{MS}^2 = (\frac{c_s^2 + V_{Ai}^2}{\Gamma^3})k^2. \tag{95}$$

This mode corresponds to the *magnetosonic*, or *fast Alfvén*, mode.

(ii) $\vec{k} \parallel \vec{B}_0$. This case divides into two subcases. First we treat the case $\vec{v} \parallel \vec{B}_0$. The equation and its solution are the same; the only difference is the frequency of the mode, which is given here by

$$\omega_{ac}^2 = (\frac{c_s^2}{\Gamma^3})k^2. \tag{96}$$

This mode corresponds to a magnetized acoustic wave in the plasma.

The second subcase is specified by $\vec{v} \perp \vec{B}_0$. Here we obtain

$$\omega_A^2 = (\frac{V_{Ai}^2}{\Gamma^3})k^2. \tag{97}$$

This is the "pure," or shear (torsional) Alfvén wave.

For an arbitrary angle of propagation θ, one can describe the dispersion relation in a graphical manner by means of the wave-normal diagram.

We see from these results that the Alfvén waves in our cosmological plasma are formally analogous to those of their special-relativistic counterparts. All these modes, the acoustic (both magnetized and unmagnetized), the magnetosonic, and the Alfvén waves, decay in time as $t^{-1/2}$, so once again we find that these plasma wave frequencies behave like the photon frequency. The *phase velocities* of the acoustic and Alfvén waves are *constant* in time for this model, however.

Equation (91) plays a significant role in cosmic dynamo theory, when diffusive terms are neglected. When cosmic fluctuations or other turbulence provides the so-called passive (or high-β) velocity fields, we can describe the velocity field $\vec{v}$ as an (almost) external function. A complete treatment of this problem is very important, but is beyond the scope of the present paper.

(e) Collisional Effects

We have so far neglected collisional effects, which for an electron-positron plasma include the creation and annihilation of particles. This is generally valid if the plasma frequency is much greater than the collision frequency. This condition should be satisfied in the early part of the $e^+ - e^-$ epoch, and all our above results are applicable to this era. In the later part of the epoch, the opposite will be true, and the plasma will establish local equilibrium. In this case, the plasma oscillations will disappear, but our results which are derived from the MHD approximation should continue to hold. The transition region between the two regimes, where the plasma frequency is comparable to the collision frequency, is likely to be quite interesting, but will also be very complicated to treat theoretically, and we shall neglect it here. Our purpose in this section is to illustrate some new physics, rather than to carry out a complete statistical analysis of the plasma.

The general-relativistic Boltzmann equation for our metric can be written in the form[19]

$$\frac{DG}{Dt} = \text{collision} \quad \text{terms}, \tag{98}$$

where the convective derivative includes transport terms in the phase space. The "conformalized" distribution function is defined as

$$G \equiv t^{3/2} f, \tag{99}$$

where f is the particle distribution function. Except for the presence of the "conformal" factor $t^{3/2}$, Equation (98) has exactly the Newtonian form.

Now take density moments of Equation (98). The presence of the "conformal" factor $t^{3/2}$ results in an equation for $\hat{n} \equiv t^{3/2} n$, rather than for n alone; this is consistent with Equation (28). Let us write the "conformalized" number densities of the electrons, positrons, and photons as, respectively, $\hat{n}_e$,$\hat{n}_+$, and $\hat{n}_\gamma$. Then if we assume a form for the collision terms analogous to their Newtonian forms, we can write

$$\dot{\hat{n}}_e = \nu_c \hat{n}_\gamma - \tilde{\nu} \hat{n}_+ \hat{n}_e \tag{100a}$$

$$\dot{\hat{n}}_+ = \nu_c \hat{n}_\gamma - \tilde{\nu}\hat{n}_e \hat{n}_+ \tag{100b}$$

and

$$\dot{\hat{n}}_\gamma = -\nu_c \hat{n}_\gamma + \tilde{\nu}\hat{n}_e \hat{n}_+. \tag{100c}$$

Here ν_c is the collision frequency for pair creation, ν is the specific frequency of annihilations, and $\tilde{\nu} \equiv t^{-3/2}\nu$. We assume that $\nu_c > 0$, $\nu > 0$. In this model; both ν_c and $\tilde{\nu}$ vary in time.

Now set $\hat{n}_e = \hat{n}_+ = \hat{n}$. Then Equations (100) simplify to

$$\dot{\hat{n}} = \nu_c \hat{n}_\gamma - \tilde{\nu}\hat{n}^2 \tag{101a}$$

and

$$\dot{\hat{n}}_\gamma = -\nu_c \hat{n}_\gamma + \tilde{\nu}\hat{n}^2, \tag{101b}$$

from which we see that the total conformalized number density of quanta is conserved:

$$\hat{n} + \hat{n}_\gamma = \hat{N} = \text{constant}. \tag{101c}$$

These may be combined to yield

$$\dot{\hat{n}} = \nu_c(\hat{N} - \hat{n}) - \tilde{\nu}\hat{n}^2, \tag{102}$$

from which we obtain

$$\int \frac{d\hat{n}\tilde{\nu}^{-1}}{\hat{n}^2 + \frac{\nu_c}{\tilde{\nu}}\hat{n} - \frac{\nu_c}{\tilde{\nu}}\hat{N}} = -\int dt. \tag{103}$$

The behavior is determined by the denominator of the integral. Since

$$\left(\frac{\nu_c}{\tilde{\nu}}\right)^2 + \left(\frac{4\nu_c \hat{N}}{\tilde{\nu}}\right) > 0, \tag{104}$$

we obtain two real roots, which we shall denote n_1 and n_2. Assuming $\hat{n} = \hat{n}_0$ at $t_0 = 0$, and carrying out an integration by partial fractions, gives

$$\hat{n} = \frac{n_1 - n_2 \frac{n_0 - n_1}{n_0 - n_2} e^{-\Gamma t}}{1 - \frac{n_0 - n_1}{n_0 - n_2} e^{-\Gamma t}}, \tag{105}$$

where

$$\Gamma \equiv \sqrt{\nu_c^2 + 4\hat{N}\tilde{\nu}\nu_c} \tag{106}$$

and

$$\hat{n}_2 = \hat{n}_1 - \frac{\Gamma}{\tilde{\nu}}. \tag{107}$$

This demonstrates that the plasma equilibrates exponentially from $\hat{n} = \hat{n}_0$ to $\hat{n} = \hat{n}_1$, with characteristic timescale Γ^{-1}. The plasma oscillations are unaffected by the annihilations and creations.

IV. Conclusions

We have developed the fundamental theory of an electron-positron plasma in a radiation-dominated, FRW background. We have derived an explicit solution for the electromagnetic wave equation in such a Universe, and have shown that it plays the role in our spacetime that is filled by the sinusoidal functions in Minkowski spacetime. We have written the electrodynamic and magnetohydrodynamic equations for such a plasma, and have studied the linearizations of these equations. From this analysis, we obtain several interesting conclusions. (1) The redshift of a photon is the same whether it is in free space or in the plasma; thus the photon is self-similar. (2) The propagator of the photon (as well as that of waves in plasmas) in this background is represented by spherical Bessel (or Hankel) functions. (3) The frequencies of the fundamental plasma oscillations all decay, due to the expansion of the background, in the same way in whch the photon redshifts. This conclusion depends on both the background model and the plasma equation of state, however, and may change later in the radiation era. (4) Any primordial magnetic field must decay like $t^{-3/2}$ during the plasma, or radiation, epoch. This is a consequence only of the background model and the general- relativistic electromagnetic field equations (assuming a perfect conductor), and so holds throughout the radiation era. This result differs from a common assumption in the literature, and demonstrates the care that must be taken when general relativity is introduced into a theory. (5) The phase velocities of the free photon, the acoustic waves, and the Alfvén waves are constant for this particular plasma in this background. (6) Plasma oscillations are unaffected by the annihilation-creation collisions of electron-positron pairs. We expect that the acoustic waves will be affected, but we have not carried out the analysis, which requires a more careful treatment of the relativistic kinetic theory.

This paper has barely begun the investigation of plasma effects in the early Universe. Subsequent papers will study the kinetic theory of cosmological plasmas in more detail, and will apply the equations derived here to the development of a computer code for the study of nonlinear effects, including the dynamo effect mentioned previously. Collisional effects (annihilation/ creation) can be inserted into this code by a phenomenological approach.

Applications can be made to the theory of the primordial intergalactic magnetic fields, as well as to the study of matter fluctuations in the early Universe. Different assumptions about the equation of state of the plasma could yield different results and will be considered later. In addition, the same general methods and coding techniques can be applied to the quark-gluon plasma believed to play a role in the very early Universe, provided appropriate changes are made in the field equations. This area is clearly a promising avenue for future research.

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