Lecture # 5

Orbits in Toroidal field

Acceleration along field Line

$$\underset{\sim}{b} \cdot \frac{d(\underset{\sim}{u} + \underset{\sim}{E}\times\underset{\sim}{b}/B)}{dt} = \frac{e}{m}E_{\parallel}$$

$$\Vert$$

$$\frac{du_{\parallel}}{dt} - \underset{\sim}{u}\cdot\frac{d\underset{\sim}{b}}{dt} - \frac{\underset{\sim}{E}\times\underset{\sim}{b}}{B}\cdot\frac{d\underset{\sim}{b}}{dt} = \frac{e}{m}E_{\parallel}$$

Average of gyro-frequency

$$\overline{\underset{\sim}{u}\cdot\frac{d\underset{\sim}{b}}{dt}} = \overline{\underset{\sim}{u}_{\perp}\cdot(\underset{\sim}{u}_{\perp}\cdot\nabla)\underset{\sim}{b}}$$

$$= \frac{u_{\perp}^2}{2}\left[I - \underset{\sim}{b}\,\underset{\sim}{b}\right]:\nabla b = \frac{u_{\perp}^2}{2}\nabla\cdot\underset{\sim}{b}$$

$$= -\frac{u_{\perp}^2\,\underset{\sim}{b}\cdot\nabla B}{2B} = -\mu(\underset{\sim}{b}\cdot\nabla)|B| \quad \left(\mu = \frac{u_{\perp}^2}{2B}\right)$$

$$\therefore \quad \frac{du_{\parallel}}{dt} = -\mu(b\cdot\nabla)B + \frac{e}{m}E_{\parallel} + \underset{\sim}{V}_E\cdot\frac{D\underset{\sim}{b}}{Dt}$$

mirror confinement force

parallel electric field force

(can lead to runaway electrons)

Dynamic mirror

Look at confinement

Neglect V_E

$$\frac{du_{\parallel}}{dt} = -\mu (\underset{\sim}{b} \cdot \nabla) B$$

If $B = B(s)$

s distance along field line

$$u_{\parallel} = \frac{ds}{dt}$$

$$\frac{d^2 s}{dt^2} = -\mu \frac{\partial B}{\partial s} = -\frac{\partial \Phi_{eff}}{\partial s}$$

$$\Phi_{eff} = \mu B$$

energy $\quad E = \frac{u_{\parallel}^2}{2} + \Phi_{eff}$

$$E = \frac{u_{\parallel}^2}{2} + \mu B(s)$$

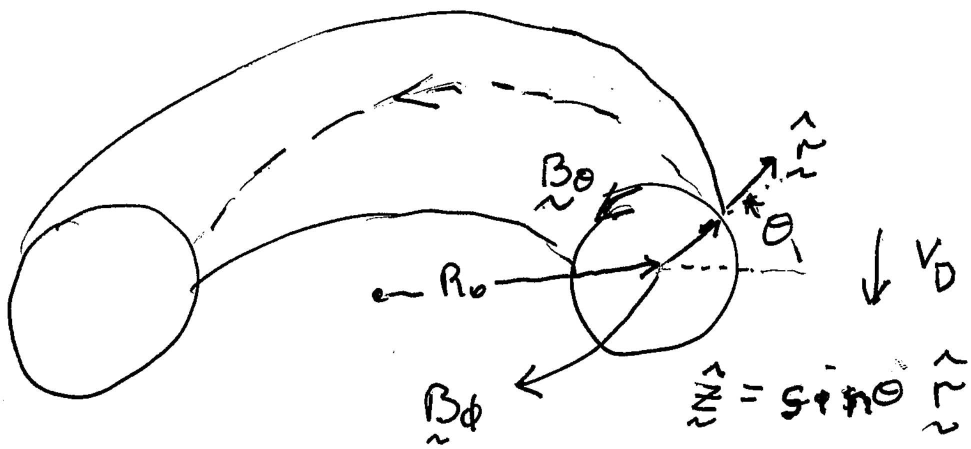
$$V_{\parallel} = \pm \left[2 (E - \mu B(z)) \right]^{1/2}$$

Near mid plane: $\quad B(s) = B_0(0) + \frac{s^2}{2} \frac{\partial^2 B}{\partial s^2}$

$$\frac{\partial B(s)}{\partial s} = s \frac{\partial^2 B}{\partial s^2}(0)$$

$$\boxed{\frac{d^2 s}{dt} \approx -\mu \frac{\partial^2 B}{\partial s^2} s} : \quad s = s_0 \cos(\omega_b t) ; \quad \omega_b^2 = \mu \frac{\partial^2 B}{\partial s^2}(0) \approx V_{\perp 0}^2 / L^2$$

To avoid continual downward shift a poloidal magnetic field is needed to confine particles



$\hat{z} = \sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}$

$$\underset{\sim}{B} = B_0\left(1 - \frac{r}{R}\cos\theta\right)\hat{\phi} + B_\theta(r)\,\hat{\theta}$$

$$\underset{\sim}{V}_D = -\left(\frac{V_\perp^2}{2} + V_\parallel^2\right)\frac{1}{\omega_c R}\left[\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}\right]$$

(assumptions) $\quad \dfrac{B_\theta}{B_\phi} \sim \dfrac{r}{R} << 1$

$$\underset{\sim}{V}_D = V_\parallel \underset{\sim}{b} + \underset{\sim}{V}_B + \underset{\sim}{V}_{\cancel{\kappa}}$$

$$= V_\parallel\hat{\phi} + \hat{\theta}\left(V_\parallel\frac{B_\theta}{B} - \frac{\left(\frac{V_\perp^2}{2} + V_\parallel^2\right)}{\omega_c R}\cos\theta\right)$$

$$- \hat{r}\,\frac{\left(\frac{V_\perp^2}{2} + V_\parallel^2\right)}{\omega_c R}\sin\theta \;; \quad \mu = \frac{V_\perp^2}{2B}, \; E = \frac{V_\parallel^2 + \mu B(r,\theta)}{2}$$

Thus: $\dfrac{dr}{dt} = -\dfrac{[2E - \mu B]}{\omega_c R}\sin\theta \;; \quad r\dfrac{d\theta}{dt} \approx [2(E - \mu B_0(1 - \frac{r}{R}\cos\theta))]^{1/2}\dfrac{B_\theta}{B} - \dfrac{[2E - \mu B_0]}{\omega_c R_0}\cos\theta$

These equations can be solved:

B)

Slick Method of Solution

We note the following

If $\underset{\sim}{B}$ - field is a constant

$$E = \frac{V_\parallel^2}{2} + \mu B(\underset{\sim}{r})$$

is a constant of motion

$$V_\parallel = \sqrt{(E - \mu B_0 (1 - \frac{r}{R}\cos\theta)}$$

If B - field has a toraidal symmetry

$$P_\phi = R^2\dot{\phi} + \frac{e}{cm} R A_\phi ;$$

a constant of motion

$$\bar{P}_\phi = \left(\underset{\sim}{V}_\parallel \frac{\underset{\sim}{b}\cdot\underset{\sim}{B}}{B} + \cancel{V}_{\underset{\sim}{D}\perp}\cdot\hat{\phi}\right) + \frac{e}{c} R A_\phi$$

$$\cong \sqrt{2(E - \mu B_0 (1 - \frac{r}{R}\cos\theta)} \left(\text{if } \frac{B_\phi}{B_\theta} \gg 1\right) + \frac{e}{mc} R A_\phi$$

∴ [illegible] $+$ [illegible] $R_0 V_\parallel$

If $r/R \ll 1$

$$\boxed{\bar{P}_\phi = (R_0 + r\cos\theta)\sqrt{2(E - \mu B_0 (1 - \frac{r}{R}\cos\theta))} + \frac{eRA_\phi(r,\theta)}{mc}}$$

$$\bar{P}_\phi = (R_0 + r\cos\theta)\left[2\left[E - \mu B_0\left(1 - \frac{r}{R}\cos\theta\right)\right]\right]^{1/2} + \frac{eRA_\phi(r,\theta)}{mc}$$

If orbit deviation is much less than r_0, the second term turns out to be the largest.

Let us choose r_0 (a reference point so that at $\theta = 0$)

$$\bar{P}_\phi = +\frac{e A_\phi R(r_0, \theta=0)}{mc} + (R_0 + r_0)\left\{2\left[E - \mu B_0\left(1 - \frac{r_0}{R}\right)\right]\right\}^{1/2}$$

Then we have $\left(\text{since } \frac{1}{R}\frac{\partial}{\partial R}(RA_\phi) = -B_\theta\right)$

$$-\frac{eRA_\phi(r,\theta)}{mc} = -\frac{eRA_\phi(r_0, \theta=0)}{mc} - \frac{eR_0 B_\theta(r_0)\,\delta r(r_0,\theta)}{mc}$$

$(\delta r = r - r_0)$ and we find

$$\delta r = \frac{1}{\omega_{c\theta}}\left[\left(1 + \frac{r}{R}\cos\theta\right)\left\{2\left[E - \mu B_0\left(1 - \frac{r_0}{R}\cos\theta\right)\right]\right\}^{1/2} - \left(1 + \frac{r_0}{R}\right)\left\{2\left[E - \mu B_0\left(1 - \frac{r_0}{R}\right)\right]\right\}^{1/2}\right]$$

If $E - \mu B_0 \gg \frac{r_0}{R}\mu B_0$

$$\delta r \simeq -\frac{(1-\cos\theta)}{\omega_{c\theta}}\left[\left[2(E-\mu B_0)\right]^{1/2} + \frac{\mu B_0}{[2(E-\mu B_0)]^{1/2}}\right] = -\frac{(1-\cos\theta)\,q\left(V_{\parallel 0}^2 + \frac{V_{\perp 0}^2}{2}\right)}{\omega_{c0} V_{\parallel 0}}$$

$$q = \frac{B_\phi(r_0)\, r_0}{B_{\theta 0} R}; \quad V_{\parallel 0}^2 = 2\left[E - \mu B_0\left(1 - \frac{r_0}{R}\right)\right]$$

$$V_{\perp 0}^2 = 2\mu_0 B_0$$

but gets large as $V_{\parallel 0} \to 0$ also $B_\theta \to 0$

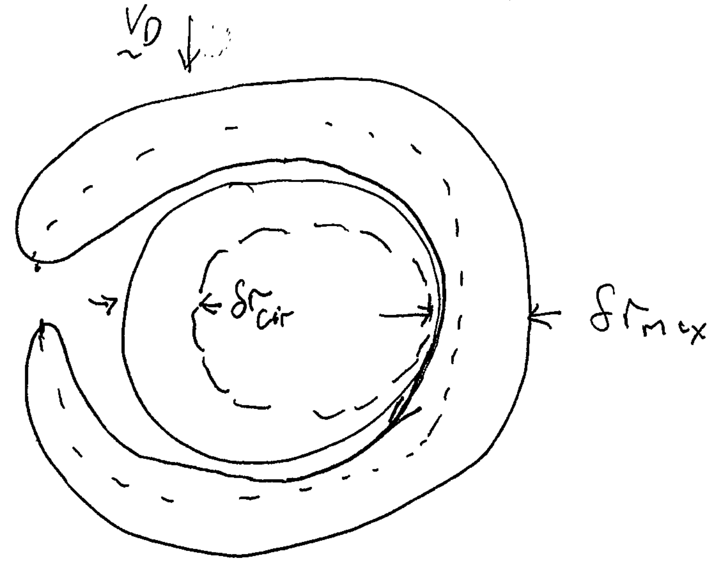
(8)

If $\frac{V_{\parallel 0}^2}{V^2} \simeq \frac{2r}{R}$. $V_{\parallel 0}$ = parallel velocity at $\theta = 0$.

$= [2(E - \mu B_0(1 - r_0/R))]^{1/2}$

Then orbit width has maximum where $E - \mu B_0 = \frac{r_0}{R}\mu B_0$

Here particle stagnates on inside of torus and turns around



Maximum parallel velocity at $\theta = 0$ that is trapped

$$V_{\parallel 0} = \sqrt{2}\left(\frac{r}{R}\right)^{1/2} V_{\perp 0}$$

$$\delta r_{max} = 2\frac{V_{\parallel 0}}{\omega_{c\theta}} \simeq 2\sqrt{2}\frac{V_{\perp 0}}{\omega_{c\theta}}\left(\frac{r_0}{R}\right)^{1/2} = 2\sqrt{2}\,q\frac{V_{\perp 0}}{\omega_{c0}}\left(\frac{R}{r}\right)^{1/2}$$

$$\delta r_{cir} \simeq \frac{2q\,(V_{\parallel 0}^2 + V_{\perp 0}^2/2)}{\omega_{c0}\, V_{\parallel 0}}$$

Trapped particles are contained within a "poloidal Larmor radius" where the velocity is the parallel velocity