Lecture # 36

Solution of Quasi Linear Equation for bump-on-tail

Wave Energy for Electrostatic Waves

We assume dissipation from resonant particles can be neglected

$$\frac{\partial f}{\partial t} = -v\frac{\partial f}{\partial x} - \frac{eE_k}{m}\frac{\partial f}{\partial v}$$

construct the time rate of change of kinetic energy, averaging over a spatial period L

$$\int \frac{dx}{L}\int dv\, \frac{mv^2}{2}\frac{\partial f}{\partial x} \equiv \frac{\partial T}{\partial t}$$

$$= -\int\frac{dx}{L}\left[\underbrace{\int dv\, \frac{mv^2}{2}\, v\frac{\partial f}{\partial x}}_{=0} - \frac{me}{m}\int E_k\frac{v^2}{2}\,dv\,\frac{\partial f}{\partial v}\left(e^{ikx-i\omega_k t}+cc\right)\right]$$

$$f = n_0F_0 - \left[\frac{e}{m}\frac{iE_k e^{i(kx-\omega_k t)}}{(\omega_k-kv)} + c.c.\right]\frac{\partial F_0}{\partial v}n_0$$

Only the non-oscillatory part persists to give, on integration by parts

$$\frac{\partial T}{\partial t} = -\frac{n_0e^2}{m}|E_k|^2\int\frac{2\gamma_k v\,\frac{\partial F}{\partial v}}{(\omega_k-kv)^2+\gamma^2}\,dv$$

(we used $\frac{|E_k|^2}{\omega_{k0}-kv+i\gamma_k} + \frac{|E_k|^2}{(\omega_{-k}+kv+i\gamma_k)} = \frac{-2i\gamma_k|E_k|^2}{(\omega_k-kv)^2+\gamma_k^2}$)

$$\boxed{\cancel{\frac{\partial T}{\partial t} = -\frac{\partial\omega_p^2}{\partial t}\int dv\,\frac{\partial F_0}{\partial v}\,v\,\frac{2|E_k|^2}{(\omega_k-kv)^2\,8\pi}} = \cancel{-\omega_p^2\int dv\,\frac{\partial F_0}{\partial v}\left[\underbrace{1}_{0}-\frac{1}{k}\frac{\omega}{(\omega-kv)}\right]\frac{2|E_k|^2}{8\pi}}}$$

$$2\gamma_k |E_k(t)|^2 \simeq \frac{\partial |E_k(t)|^2}{\partial t}$$

$$\frac{\partial T}{\partial t} = -\frac{\partial}{\partial t}\left(\omega_p^2 \frac{2|E_k|^2}{8\pi k}\int \frac{dv\, vk \frac{\partial f}{\partial v}}{(\omega_k - kv)^2}\right)$$

$$= \frac{\partial}{\partial t}\frac{2|E_k|^2}{8\pi}\,\omega_p^2 \frac{\partial}{\partial \omega_k}\int \frac{dv\, kv\, \frac{\partial f}{\partial v}}{(\omega_k - kv)}$$

$$\frac{kv}{\omega - kv} \rightarrow \underset{0}{\cancel{-1}} + \frac{\omega}{\omega - kv}$$

$$= \frac{\partial}{\partial t}\left[\frac{2|E_k|^2}{8\pi}\,\omega_p^2 \frac{\partial}{\partial \omega_k}\left(\omega_k \int \frac{dv\, \frac{\partial f}{\partial v}}{(\omega_k - kv)}\right)\right]$$

$$= \frac{\partial}{\partial t}\left(\frac{2|E_k|^2}{8\pi}\frac{\partial \omega_k \left(\epsilon_R(\omega_k, k) - 1\right)}{\partial \omega_k}\right)$$

or

$$\frac{\partial}{\partial t}\left(T + \frac{2|E_k|^2}{8\pi}\right) = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial \omega_k}\left(\omega_k\, \epsilon_R(\omega_k, k)\right) \frac{2|E_k|^2}{8\pi}\right)$$

If ω_k is a natural mode of the system

$$\epsilon_R(\omega_k, k) \simeq 0$$

$$T + \frac{2|E_k|^2}{8\pi} = \omega_k \frac{\partial \epsilon_R(\omega_k, k)}{\partial \omega}\, 2\frac{|E_k|^2}{8\pi} \equiv W_{E_k}$$

excited kinetic energy + field energy $\equiv$ wave energy (or coherent energy) $\equiv W_{E_k}$

In a similar way we can construct

Wave Momentum (electrostatic wave has no field momentum)

$$\frac{\partial}{\partial t}\int dv\, m v f = \frac{\partial}{\partial t}\left(k \frac{\partial \epsilon_R(\omega_k)}{\partial \omega} \frac{|E_k|^2}{8\pi}\right)$$

$$\parallel$$

$$\frac{\partial P_k}{\partial t}$$

$$P_k = k \frac{\partial \epsilon_R}{\partial \omega} \frac{|E_k|^2}{8\pi} \equiv WM_k$$

Note quantization-like relation

$$\frac{WE_k}{P_k} = \frac{\omega}{k} = \frac{\hbar\omega}{\hbar k}$$

If we have a spectrum of waves, the wave momentum goes over to

$$WE_k \to \int \frac{dk}{2\pi} WE_k = \int \frac{dk}{2\pi} \frac{2|E_k|^2}{8\pi} \omega_k \frac{\partial \epsilon(\omega_k, k)}{\partial \omega}$$

$$WM_k \to \int \frac{dk}{2\pi} WM_k = \int \frac{dk}{2\pi} \frac{2|E_k|^2}{8\pi} k \frac{\partial \epsilon(\omega_k, k)}{\partial \omega}$$

No let us go back to our quasi-linear equation we derived ($F_0 \to n_0 F_0$)

$$\frac{\partial}{\partial t}\left[n_0 F - \frac{\partial}{\partial V}\int \frac{dk}{2\pi} \frac{\omega_p^2}{(\omega - kV)^2} \frac{\partial F}{\partial V} \frac{2|E_k|^2}{8\pi}\right]$$

$$- \frac{\partial}{\partial V}\pi\int \frac{dk}{2\pi} \frac{4\pi n_0 e^2}{m^2} \delta(\omega_k - kv) \frac{2|E_k|^2}{8\pi} \frac{\partial F(v)}{\partial V} = 0$$

/// (performing k - integration)

$$- \frac{\partial}{\partial V}\pi\left(\frac{\omega_p^2}{2\pi m} \frac{1}{\left|V - \frac{\partial \omega_k}{\partial k}\right|} \frac{2|E_{\omega/v}|^2}{8\pi} \frac{\partial F}{\partial V}\right)$$

In addition,
Wave evolution equation is

$$\frac{\partial}{\partial t}\left(2\frac{\partial \epsilon_R(\omega_k, k)}{\partial \omega}\frac{|E_k|^2}{8\pi}\right) = -2\epsilon_I(\omega_k)\frac{2|E_k|^2}{8\pi}$$

$$= 2\pi\frac{\omega_p^2}{k^2}\frac{\partial F}{\partial V}\left(\frac{\omega_k}{k}\right)\frac{2|E_k|^2}{8\pi}$$

→ use this relation in QL equation

Thus we have:

$$\frac{\partial}{\partial t}\left(n_0 F - \frac{\partial}{\partial V}\int \frac{dk}{2\pi}\frac{\omega_p^2}{m(\omega - kv)^2}\frac{\partial F}{\partial V}\frac{2|E_k|^2}{8\pi}\right)$$

$$- \frac{1}{2\pi}\frac{\partial}{\partial t}\left\{\frac{\partial}{\partial V}\left[\left(\frac{\frac{\partial \epsilon_R(\omega_k, \omega_k/v)}{\partial \omega_k} k^2}{m\left|V - \frac{\partial \omega_k}{\partial k}\right|}\right)\frac{2|E_{\frac{\omega}{v}}|^2}{8\pi}\right]\right\} = 0$$

Now let us ask How the resonant region, where the source of instability resides, changes

We can neglect the middle term which is just the non-resonant contribution, & therefore does not involve resonant terms

We have then that

$$\frac{\partial}{\partial t} n_0 F = \frac{\partial}{\partial t}\frac{\partial}{\partial V}\left[\frac{\frac{\partial \epsilon_R(\omega_k, \omega_k/V)}{\partial \omega} k^2}{m\left|V - \frac{\partial \omega_k}{\partial k}\right|} \frac{2\left|E_{\left(\frac{\omega}{k}\right)}\right|^2}{8\pi}\right]$$

integrate in time

$$n_0 F(V,t) = n_0 F_0(V) + \frac{\partial}{\partial V}\frac{\partial}{\partial \omega} \frac{\epsilon_R(\omega_k, k=\omega_k/V) k^2}{\left|V - \frac{\partial \omega_k}{\partial k}\right| 2\pi} \frac{2\left|E_{k=\frac{\omega}{V}}\right|^2(t)}{8\pi}$$

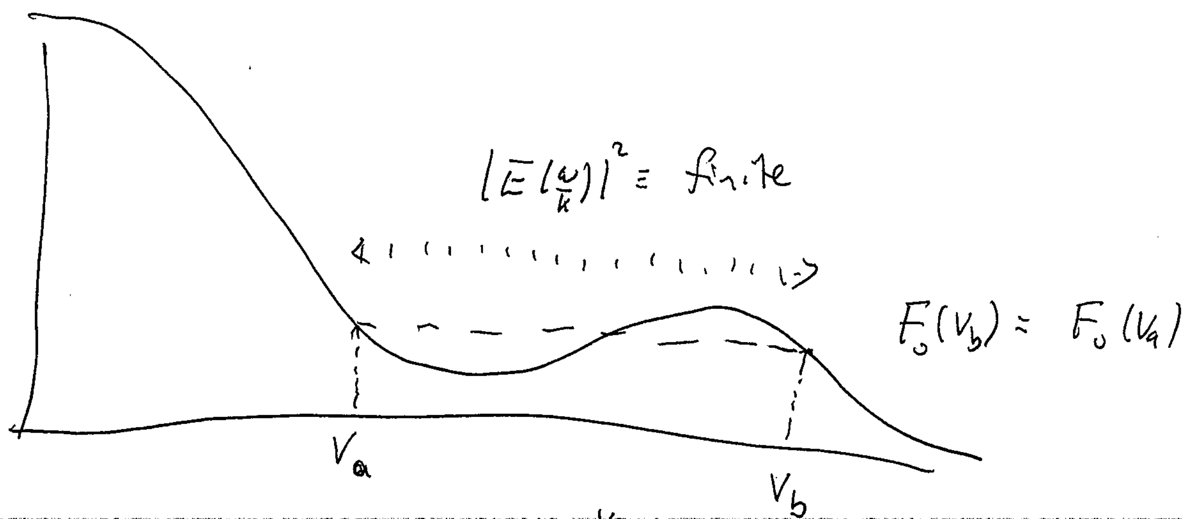
$F_0(V) \equiv$ constant initial value

If we look for a steady solution and go back to the quasilinear Equation and multiply by F and integrate over V

$$n_0 \int \frac{\partial}{\partial t}\frac{F^2}{2} dV = -\pi \int dV \left(\frac{\partial F}{\partial V}\right)^2 \frac{\omega_p^2}{2\pi m} \frac{1}{\left|V - \frac{\partial \omega}{\partial k}\right|} \frac{2\left|E_{k=\omega/V}\right|^2}{8\pi}$$

It tells us that a steady solution requires $\frac{\partial F}{\partial V} = 0$ over region where $|E_k|^2$ is finite!

Thus we have a solution of the form



$$(V_b - V_a)F_0(V_a) = \int_{V_a}^{V_b} dV \, F_0(V)$$: particle conservation that follows from kinetic equation

$$F_0(V_b) = F_0(V_a)$$

two equations for two unknowns

Integrating $\frac{\partial}{\partial V}\left(|E_{k=\omega_k/V}|^2 \cdots\right)$ equation

$$\boxed{\frac{k^2 \frac{\partial \epsilon_R(\omega_k, k = \frac{\omega_k}{V})}{\partial \omega}}{|V - \frac{\partial \omega_k}{\partial k}| m} \; 2 \frac{|E_k|^2(V)}{8\pi} = n_0 \int_{V_a}^{V} dV' \left[F_0(V_a) - F_0(V')\right]}$$

This relation determines the saturated field amplitude, since in principle V_a is known.

Now we show that the difference of the momentum of the final distribution to the initial distribution is just the excited wave momentum.

We found

$$n_0\left((V-V_a)\,F_0(V_a)-\int_{V_a}^{V} dv'F_0(v')\right)=\frac{k^2\,\frac{\partial\epsilon_R(\omega_k,k=\frac{\omega_k}{V})}{\partial\omega}}{2\pi m\left|V-\frac{\partial\omega_k}{\partial k}\right|}\,2\,\frac{\left|E_{k=\frac{\omega}{V}}\right|^2}{8\pi}$$

$$\text{''}$$

$$n_0\int_{-\infty}^{V} dv\left(F_f(v')-F_i(v')\right)$$

since $F_f(v')=F_i(v)$ for $v<V_a$

Now

$$n_0 m_0\int_{-\infty}^{V_b} dv\int_{-\infty}^{V}(F_f(v')-F_i(v'))dv'= n m_0\int_{-\infty}^{V_b} dv\,\frac{d}{dv}v\int_{-\infty}^{v} dv'(F_f(v')-F_i(v'))$$

integrate by parts

$$=-n_0 m_0\int_{-\infty}^{V_b} dv(F_f(v)-F_i(v'))v=(P_i-P_f)$$

(we shall use $d(\omega_k-kv)=0$

$\therefore\ dv=\frac{dk}{k}\left|V-\frac{\partial\omega_k}{\partial k}\right|$)

$$P_i-P_f=\int_{V_a}^{V_b} dv\,\frac{k^2\,\frac{\partial\epsilon_R(\omega_k,k=\frac{\omega_k}{V})}{\partial\omega}}{|V-\partial\omega_k/\partial k|}\,2\,\frac{\left|E_{k=\omega_k/V}\right|^2}{8\pi}$$

$$=\int_{k_b}^{k_a}\frac{dk}{2\pi}\,\frac{\partial\epsilon_R(\omega_k,k)}{\partial\omega}\,k\,\frac{|E_k|^2}{8\pi}\equiv\sum_k WM_k$$

QED