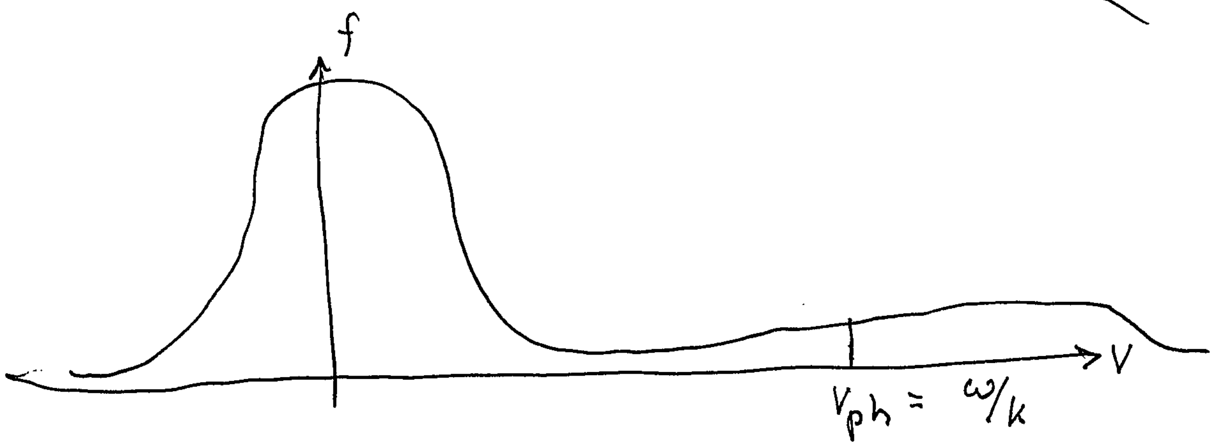
Lecture # 35

Quasi-Linear Theory

Let us return to the problem of the bump-on-tail instability



$$\frac{\partial f}{\partial v}(\omega/k) > 0$$

Waves grow at a rate γ_L given by

$$\epsilon(\omega, k) \equiv 1 + \frac{\omega_p^2}{k^2}\int dv \frac{k \frac{\partial f}{\partial v}}{\omega - kv} = 0$$

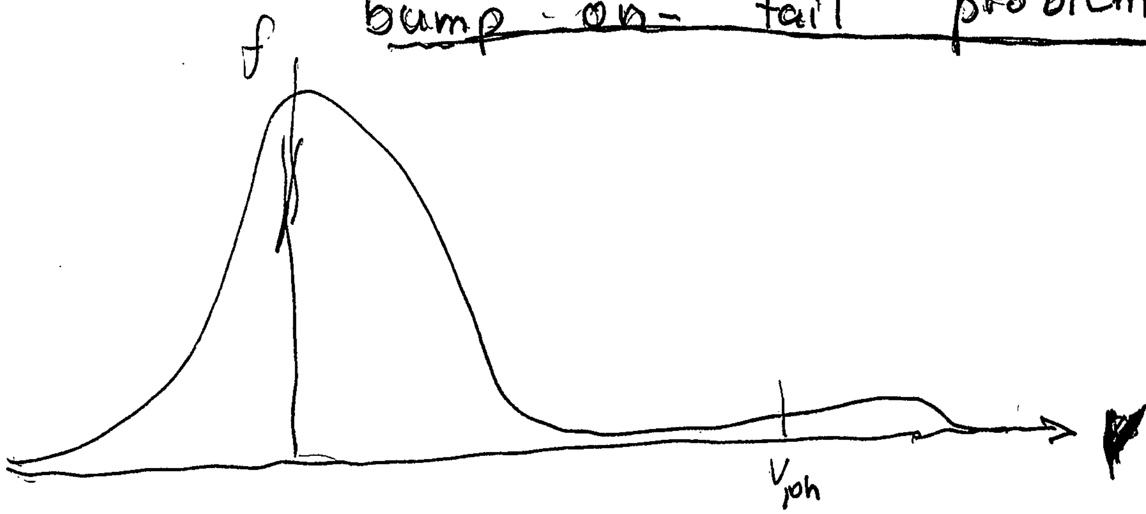
$$\epsilon(\omega, k) = \epsilon_R(\omega_0 + \delta\omega, k) + i\,\epsilon_I(\omega_0 + \delta\omega, k) = 0$$

If $\left(\epsilon_R, \ \omega \frac{\partial \epsilon_R}{\partial \omega}\right) \gg \epsilon_I(\omega, k)$

$$\underset{0}{\underset{\shortparallel}{\epsilon_R(\omega_0, k)}} + \delta\omega \frac{\partial \epsilon_R}{\partial \omega}(\omega_0, k) + i\,\epsilon_I(\omega_0, k) \simeq 0$$

$$\therefore \ \delta\omega = \frac{-i\epsilon_I(\omega_0, k)}{\frac{\partial \epsilon_R}{\partial \omega}(\omega_0)}$$

Let us return to the bump-on-tail problem



$$\frac{\partial f}{\partial v}\left(\frac{\omega}{k} = V_{ph}\right) > 0$$

Waves grow at a rate γ_L given by: $D(\omega,k) = 1 + \frac{k\omega_p^2}{k^2}\int dv \, \frac{\partial f}{\partial v}(v) \Big/ \omega - kv = 0$

$$D(\omega, k) \equiv D_R(\omega_0 + \delta\omega, k) + i D_I(\omega_0 + \delta\omega, k) = 0$$

If $\left(D_R, \frac{\partial D_R}{\partial \omega}(\omega_0)\right) \gg D_I(\omega_0)$

$$\underset{\overset{\parallel}{0}}{D_R(\omega_0)} + \delta\omega \frac{\partial D_R}{\partial \omega}(\omega_0) + i D_I(\omega_0) = 0$$

$$\delta\omega = -i \frac{D_I(\omega_0)}{\frac{\partial D_R}{\partial \omega}(\omega_0)} = + \frac{i\pi\omega_p^2}{|k|k} \frac{\partial f}{\partial v}\left(\frac{\omega_0}{k}\right)$$

There are two extreme limits to the evolution of the non-linear wave. One is particle trapping of a single mode.

The second is the random phase effect of many "over lapping" modes.

Let's briefly discuss the first limit when there is a single mode:

We look at a particle orbit in an electrostatic wave of amplitude

$$\phi(x,t) = 2\hat{\phi}\ \cancel{\cos}(kx-\omega t)$$

$$\frac{d^2x}{dt^2} = -\frac{e}{m}\frac{\partial\phi(x,t)}{\partial x} = -2e\frac{k\hat{\phi}}{m}\sin(kx-\omega t)$$

let $\psi = kx-\omega t$; $\omega_b^2 = 2ek^2\hat{\phi}/m$

$$\frac{d^2\psi}{dt^2} + \omega_b^2\sin\psi = 0$$

This is an equation for pendulum with trapping frequency (square) ω_b^2 for deeply trapped particle. In fact one can show that the interaction of nearly any Hamiltonian system will have this form for sufficient low amplitude wave.

The linear theory for the motion is

$$\psi = \psi_0 + \dot{\psi}_0 t + \delta\psi$$

$$\dot{\psi}_0 = kv_0 - \omega$$

$$\frac{d^2\delta\psi}{dt^2} + \omega_b^2 \sin(\psi_0 + \dot{\psi}_0 t) \simeq 0 \quad \left(\begin{array}{c}\text{assumption} \\ \delta\psi \ll 1 \\ \text{as neglected here}\end{array} \right)$$

$$\delta\psi(t) = \frac{\omega_b^2 \sin(\psi_0 - \dot{\psi}_0 t)}{(\omega - kv_0)^2} \simeq \frac{\omega_b^2 \sin\psi}{(\omega - kv_0)^2}$$

We therefor require $\delta\psi \lesssim \frac{\omega_b^2}{(\omega - kv_0)^2} \ll 1$

$$\cancel{\delta\dot{\psi} = -\frac{\omega_b^2 \dot{\psi}_0}{(\omega - kv_0)^2} \cos(\psi) \simeq \frac{\omega_b^2 \cos\psi}{(\omega - kv_0)}}$$

$$\cancel{\text{validity}: \quad \delta\dot{\psi} \ll \frac{\omega_b^2}{|\omega - kv_0|} \ll \omega_b}$$

$$\cancel{\dot{\psi}_0 = \omega - kv_0 \gg \delta\psi}$$

$$\cancel{\delta\dot{\psi} \ll \frac{\omega_b^2}{|\omega - kv_0|} \ll \frac{\omega_b^2}{\delta\dot{\psi}}}$$

$\therefore \ \omega_b \ll |\omega - kv|$ required for validity of this linear expansion

There is always a group of particles that cannot satisfy this condition. These are the near resonant particles:

$$\omega - k_0 v_0 \lesssim \omega_b$$

We can still apply linear theory, but this theory has validity for a limited time

$$\psi = \psi_0 + \dot{\psi}_0 t + \delta\psi_0$$

$$\frac{d^2\delta\psi}{dt^2} = -\omega_b^2 \sin(\psi_0 + \delta\psi)$$

$$= -\omega_b^2 \sin\psi_0 + \omega_b^2 \delta\psi \cos(\psi_0) + \dots$$

Thus:

$$\frac{d^2\delta\psi_0}{dt^2} \simeq -\omega_b^2 \sin\psi_0$$

$$\frac{d\delta\psi_0}{dt} = -\omega_b^2 t \sin\psi_0 < \omega_b \sin\psi_0 < \omega_b$$

$\delta\psi_0 < \omega_b t \ll 1$; required for validity

certainly for $t < \frac{1}{\omega_b}$ is required

Thus the particles with

$$\omega_0 - k \cdot v_0 < \omega_b$$

can only be followed by linear theory for a limited time, and as $\omega_b \to 0$

these particles take the form of a dissipative response, extracting $\left(\frac{\partial f}{\partial v}<0\right)$ or giving $\left(\frac{\partial f}{\partial v}>0\right)$ energy to the wave excitation.

After this time the form of the response becomes ~~entirely~~ different:

Two major models

(1) Single mode model:

Resonant Particles become trapped and these particles cease acting in a dissipative manner.

If time permits we will discuss this case further in later lectures

(2) Multimode model

The phase mixing due to exciting many mode causes mode saturation due to stochastic motion of resonant particles (Random phase approximation)

The standard quasi-linear theory deals with this case.

For electron plasma oscillations
$(\omega / k v_{th} \gg 1)$

$$D_p(\omega, k) \simeq 1 - \frac{\omega_p^2}{\omega^2}\left(1 + \frac{3k^2 v_{th}^2}{\omega^2}\right) = 0$$

$$\omega_0 \simeq \omega_p\left(1 + \frac{3k^2 v_{th}^2}{2\omega_p^2}\right)$$

What happens to the waves as they grow?
In particular, how does it saturate?

The description of this process is called quasi-linear theory
In 1-d

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e\,\partial\phi}{m\,\partial x}\frac{\partial f}{\partial v} = 0 \quad , \quad k^2\phi_k = 4\pi e\int f_k\, dv$$

$$\phi = \int dk\, \phi_k(t)\, e^{-i\omega_k t + ikx} \quad , \quad e^{-i\omega_k t}\phi_k(t) = \phi_k^* e^{i\omega_k t}$$

Substitute in $f = f^{(0)} + f^{(1)} + f^{(2)}$

$$\phi = 0 + \phi_k$$

$$\frac{\partial f_0}{\partial t} \propto |\phi|^2$$

$$f^{(0)} = F(v) \quad , \quad \phi_k^{(0)} = 0$$

$$\frac{\partial f_k^{(1)}}{\partial t} + ikv f_k^{(1)} - i\frac{e}{m}k\phi_k^{(1)}\frac{\partial F}{\partial v} = 0$$

$$\frac{\partial f_k^{(2)}}{\partial t} + ikv f_k^{(2)} - i\frac{e}{m}k\phi_k^{(2)}\frac{\partial F_0}{\partial v} = i\int\frac{dk'}{2\pi}k'\phi_{k'}^{(1)}\,\frac{e}{m}\frac{\partial f_{k-k'}^{(1)}}{\partial v}$$

Now: We look for the long-wavelength response, i.e. $k \to 0$

$$\frac{\partial f}{\partial t} = -\frac{e}{m}\int_{-\infty}^{\infty}\frac{dk'}{2\pi}(+ik')\phi_{k'}\frac{\partial f_{-k'}}{\partial v}$$

$$= \frac{e}{m}\int_{0}^{\infty}\frac{dk'}{2\pi}\left[ik'\phi_{k'}\frac{\partial f_{-k}}{\partial v} - ik'\phi_{-k}\frac{\partial f_{k'}}{\partial v}\right]$$

Now $\displaystyle f_{k'} = -\frac{e}{m}\frac{k\phi_{k'}\frac{\partial F_0}{\partial v}}{\omega_{k'}-k'v}\quad\frac{\omega_k^{*}-k'v}{(\omega_k^{*}-kv)}$

$$= \frac{ie}{m}k'\phi_{k'}\frac{\partial F_0}{\partial v}\frac{\gamma_{k'}-(\omega_{kr}-k'v)}{(\omega_{kr}-k'v)^2+\gamma_{k'}^2}$$

and

$$-k'\phi_{k'}\frac{\partial f_{-k}}{\partial v}\frac{e}{m} + ik'\phi_{-k'}\frac{\partial f_{k'}}{\partial v}\frac{e}{m}\quad,\quad(\text{using } \omega_{-k} = -\omega_k)$$

$$= 2\left(\frac{e}{m}\right)^2 k'^2\frac{|\phi_{k'}|^2\gamma_{k'}}{(\omega_k-k'v)^2+\gamma_k^2}\quad,\quad 2\gamma_k|\phi_k|^2 = \frac{\partial|\phi_k|^2}{\partial t}$$

$$\xrightarrow{\gamma_k\to 0} -\frac{e^2}{m^2}k'^2\frac{\partial}{\partial t}|\phi_k|^2\frac{\partial}{\partial\omega}\frac{P}{(\omega_k-k'v)}$$

$$+ 2\left(\frac{e}{m}\right)^2|k'\phi_k|^2\pi\delta(\omega_{k'}-kv)$$

$$\omega_{bk}^2 \equiv 2k^2|\phi_k|\frac{e^2}{m^2}$$

Single mode satisfies

$$\frac{d^2(kx)}{dt^2} = -\omega_{bk}^2 \sin(kx - \omega_k t)$$

$$\omega_{bk}^2 = \sqrt{2}\, k^2 \phi_k \frac{e}{m} \equiv$$ "trapping frequency of deeply trapped particle"

Then

$$\frac{\partial}{\partial t}\left(F_0 - \int_0^\infty \frac{dk'}{2\pi} \frac{\partial |\omega_{bk'}|^4}{\partial kV} \left(\frac{\partial}{\partial \omega} P \frac{1}{(\omega_{k'} - k'v)} \right) \frac{\partial F_0}{k\partial V} \right)$$

$$- \pi \int_0^\infty \frac{dk'}{2\pi} \frac{\partial |\omega_{bk'}|^4}{k\partial V} \delta(\omega_{k'} - k'v) \frac{\partial F_0}{k\partial V} = 0$$

We also have an equation for wave evolution: $\phi_k = \phi_k(t) e^{-i\omega_k t + ikv}$

It can be written as

$$k^2 D_R\left(\omega_{k0} + i\frac{\partial}{\partial t}, k\right) \phi_k(t) + i D_I(\omega) \phi_k = 0$$

$$D_R(\omega_{k0}) = 0$$

$$\phi_{-k}: \quad i\frac{\partial D}{\partial \omega}(\omega_k, k) \frac{\partial \phi_k}{\partial t}(t) + i D_I(\omega) \phi_k = 0$$

$$\frac{\partial D}{\partial \omega}(\omega, k) \frac{\partial}{\partial t} |\phi_k|^2 = -2 D_I(\omega) |\phi_k|^2 = \frac{2\pi \omega_p^2}{|k|} \int k\, dv\, \delta(\omega_k - kv) \frac{\partial F}{\partial kV} \cdot |\phi_k|^2 \quad (8)$$

using

$$\gamma_k = \frac{\omega_p^2}{k^2 \frac{\partial D_R}{\partial \omega}} \int k\,dv\, \delta(\omega_k - kv) \frac{\partial F}{\partial v}$$

we have, with

$$|\omega_{bk}^2| = \left|\frac{e k^2 \phi_k}{m}\right|^2$$

$$\frac{\partial |\omega_{bk}^2|}{\partial t} = 2\gamma_k |\omega_{bk}|^2$$

We now have a closed system with evolution of distribution function

$$\frac{\partial}{\partial t}\left[F_0 - \int_0^\infty \frac{dk}{2\pi} \frac{\partial}{\partial \Omega_k} |\omega_{bk}|^4 \left(\frac{\partial}{\partial \omega} P \frac{1}{(\omega_k - \Omega_k)}\right) \frac{\partial F}{\partial \Omega_k}\right]$$

$$- 2\int_0^\infty \frac{dk}{2\pi} \pi \frac{\partial}{\partial \Omega_k} |\omega_{bk}|^4 \delta(\omega_k - \Omega_k) \frac{\partial F}{\partial \Omega_k} = 0$$

$\Omega = kv$

$$\frac{\partial D(\omega_k)}{\partial \omega} \frac{\partial |\omega_{bk}^2|^2}{\partial t} - 2\gamma_k \frac{\partial D(\omega_k)}{\partial \omega} |\omega_{bk}^2|^2 = 0$$

Quasi-Linear equation, (in a not too frequently shown notation) This form generalizes to nearly all physical systems! (11)