Lecture # 38

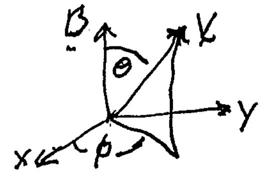
Resistivity &

Quasi-Linear theory

for radial diffusion

Return to collisional transport and Lorentz collisions operator (model)

If accurately describes collisions of electron by ions.



$$\frac{\partial f}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla} f + \frac{e\underset{\sim}{E}}{m} \cdot \frac{\partial f}{\partial v} + \omega_c \frac{\partial}{\partial \underset{\sim}{v}} \cdot (\underset{\sim}{v} \times \underset{\sim}{b}\, f)$$

$$= + \frac{\nu(v)}{2} \left[\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \sin\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} \right] f$$

$$\nu(v) = 4\pi Z \frac{e^4 n_0}{m_e^2 v^3} \ln\Lambda \qquad = -\frac{\nu}{2} \hat{C} f$$

$$\hat{C}\, Y_{lm}(\theta, \phi) = -l(l+1)\, Y_{lm}(\theta, \phi)$$

Let us consider a plasma that varies in the x-direction and has a B-field in the z-direction

$\frac{\nu}{\omega_c} \ll 1$ is assumed

but this is taken as a subsidiary ordering

Let $F = \frac{n(x)}{(2\pi T(x)/m)^{3/2}} \exp\left(-\frac{mv^2}{2T(x)}\right) + f$

$$\overset{\displaystyle \|}{F_M}$$

If we look for a steady solution order $\frac{\nu}{\omega_c} \approx O(1)$, $\underset{\sim}{E} = E_z \hat{z} + E_y \hat{y}$

$$+ v \sin\theta \left(\frac{\cos\phi}{n} \frac{\partial n}{\partial x} + \frac{\cos\phi}{T} \frac{\partial T}{\partial x} \left(\frac{mv^2}{2T} - \frac{3}{2} \right) \right) F_M$$

$$\left(\frac{e E_z v \cos\theta}{T} + \frac{e E_y v \sin\theta \cos\phi}{T} \right) F_M$$

$$- \omega_c \frac{\partial f}{\partial \phi} = - \frac{\nu}{2} \hat{C} f \; ; \quad \nu = \frac{4\pi Z n_0 e^4}{m^2 v^3} \ln\Lambda$$

Let us go over how we would solve for resistivity $\perp$ to B field.

For final, I am asking you to solve for D_{ij}, where

$$\Gamma_x = \int d\underset{\sim}{v}\, v_x f = -D_{11} \frac{\partial n}{\partial x} - D_{12} \frac{n}{T} \frac{\partial T}{\partial x}$$

$$Q_x = \int d\underset{\sim}{v}\, v_x \frac{mv^2}{2} f - \frac{3}{2} T \Gamma_x = -D_{21} T \frac{\partial n}{\partial x} - D_{22}\, n \frac{\partial T}{\partial x}$$

(when $\underset{\sim}{E} = 0$)

and $\omega_c \gg \nu$

$$f = f_{\parallel} + f_{\perp}$$

For electric field problem

$$-e\frac{E_z}{T} v\cos\theta F_M = -\frac{\nu}{2}\hat{C} f_{\parallel} = -\nu f_{\parallel} \quad ; \text{ parallel}$$

$$-\omega_c \frac{\partial f_{\perp}}{\partial \phi}$$

$$-e E_y \frac{v \sin\theta}{T} \sin\phi F_M = -\frac{\nu}{2}\hat{C} f_{\perp} = -\nu f_{\perp} + \omega_c \frac{\partial f_{\perp}}{\partial \phi}$$

perp.

$$f_{\perp} = f^{+} e^{i\phi} + e^{-i\phi} f^{-}$$

$$\pm i \frac{e E_y}{2} \frac{v \sin\theta}{T} F_M = -\nu f^{\pm} \pm i \omega_c f^{\pm}$$

$$f^{\pm} = \frac{e E_y v \sin\theta}{2\omega_c T \omega_c \pm i\nu} F_M$$

$$j_x \pm i j_y \equiv j^{\pm} = \int e \sin\theta\, v(\cos\phi \pm i \sin\phi)\, d^3v \left(f^{+} e^{i\phi} + f^{-} e^{-i\phi}\right)$$

$$= \int d^3v\, f^{\mp} \sin\theta\, v e$$

$$= 2\pi \int_0^{\infty} dv\, v^2 \int_0^{\pi} \sin^2\theta\, d\theta \; \frac{e^2 E_y \sin\theta}{2\omega_c T(\omega_c \mp i\nu(v))} \; n_0 \frac{e^{-v^2/2v_{th}^2}}{(2\pi v_{th}^2)^{3/2}}$$

$$= \frac{e^2 E_y}{2(2\pi)^{1/2} \omega_c T} \int_0^{\infty} dv \frac{v^4 e^{-\frac{v^2}{2v_{th}^2}}}{v_{th}^3} \int_0^{\pi} d\theta \frac{\sin^3\theta}{\omega_c \mp i\nu(v_{th}) v^3/v_{th}^3}$$

$$\nu(v_{th}) = \frac{4\pi Z n e^4 \ln\Lambda}{m^2 v_{th}^3} \qquad \int_0^{\pi} d\theta \sin\theta (1-\cos^2\theta) = \frac{4}{3}$$

$$j_x \pm i j_y \simeq \frac{2e^2 n_0 E_y}{3(2\pi)^{1/2} m\omega_c} \int_0^\infty dx\, x^4 \frac{e^{-x^2/2}}{\left(1 \mp \frac{i\nu(v_{th})}{\omega_c x^3}\right)}$$

If $\frac{\nu}{\omega_c} \ll 1$

$$j_x \pm i j_y = \frac{2e^2 n_0 E_y}{3(2\pi)^{1/2} m\omega_c} \int_0^\infty dx\, x^4 e^{-x^2}\left(1 \pm \frac{i\nu(v_{th})}{\omega_c x^3}\right)$$

$$= \frac{e^2 n_0}{m\omega_c} E_y \left(1 \pm \frac{i\,2\nu_0(v_{th})}{3(2\pi)^{1/2}\omega_c}\right)$$

To get resistivity, invert

$$E_y = \frac{4\pi\omega_c}{\omega_p^2}(j_x \pm i j_y)\left(1 \mp \frac{i\,2\nu}{3(2\pi)^{1/2}\omega_c}\right)$$

The resistive electron current along y is (take real part)

$$E_x = \frac{4\pi}{\omega_{pe}^2}\frac{2\nu}{3(2\pi)^{1/2}} j_y \equiv \eta_\perp j_y \qquad \nu = \frac{n_0 e^4 Z 4\pi \ln\Lambda}{m_e^2 v_{th}^3}$$

$$\eta_\perp = \frac{E_y}{j_y} = \frac{8\pi}{3(2\pi)^{1/2}}\frac{\nu(v_{th})}{\omega_{pe}^2}$$

Parallel resistivity is found from

$$-\frac{vE}{T}\cos\theta \; F_M = -\nu(v)\, f_{\parallel}$$

$$f_{\parallel} = \frac{e v E \cos\theta}{T\,\nu(v)}$$

$$j_{\parallel} = e\int d^3v\, v\cos\theta\, f_{\parallel} = \int d^3v\, v^2 \frac{\cos^2\theta\, v^3 e^2 F_M E}{\nu(v_{th})\, v_{th}^3\, T}$$

$$\nu(v_{th}) = \nu(v = v_{th}) \qquad \int_0^{\pi} d\theta \sin\theta\cos^2\theta = \frac{2}{3}$$

substitute for F_M

$$= \frac{n_0 e^2\, 4\pi}{3 m \nu_0}\int_0^{\infty} dv\, v^7 \frac{e^{-v^2/2v_{th}^2}\, E}{(2\pi)^{3/2}\, v_{th}^8}$$

$$= \frac{2 n e^2 E}{m (2\pi)^{1/2}} \cdot \frac{6\cdot 4\cdot 2}{3\,\nu(v_{th})} = \frac{32\,\omega_p^2}{4\pi (2\pi)^{1/2}} E$$

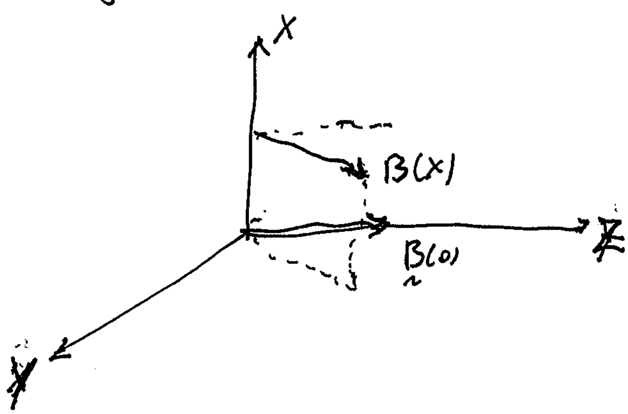
$$\eta_{\parallel} = \frac{E}{j_{\parallel}} = \frac{(2\pi)^{1/2}\pi}{8}\,\frac{\nu(v_{th})}{\omega_p^2};$$

recall $\eta_{\perp} = \frac{8\pi}{3(2\pi)^{1/2}}\frac{\nu(v_{th})}{\omega_p^2}$

$$\frac{\eta_{\perp}}{\eta_{\parallel}} = \frac{64}{6\pi} \simeq 3.4$$

Expression for $\eta_{\parallel}$ is accurate for large Z

Consider waves is a sheared magnetic field



$$\underset{\sim}{B} = B_0 \left[\left(1 - \frac{x^2}{L_s^2}\right)\hat{\underset{\sim}{z}} + \frac{x}{L_s}\hat{\underset{\sim}{y}} \right]$$
$$\simeq B_0 \left[\hat{\underset{\sim}{z}} + \frac{x}{L_s}\hat{\underset{\sim}{y}} \right]$$

We look for an electrostatic wave of the form

$$\phi = \phi_0 \exp[i k_y y + i k_z z]$$

Focus on a point $x = x_0$ (choose it to be where $x_0 = 0$) where $\underset{\sim}{k} \cdot \underset{\sim}{b}(x) = 0$

Recall for drift wave when

$$f_e = \frac{n_0(x)}{(2\pi T_e(x))^{3/2}} \exp\left(-\frac{m_e v^2}{2 T_e(x)}\right)$$

$$f_i = \frac{n_0(x)}{(2\pi T_i(x))^{3/2}} \exp\left(-\frac{m_i v^2}{2 T(x)}\right)$$

$f_j = F_j + \delta f_j e^{-i\omega t}$ perturbed distributions satisfies

$$\boxed{\cancel{\delta f_j = -e_j \frac{\phi F_{Mj}}{T_j} \quad i\omega \int_{-\infty}^{t} dt' \exp(-i\omega t' + ik}}$$

$$\delta f_j = e_i \phi_0 \frac{\partial F_{Mj}}{\partial E} e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} + i\phi_0 e_j \left(\omega \frac{\partial F}{\partial E}(E, x_g) + \frac{k_y}{\omega_c}\frac{\partial F}{\partial x_g}(E, x_g)\right)\left(1 - \frac{k_\perp^2 v_\perp^2}{2\omega_{cj}^2}\right)$$

$$\int_{-\infty}^{t} dt' \exp\left[\begin{array}{l} -i\omega(t'-t) + i k_y y(t') \\ + i k_z z(t') \end{array} \right]$$

Now $y(t') = y + v_\parallel \frac{B_y}{B}(t'-t) = y + v_\parallel \frac{x}{L_s}(t'-t)$

$z(t') = z + v_\parallel \frac{B_z}{B}(t'-t) = z + v_\parallel (t'-t)$

We need to integrate

$$\int_{-\infty}^{t} dt' \exp\left[-i\left(\omega - k_z v_\parallel - v_\parallel k_y \frac{x_g}{L_s}\right)(t'-t) + i k_y y + i k_z z\right]$$

to obtain

$$\frac{-i \exp(i k_y z + i k_z z)}{\left[\omega - \left(k_z + k_y \frac{x}{L_s}\right) v_\parallel\right]} \qquad \text{Let } k_z = -\frac{k_y x_0}{L_s}$$

$\therefore$

$$e^{-i\underset{\sim}{k}\cdot r}\, \delta f_j = \phi_0 e_j \frac{\partial F}{\partial E_j} - e\phi_0 \frac{\omega \frac{\partial F}{\partial E} + \frac{k_y}{\omega_{cj}}\frac{\partial F}{\partial x_g}}{\omega - k_y \frac{(x_g - x_0)}{L_s} v_\parallel}\left(1 - \frac{k_\perp^2 v_\perp^2}{2\omega_c^2}\right)$$

For ions assume $\omega \gg k_\parallel v_{thi} \simeq k_y \frac{x_s v_{thi}}{L_s}$

For electrons assume $\omega \ll k_\parallel v_{the} \simeq \frac{k_y x_s v_{the}}{L_s}$

Thus to leading order

$$e_e = -e \quad , \quad \frac{\partial F_j}{\partial E} = -F_j/T_j$$

$$e^{-i\underline{k}\cdot\underline{r}}\,\delta f_e = \frac{e F_{me}}{T_e}\phi_0 + \text{extra}_e$$

$$e^{-i\underline{k}\cdot r}\,\delta f_i = -\frac{eF_i}{T_i}\phi_0 + \frac{eF_i}{T_i}\left(1 - \frac{k_\perp^2 v_\perp^2}{2\omega_{ci}^2}\right) - \frac{\hat{\omega}_i^*}{\omega}\frac{eF_i}{T}\left(1 - \frac{k_\perp^2 v_\perp^2}{\omega_{ci}^2}\right) + \text{extra}_i$$

$$\hat{\omega}_j^* = -\frac{k_y}{n_0\omega_{cj}}\frac{\partial}{\partial x}n_0 v_{thj}^2\left[1 - \left(\frac{3}{2} + \frac{mv^2}{2T}\right)\eta\right]$$

$$= -\frac{eF_i}{T_i}\frac{k_\perp^2 v_\perp^2}{\omega_{ci}^2} + \frac{\hat{\omega}_i^*}{\omega}\left(1 + \frac{k_\perp^2 v_\perp^2}{\omega_{ci}^2}\right)\frac{eF_i}{T_i}$$

Requiring Quasi-neutrality

$$-e n_e + e n_i = 0$$

$$-e\int d^3v\,\delta f_e + e\int d^3v\,\delta f_i = 0$$

Leads to $\left(\rho_i^2 = v_{thi}^2/\omega_{ci}^2\right)$

$$D_R(\omega) = \frac{\omega_{pi}^2}{v_{thi}^2}\left[\frac{1}{\tau} + k_\perp^2\rho_i^2 + \frac{\omega_i^*}{\omega}\left(1 - k_\perp^2\rho_i^2(1+\eta_i)\right)\right] = 0$$

$$\omega_e^* = -\omega_i^* T_e/T_i \quad , \quad \tau = T_e/T_i$$

$$\omega_0 \simeq \omega_e^*\left[1 - k_\perp^2\rho_i^2(\tau + 1 + \eta_i)\right]$$

Destabilization comes from electrons

$$D_R(\omega) + i\,D_I(\omega) = 0$$

$$D_R(\omega_0 + \delta\omega) + i\,D_I(\omega) \simeq \delta\omega\frac{\partial D_R}{\partial\omega}(\omega_0) + i\,D_I(\omega_0)$$

$$\delta\omega = -i\,D_I \Big/ \frac{\partial D_R}{\partial\omega}$$

$$\frac{\partial D_R}{\partial \omega} \simeq -\frac{\omega_{pi}^2}{V_{thi}^2} \qquad \frac{\omega_*^*}{\omega^2} = \frac{\omega_{pi}^2}{V_{the}^2}\frac{1}{\tau}\,\omega_e^*$$

Now D_I is given by

$\int$ extra $e\, d^3v$

$$= e\phi\,\pi\int d^3v\ \delta\left(\omega - k_y\frac{(X_g-X_0)}{L_s}\right)\left(\omega\frac{\partial F_e}{\partial E} + \frac{k_y}{m\omega_{ce}}\frac{\partial F_e}{\partial X_g}\right)$$

$$\sim \frac{\omega}{T}F_{Me}\left(1-\frac{\omega}{\omega_e^*}\left(1+\frac{\eta_e}{2}\right)\right)$$

Instability for

$$\omega = \omega_e^*\left(1-k_\perp^2\rho_i^2(\tau+1+\eta_i)\right) < \omega_e^*\left(1+\frac{\eta_e}{2}\right)$$

The drive involves two dimensions:

E and $P_y = -\frac{e}{c}\int_{x_0}^{x} B_z\,dx + mv_y \simeq \omega_{ce}(X_g - X_0)$

This can be reduced to one dimension by defining new equilibrium variable

$$E' = E - \frac{\omega}{k_y}P_y = E - \frac{m\omega}{k_y}\omega_{ce}(X_g - X_0)$$

Then:

$$\omega\frac{\partial F_e}{\partial E}\Big|_{X_g} + \frac{k_y}{m\omega_{ce}\omega}\frac{\partial F_e}{\partial X_g}\Big|_E = \frac{k_y}{m_e\omega_{ce}\omega}\frac{\partial F}{\partial X_g}\Big|_{E'}$$

Then the D_I can be written

$$= -i \frac{e\phi}{m_e} \pi\omega \int d^3v \, \delta\left(\omega - \frac{k_y(x_g - x)}{L_s} v_\parallel\right) \cdot \frac{k_y \, \partial F_e/\partial x_g}{\omega \, \omega_{ce}}\bigg|_{E'}$$

$$= -i\pi \int d^3v \, \delta\left(\omega - \frac{k_y(x_g - x)}{L_s} v_\parallel\right) c \frac{k_y \phi}{B} \frac{\partial F_e}{\partial x_g}\bigg|_{E'}$$

Now let us consider the y - average quasi-linear theory for the electrons

$$\bar{G} = \int_0^{L_y} dy \, G / L_y$$

$$\frac{\partial \bar{F}}{\partial t} = \overline{\sum_{k_y} c \frac{\mathbf{b} \times \nabla\phi}{B} \cdot \nabla (F_0 + f_k)}\bigg|_{E'}$$

$$= \overline{\sum_{k_y} \nabla \cdot \frac{c(\mathbf{b} \times \nabla\phi)_{-k_y}}{B} f_{k_y}} \; ; \quad f_{k_y} = \frac{\frac{c k_y \phi_k}{B} \frac{\partial F_0}{\partial x_g}\Big|_{E'}}{\left(\omega - \frac{k_y (x_g - x_0)}{L_s}\right)}$$

$$\boxed{\frac{\partial \bar{F}}{\partial t} = 2\pi \sum_{k_y > 0} \frac{\partial}{\partial x_g}\bigg|_{E'} \left| \frac{c k_y \phi_k}{B} \right|^2 \delta\left(\omega - k_y \frac{(x_g - x_0)}{L_c}\right) \frac{\partial \bar{F}}{\partial x_g}\bigg|_{E'}}$$

This is spatial diffusion equation exactly as $\omega \to 0$

$$\frac{\partial}{\partial x_g} \equiv \frac{\partial}{\partial x_g}\bigg|_{E' = E - \frac{\omega}{k_y} \omega_{ce} (x - x_g) \xrightarrow[\omega \to 0]{} E}$$