Lecture # 34

Plasma Transport Theory

Classical Chapman - Enskog expansion

$L \gg \lambda_{mfp}$, $\nu T \gg 1$, $L \ll \frac{v_{th}}{\nu} = l_{mfp}$

T macroscopic time scale

$$\frac{Df}{Dt} = -\nu(\hat{f} - F_M) \quad , \quad f = f_0 + f_1 + \ldots$$

steady solution

$$v\frac{\partial f}{\partial x} + \frac{e}{m}\underset{\sim}{E}\cdot\frac{\partial f}{\partial \underset{\sim}{v}} = -\nu(f_0 - F_M)$$

Apply expansion

$$0 = \hat{\nu}(f_0 - F_M) ; \quad \therefore f_0 = F_M$$

$$v\frac{\partial f_0}{\partial x} + \frac{e}{m}E\frac{\partial f_0}{\partial \underset{\sim}{v}} = -\hat{\nu} f_1$$

where $f_0 = F_M = \dfrac{n_0(r)}{(2\pi T(r)/m)^{3/2}}\exp\left[-\dfrac{m(\underset{\sim}{v} - \underset{\sim}{u}(\underset{\sim}{r}))^2}{2T}\right]$

For simplicity take $u(\underset{\sim}{r}) = 0$

Then:

$$\underset{\sim}{v}\cdot\left[\frac{\nabla n_0}{n_0} - \frac{\underset{\sim}{\nabla} T}{T}\left(\frac{3}{2} - \frac{mv^2}{2}\right)\right]F_M = -\nu f_1$$

$$f_1 = -\frac{1}{\nu}\left[\quad '' \quad\quad '' \quad\right]F_M$$

$$\Gamma = \int d^3x\, f_1(\underset{\sim}{v})\underset{\sim}{v} = -\frac{1}{\nu}\int d^3v\, \underset{\sim}{v}\,\underset{\sim}{v}\cdot\left[\frac{\underset{\sim}{\nabla} n}{n} - \frac{\nabla T}{T}\left(\frac{3}{2} - \frac{mv^2}{2}\right)\right]F_M$$

$$\overline{\underset{\sim}{v}\underset{\sim}{v}} = v^2\,\underset{\approx}{I}$$

$$= -v_{th}^2\frac{\partial n_0}{\partial r}/2\nu = -D\frac{\partial n_0}{\partial r}; \quad D = \frac{v_{th}^2}{\nu} = \nu l_{mfp}^2$$

$$Q = \int d^3v\, f_1 \left[v_z \; \frac{1}{2} m v^2 \right] - \underset{\sim}{!}^{7}$$

$$= \int \frac{d^3v}{\nu} \left(\frac{\partial T}{T \partial \underset{\sim}{r}} \cdot \left(\frac{3}{2} - \frac{1}{2} \frac{mv^2}{T} \right) \right) F_m \, \underset{\sim}{v}\,\underset{\sim}{v}$$

$$\overline{\underset{\sim}{v}\,\underset{\sim}{v}} = v^2 \underset{\approx}{I}$$

$$= -\frac{1}{T} \frac{\partial T}{\partial \underset{\sim}{r}} \frac{1}{\nu} \int d^3v \, v^2 \left(-\frac{3}{2} + \frac{1}{2} \frac{mv^2}{T} \right) F_m$$

$$= - \frac{\partial T}{T \partial \underset{\sim}{r}} \frac{1}{\nu} n_0 v_{th}^2 \left[-\frac{\cancel{9}}{2} + \frac{15}{2} \right]$$

$$Q = -\frac{1}{T} \frac{\partial T}{\partial \underset{\sim}{r}} 3 \nu l_{mfp}^2 n_0$$

$$\equiv - \chi\, n_0 \frac{1}{T} \frac{\partial T}{\partial \underset{\sim}{r}}$$

(relatively more heat than particles lost because extra v^2 weighting in flux)

Ambipolar Diffusion

$$\nu_e \simeq \left(\frac{M}{m} \right)^{1/2} \nu_i$$

electrons diffuse more rapidly than ions

Electric field springs up to force the flow to be ambipolar

$$J_e = J_i$$

$$F_M = n g(x) \exp\left[-\frac{m_j v^2}{2T} - \frac{e_j \Phi}{T} \right]$$

$$-\nu f_j = v \cdot \nabla F_M = -\left\{ \frac{\partial n_j}{n_j \partial \underset{\sim}{x}} + \frac{e_j \nabla \Phi}{T_j} \right\} F_M$$

$$\underset{\sim}{F_j} = -\int d^3\underset{\sim}{r}\ v\ \frac{v\cdot\nabla F_M}{\nu}$$

$$= -\nu_j\, l_{mfp}^2 \left[\frac{\partial n_j}{n_j\,\partial \underset{\sim}{r}} + \frac{e_j \underset{\sim}{\nabla\Phi}}{T_j}\right]$$

$z_i = 1 \qquad n_e = n_i = n$

$$\Gamma_e = -\nu_e\, l_{mfp}^2 \left[\frac{1}{n_0}\frac{\partial n_0}{\partial r} + \frac{e\nabla\Phi}{T_e}\right]$$ normally

larger than $\nu_i\, l_{mfp\,i}\left[\frac{\partial n}{n\,\partial r} - \frac{e\nabla\Phi}{T_i}\right]$

unless electron flux nearly blocked

by compensation of $\underset{\sim}{\nabla} n_e$ and $\underset{\sim}{\nabla}\Phi_e$

$$\frac{1}{n_0}\frac{\partial n_0}{\partial r} \simeq -\frac{e\nabla\Phi}{T_e}$$

$$\therefore\ \Gamma_i = -\nu_i\, l_{mfp\,i}^2\left[\left(1 + \frac{T_e}{T_i}\right)\right]\frac{\partial n_0}{n_0\,\partial r}$$

Electric field enhances flux of ions

Electric field inhibits fluxes of electrons

Now let us consider transport with a magnetic field present

$\frac{eB}{mc} = \omega_c$ can readily exceed ν

Ordering has to be modified

$$v \frac{\partial f}{\partial \underset{\sim}{r}} + \frac{e}{m} \underset{\sim}{E} \cdot \frac{\partial f}{\partial \underset{\sim}{v}} + \underset{\sim}{v} \times \underset{\sim}{b} \cdot \frac{\partial f}{\partial \underset{\sim}{v}} = -\hat{\nu} f$$

Lowest order $\underset{\sim}{v} \times \underset{\sim}{b} \cdot \frac{\partial f}{\partial \underset{\sim}{v}} = -\frac{\partial f}{\partial \phi}$

$$\underset{\sim}{v} = v \cos\theta \, \hat{\underset{\sim}{z}} + v \sin\theta \left[\cos\phi \, \hat{\underset{\sim}{x}} + \sin\phi \, \hat{\underset{\sim}{y}}\right]$$

Take space gradient in x-direction

$\underset{\sim}{b}$ in z-direction

$$-\hat{\nu} f^{(0)} + \omega_c \frac{\partial f^{(0)}}{\partial \phi} = 0$$

$$-\hat{\nu} f^{(1)} + \omega_c \frac{\partial f^{(1)}}{\partial \phi} = +\underset{\sim}{v} \cdot \frac{\partial f^{(0)}}{\partial \underset{\sim}{r}} + \frac{e}{m} \underset{\sim}{E} \cdot \frac{\partial f^{(0)}}{\partial \underset{\sim}{v}}$$

$f^{(0)} = F_{max}$ satisfies lowest order equation

together with $F_M(\psi)$

with ψ a magnetic flux function

(in this simple case, all spatial gradients are $\perp$ to magnetic field)

Thus we have

(with Krook operator)

Take $E_x = 0$

$$-\nu f - \omega_c \frac{\partial f}{\partial \phi}$$

$$= \left[\frac{v_x}{n}\frac{\partial n}{\partial x} - \frac{eE_y v_y}{T} - \frac{eE_z v_z}{T} \right] F_{max}$$

$$= v \left[\sin\theta \frac{\partial n}{n\partial x} \cos\phi - \frac{e\sin\theta\sin\phi E_y}{T} - \frac{e\cos\theta E_z}{T} \right] F_m$$

let $\quad f = \sum_{n=-1}^{1} e^{im\phi} f_m$

$$-\nu f_0 = -\frac{e}{T}\cos\theta\, E_z v F_{max} \quad ; \quad f_0 = +\frac{e}{T}\cos\theta \frac{E_z v}{\nu} F_{max}$$

$$(-\nu \pm i\omega_c) f_{\pm 1} = \frac{v}{2}\sin\theta \left[\frac{1}{n}\frac{\partial n}{\partial x} \pm e\frac{\sin\theta}{iT} E_y \right] F_m$$

$$f_\pm = \frac{v\sin\theta}{2(-\nu \pm i\omega_c)} \left[\frac{\partial n}{n\partial x} \mp i e\frac{\sin\theta}{T} E_y \right] F_M$$

The non-zero moments are:

$$j_z = e\int d^3v\, v\cos\theta\, f_0 = \frac{n_0 e^2}{T}\frac{E_z}{\nu}\int d^3v\, v^2\cos^2\theta\, F_{max}$$

$$= \frac{ne^2}{m}\frac{E_z}{\nu} = \frac{\omega_p^2}{4\pi\nu} \cdot E_z \equiv \sigma_R \cdot E_z$$

or

$$\boxed{\sigma_R = \frac{\omega_{pe}^2}{4\pi\nu}}$$

$$\boxed{\eta_R = \frac{4\pi\nu}{\omega_{pe}^2}}$$

ion have negligible mobility compared to electrons along field line