Confinement of a rotating plasma

suppose $\mu = 0$

$$\frac{du_\parallel}{dt} = \underset{\sim}{V_E} \cdot \frac{d\underset{\sim}{b}}{dt} = \underset{\sim}{V_E} \cdot (\underset{\sim}{V_E} \cdot \underset{\sim}{\nabla})\, \underset{\sim}{b} + u_\parallel \underset{\sim}{b} \cdot \underbrace{(b \cdot \nabla) b}_{=0}$$

Can $\underset{\sim}{V_E} \cdot (\underset{\sim}{V_E} \cdot \nabla)\, \underset{\sim}{b}$ supply a confining force

Consider a mirror-field with

$$\underset{\sim}{J} = \underset{\sim}{\nabla} \times \underset{\sim}{B} = 0$$

$$\underset{\sim}{B} = \underset{\sim}{\nabla} \Phi_m (r, z)$$

$$\Phi_m = \text{[illegible]}\, f_0(z) + g(z, r)$$

Take $f_0(z) = \int^{z} dz'\, B_0(z', r=0)$

$$\underset{\sim}{B} = \underset{\sim}{\nabla} \Phi + \underset{\sim}{\nabla} g(z, r) = \hat{z} B_0(z) + \hat{r} \frac{\partial g}{\partial r} + \hat{z} \frac{\partial g}{\partial z}$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0$$

$$\frac{\partial B_0}{\partial z} = -\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial g}{\partial r} - \frac{\partial^2 g}{\partial z^2}$$

Thus:

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial g_1}{\partial r} = -\frac{\partial B_0}{\partial z}$$

$$\vdots$$

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial g_{n+1}}{\partial r} = -\frac{\partial^2 g_n}{\partial z^2}$$

$$g_1 \simeq -\frac{\partial B_0(z)}{\partial z}\frac{r^2}{4} \qquad \text{(accurate enough)}$$

$$\underset{\sim}{B} = \hat{\underset{\sim}{z}} B_0(z) + \nabla g_1(r,z) = \hat{z} B_0(z) - \frac{\partial^2 B_0}{\partial z^2}\frac{r^2}{4}\hat{z} - \hat{r}\frac{r}{2} B_0'(z)$$

$$\underset{\sim}{b} \simeq \hat{\underset{\sim}{z}} - \hat{r}\frac{r}{B_0 2}\frac{\partial B_0}{\partial z} + \mathcal{O}\left(\frac{r^2}{L^2}\right)$$

$$v_E \cdot \nabla \underset{\sim}{b} = \dot{\theta}\,\hat{\underset{\sim}{\theta}} \cdot \frac{\partial}{\partial r}\left(-\hat{r}\frac{r}{B_0 2}\frac{\partial B_0}{\partial z}\right) \qquad \left[\text{note } \frac{\partial}{\partial\theta}\hat{r} = \hat{\theta}\right]$$

$$= -\frac{r}{2B_0}\frac{\partial B_0}{\partial z} \quad ; \quad V_E = \dot{\theta} r$$

$$\therefore \quad \frac{d v_{\parallel}}{dt} \simeq \frac{d^2 z}{dt^2} = -\frac{r^2\dot{\theta}^2}{2B_0}\frac{\partial B_0(z)}{\partial z} \simeq -\frac{r^2\dot{\theta}^2}{2B_0}\frac{\partial^2 B_0(0)}{\partial z^2} z$$

$$\boxed{\frac{d^2 z}{dt^2} = -\frac{V_E^2}{2}\frac{\partial^2 B_0 z}{B_0 \partial z^2}} \qquad \omega_b^2 = \frac{V_E^2}{2B_0}\frac{\partial^2 B_0}{\partial z^2}$$

$$z \simeq z_0 \cos(\omega_b t + \phi)$$

looks almost like magnetic moment: (2)

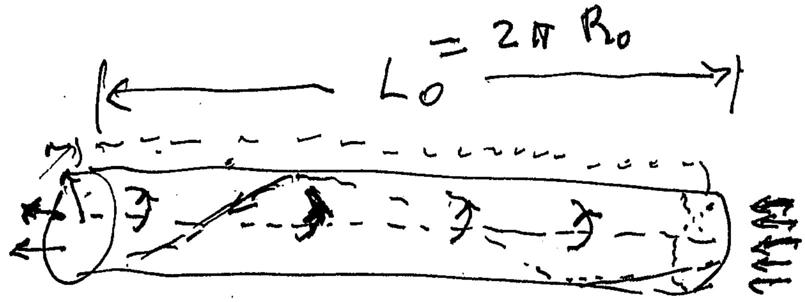
Toroidal Confinement

We have seen that confinement requires a poloidal field, and that mirror trapped particles deviate from magnetic flux surface by an amount $V_{\parallel}/\omega_{c\theta}$ (Larmor radius in a poloidal field)

Lets look more carefully at concept of flux surface

Consider a cylindrical plasma, periodic in L_0, which mocks a toroidal field periodic in $2\pi R_0$

$$L_0 = 2\pi R_0$$



$dz = \frac{L\, d\varphi}{2\pi} = R_0\, d\varphi$

$R_0 = \frac{L}{2\pi}$

There is a special axial line (magnetic axis)

The poloidal magnetic field produces a poloidal flux at a surface $r = \text{constant}$

$$\Psi_p \equiv \int_0^r B_\theta(r') L\, dr' \equiv \Psi_p(r)$$

This is addition "toroidal" flux enclosed by $r = \text{constant}$

$$\Psi_T \equiv 2\pi \int_0^r B_z\, r\, dr$$

Note $d\Psi_p = B_\theta L\, dr$

$d\Psi_T = 2\pi B_z\, r\, dr$; $B_z = B_\varphi$

$$\frac{d\Psi_p}{d\Psi_T} = \frac{B_\theta L\, dr}{2\pi B_z\, r\, dr} = \frac{B_\theta L}{2\pi B_z r} = \frac{B_\theta R_0}{B_z r} = \frac{1}{q}$$

It is now shown that as we move along a field line, that $\frac{d\Psi_p}{d\Psi_t}$ is the ratio, ι, of the number of times the field line moves around the magnetic axis, ΔN_p, to the number of times it goes an axial period ΔN_z

$$\frac{ds}{B} = \frac{r\,d\theta}{B_\theta} = \frac{dz}{B_z}$$

$$\therefore \quad \frac{d\theta}{dz} = \frac{B_\theta}{r B_z} = \frac{\Delta\theta}{\Delta z}$$

$$\Delta N_p \equiv \frac{\Delta\theta}{2\pi}, \qquad \Delta N_z \equiv \frac{\Delta z}{L}$$

$$\iota \equiv \frac{\Delta N_p}{\Delta N_z} = \frac{\Delta\theta/}{2\pi/\Delta z/L} = \frac{L}{2\pi}\frac{B_\theta}{r B_z} = \frac{d\Psi_p}{d\Psi_z} = \frac{R_0 B_\theta}{r B_z}$$

QED

Now we know that

$$\underset{\sim}{B}_p = \underset{\sim}{\nabla} \times \underset{\sim}{A}_p = \underset{\sim}{\nabla} \times \hat{\underset{\sim}{z}} A_z$$

$$= \underset{\sim}{\nabla} \times (\underset{\sim}{\nabla} z A_z) \qquad \left(\begin{array}{l} dz = R_0 d\varsigma \\ \underset{\sim}{\nabla}\varsigma = \cancel{\nabla z}/R_0 \end{array} \right)$$

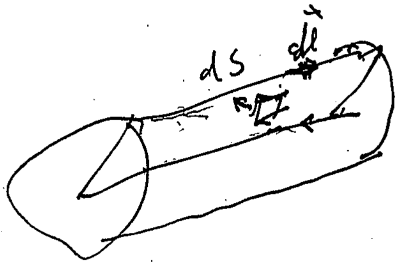
$$= \underset{\sim}{\nabla} \times (\underset{\sim}{\nabla} \varsigma R_0 A_z) = \nabla(R_0 A_z) \times \underset{\sim}{\nabla}\varsigma$$

Observe that $R_0 A_z = \Psi_p / 2\pi \equiv \psi_p$ as

$$\Psi_p \equiv \int \underset{\sim}{B}_p \cdot d\underset{\sim}{S} = \int d\underset{\sim}{S} \cdot \underset{\sim}{\nabla} \times (\underset{\sim}{\nabla}\varsigma R_0 A_z)$$

$$= \oint d\underset{\sim}{l} \cdot \underset{\sim}{\nabla}\varsigma R_0 A_z = \int_0^{2\pi} d\varsigma\, R_0 A_z(r')\Big|_{r'=r}$$

$$- \underbrace{\int_0^{2\pi} d\varsigma\, R_0 A_z(r'=0)}_{=0}$$



$$= 2\pi R_0 A_z(r)$$

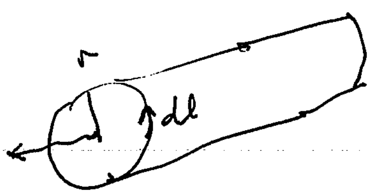
$$\therefore \quad \boxed{\psi_p = \frac{\Psi_p}{2\pi} = R_0 A_z}$$

Similarly $\underset{\sim}{B}_T = \underset{\sim}{\nabla} \times \hat{\underset{\sim}{\theta}} A_\theta \cancel{B_{\theta 0}} = \underset{\sim}{\nabla} \times (\underset{\sim}{\nabla}\theta (r A_\theta)) / 2R^2$

$$= -\underset{\sim}{\nabla}\theta \times \underset{\sim}{\nabla}(r A_\theta) = -\underset{\sim}{\nabla} \times (\underset{\sim}{\nabla}\theta\, r A_\theta)$$

$$\Psi_T = \int \underset{\sim}{B}_T \cdot d\underset{\sim}{S} = -\int d\underset{\sim}{S} \cdot \underset{\sim}{\nabla} \times (\underset{\sim}{\nabla}\theta\, r A_\theta) = -\int r\, d\theta \frac{1}{r} A_\theta r$$

$$= -2\pi r A_\theta$$



$$\boxed{\frac{\Psi_T}{2\pi} = -r A_\theta \equiv \psi_T}$$

The point of this discussion is that we have found that

$$\underset{\sim}{B} = \underset{\sim}{\nabla}\theta \times \nabla\psi_T - \underset{\sim}{\nabla}\zeta \times \underset{\sim}{\nabla}\psi_p$$

$$= \underset{\sim}{\nabla}\underset{\sim}{\theta} \times \nabla\psi_p \frac{\partial\psi_T}{\partial\psi_p}(\psi_p) - \nabla\zeta \times \nabla\psi_p$$

$$\boxed{\frac{\partial\psi_T(\psi)}{\partial\psi_p} = q} \qquad = \underset{\sim}{\nabla}\psi_p \times \nabla\left(q(\psi)\theta - \zeta\right)$$

with $q(\psi_p) \equiv \frac{1}{\iota(\psi_p)} = \frac{\partial\psi_T}{\partial\psi_p}(\psi_p)$

This form is robust to any magnetic field that contains nested flux surfaces, and $\underset{\sim}{B}$ can be written as

$$\underset{\sim}{B} = \underset{\sim}{\nabla}\theta \times \nabla\psi_T - \underset{\sim}{\nabla}\zeta \times \psi_p$$

$$\underset{\sim}{B} = \underset{\sim}{\nabla}(\theta\, q(\psi) - \zeta) \times \nabla\psi_p$$

with $\theta \equiv$ poloidal angle, $\zeta \equiv$ toroidal angle } magnetic coordinates

$$B(\psi_T, \theta, \zeta) = B(\psi_T, \theta + 2\pi, \zeta) = B(\psi_T, \theta, \zeta + 2\pi)$$

$\underset{\sim}{\nabla}\theta(\underset{\sim}{r})$, $\underset{\sim}{\nabla}\phi(\underset{\sim}{r})$ periodic in θ and ζ.