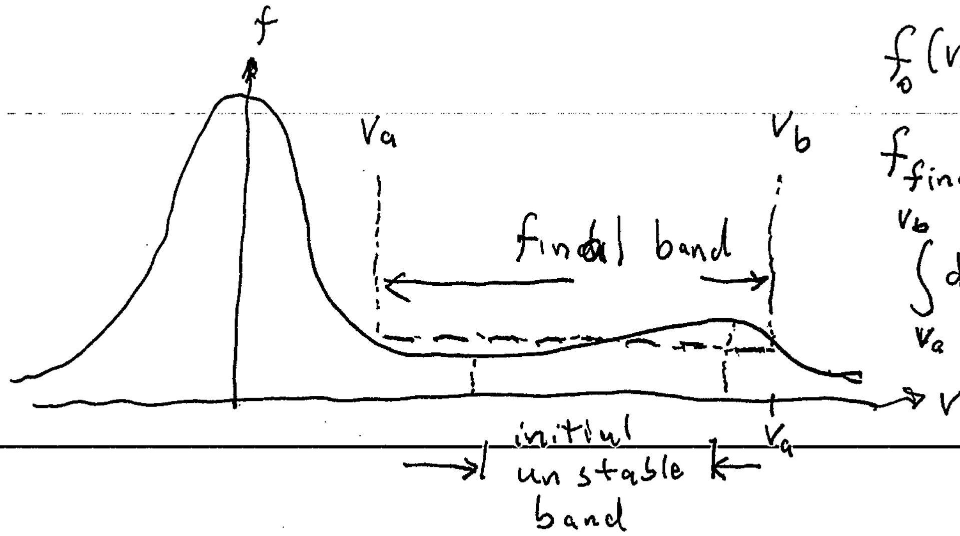
Lecture # 37

Implications of Quasi-Linear Theory

Let us discuss the mode saturation in the bump-on-tail instability in some more detail

We saw that the prediction, when there is a continuum of modes, the distribution flattens

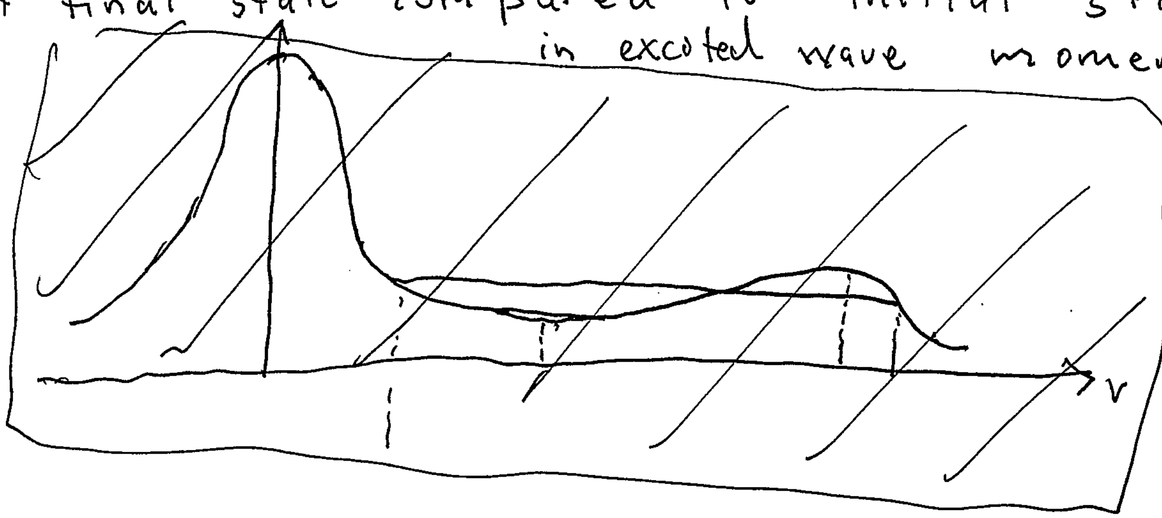


$$f_0(v_a) = f_0(v_b)$$

$$f_{final} = f_0(v_b)\,\Theta(v_b - v)\,\Theta(v - v_a)$$

$$\int_{v_a}^{v_b} dv\, f_0(v) = f_0(v_b)(v_b - v_a)$$

Further the difference in momentum of final state compared to initial state resides in excited wave momentum



$$\Delta P = P_i - P_f$$

$$= m\int_{v_a}^{v_b} dv\,(v - v_a)\left(f_0(v) - f_0(v_a)\right)$$

$$= \sum_k k\,\frac{\partial \epsilon(\omega_k, k)}{\partial \omega}\,\frac{2|E_k|_f^2}{8\pi}$$

$$E = \sum_k E_k \exp[-i\omega_k t + ikx] + c.c.$$

Also recall that quasi-linear equation for the tail particles was

$$\frac{\partial f}{\partial t} - \frac{\partial}{\partial v}\int \frac{dk}{2\pi}\left|\frac{eE_k}{m}\right|^2 \pi\delta(\omega_k - kv)\frac{\partial f}{\partial v} = 0$$

Recall that $\pi\delta(\omega_k - kv)$

came from $\lim_{\gamma_k \to 0} \dfrac{\gamma_k}{(\omega_k - kv)^2 + \gamma_k^2}$

We will come back to this discussion in a short while.

Now let us consider what happens with a discrete mode

If we had a single mode

$$\frac{d^2x}{dt^2} = -\frac{ZeE}{m}\sin(kx-\omega t)$$

$$\frac{d^2(kx-\omega t)}{dt^2} = -\frac{Ze}{m}kE\sin(kx-\omega t)$$

Let $\psi = kx - \omega t$

$\omega_b^2 = Ze\frac{kE}{m}$ ≡ square of trapping frequency of a deeply trapped particle

$$\frac{d^2\psi}{dt^2} + \omega_b^2 \sin\psi = 0$$

$$\frac{d\dot\psi}{dt} = \frac{d\psi}{dt}\frac{d\dot\psi}{d\psi} = \frac{1}{2}\frac{d}{d\psi}\dot\psi^2 = -\omega_b^2\sin\psi = \frac{d}{d\psi}\omega_b^2\cos\psi$$

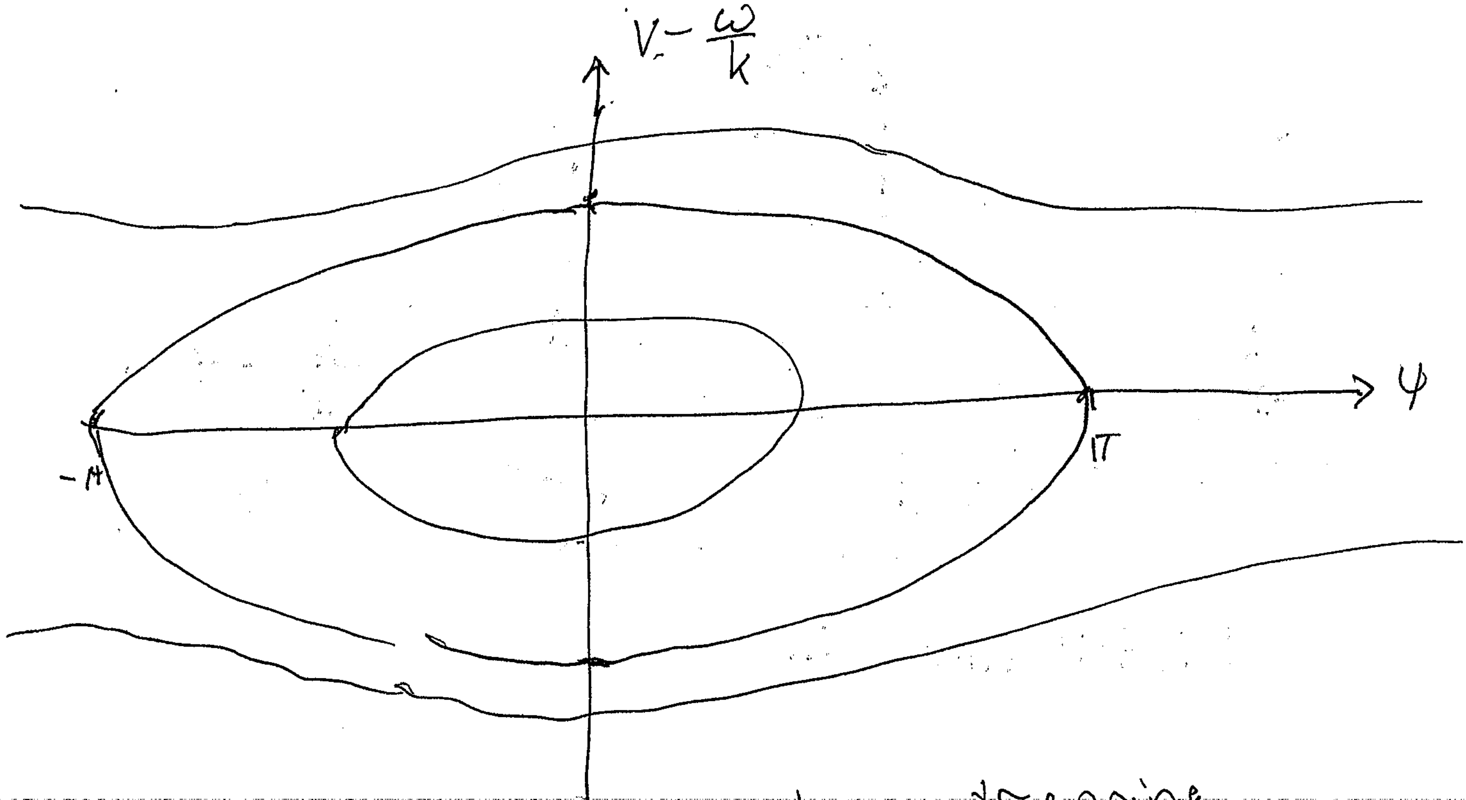
$$\therefore \quad \frac{\dot\psi^2}{2} - \omega_b^2\cos\psi = \mathcal{E} \equiv \text{constant}$$

$$\dot\psi^2 = 2(\mathcal{E} + \omega_b^2\cos\psi) > 0$$

Note that if $-\omega_b^2 < \mathcal{E} < \omega_b^2$

Particle is trapped in ψ

$$\psi_{max} = \cos^{-1}\left(-\frac{\mathcal{E}}{\omega_b^2}\right)$$



Distribution within trapping region mixes

Distribution outside trapping regions just distorts without mixing.

We can estimate saturation level of a single mode by comparing the momentum lost from mixing, compared with momentum of wave that is excited

Momentum difference in mixed region before mixing compared with after mixing

$$\psi = kv - \omega = [2(\omega_b^2 \cos\psi - \varepsilon)]^{1/2}$$

$$\sqrt{2}(\omega_b^2(1+\cos\psi))^{1/2}/k + \omega/k = \text{upper}$$

$$\psi = kx - \omega t$$

$$m\int dx \int dv \left(f_0(v) - f_0\left(\frac{\omega}{k}\right)\right) v$$

$$-\sqrt{2\omega_b^2(1+\cos\psi)}/k + \omega/k = \text{lower}$$

$$= \frac{m}{k}\int_{-\pi}^{\pi} d\psi \int_{\text{lower}}^{\text{upper}} d\left(v-\frac{\omega}{k}\right)\left[f_0\left(\frac{\omega}{k}\right) + \left(v-\frac{\omega}{k}\right) f_0'\left(\frac{\omega}{k}\right) - f_0\left(\frac{\omega}{k}\right)\right]\left[\left(v-\frac{\omega}{k}\right)\right]$$

only term to persist

integrates to zero

$$= \frac{2}{k^4} m \int_0^{\pi} \frac{2}{3}\left[\sqrt{2\omega_b^2(1+\cos\psi)}\right]^3 d\psi \, \frac{\partial f}{\partial v}\left(\frac{\omega}{k}\right)$$

$$= \frac{8\sqrt{2}}{3k^4}\int_0^{\pi} d\psi \, (1+\cos\psi)^{3/2}\, \omega_b^3 \, m \frac{\partial f(\frac{\omega}{k})}{\partial v} \qquad \left(1+\cos\psi = 2\cos^2\frac{\psi}{2}\right)$$

$$= \frac{\partial f}{\partial v}\frac{32}{3k^4} m \int_0^{\pi} d\psi \cos^3\frac{\psi}{2}\,\omega_b^3 = \frac{64}{3}\frac{m\omega_b^3}{k^4}\frac{\partial f(\frac{\omega}{k})}{\partial v}\int_0^{\pi/2} d\theta \cos^3\theta$$

$$\int_0^{\pi/2} d\theta \cos^3\theta = \frac{2}{3}$$

$$= \frac{128}{9}\frac{m\omega_b^3}{k^4}\frac{\partial f}{\partial v}\left(\frac{\omega}{k}\right) = \frac{128}{36\pi^2}\frac{m^2}{e^2}\frac{\partial \varepsilon_R}{\partial \omega}\gamma_L \omega_b^3$$

$$= WM = \frac{k^2|E_k|^2}{8\pi}\frac{\partial \varepsilon_R}{\partial \omega}\frac{2\pi}{k_0} = \frac{m^2}{e^2}\frac{1}{8}\omega_b^4 \frac{\partial \varepsilon_R}{\partial \omega}$$

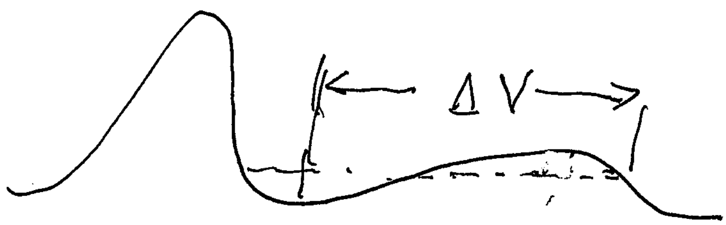
Thefore $\underbrace{\omega_b = \frac{256}{9\pi^2}\gamma_L \simeq 2.9\,\gamma_L}$

There is a major difference from momentum released by "non-overlapped" modes and "over lapped modes

From our estimate of momentum released from single modes (non-overlapped)

$$WM_{SM's} \simeq m \frac{\partial f}{\partial V}\left(\frac{\omega_b}{k}\right)^3 N < m \frac{\partial f}{\partial V}\left(\frac{\Delta V}{N}\right)^3 N$$

$N \equiv$ # of Modes



$$WM_{QL} \simeq m \frac{\partial f}{\partial V}(\Delta V)^3$$

$$\frac{WM_{SM's}}{WM_{QL}} < \frac{1}{N^2} \approx \left(\frac{\gamma_L}{k\Delta V}\right)^2$$

$N \equiv$ # of Modes filling unstable space

This disparity actually infers that explosive behavior should occur in a system whose sources produce ~~instability~~ linearly unstable distribution functions, and where additional damping mechanisms, such as due to collisions, exist.

In quasi-linear theory, the wave energy that is present after a flattened distribution arises, damps away. The sources then slowly build up the ustable ~~distribution~~ function. However, initially, ~~when~~ only a finite # of modes can be excited, quasilinear theory not appliable since mode overlap doesn't occur. ~~However,~~ Modes saturate at single mode limits. However, when overlap is triggered, modes may suddenly grow, which causes dump of accumulated "free" energy.

We also note that a universal way of writing Quasi-linear Equation is

$$\frac{\partial f}{\partial t} - \frac{\partial}{\partial v} \sum_k \pi\delta(\omega - kv) \left|\frac{eE_k}{m}\right|^2 \frac{\partial}{\partial v} f = 0$$

$\pi\delta(\omega - kv)$ came from $\dfrac{\gamma_L}{(\omega_k - kv) + \gamma_L^2}$

$$= \pi Q(\Omega - \omega_k)$$

where $\int d\Omega \; Q(\Omega - \omega_k) = 1$

with $kv = \Omega \qquad \dfrac{ze\,kE_k}{m} = \omega_b^2$

$kv = \Omega$

$$\frac{\partial f}{\partial t} - \frac{\partial}{\partial \Omega} \sum_j \pi Q(\Omega - \omega_j) \, |\omega_{bj}^2|^2 \frac{\partial}{\partial \Omega} f$$

This is the uniform form that is applicable to most physical systems, when appropriate wave trapping frequency ω_b^2 is defined.

We will see this shortly (next lecture) in drift wave problem