Lecture 33

Landau Fokker-Planck Equation

Fokker - Planck Equation

$$\left.\frac{\partial f_i}{\partial t}\right|_c = \frac{\Gamma_i}{2}\sum_j \left(\frac{e_j}{e_i}\right)^2 \frac{\partial}{\partial \underset{\sim}{v}}\cdot \int d^3 v' \, \frac{\partial^2 |\underset{\sim}{g}|}{\partial \underset{\sim}{v}' \partial \underset{\sim}{v}'} \cdot$$

$$\left[f_j(\underset{\sim}{v}') \frac{\partial f_i(\underset{\sim}{v})}{\partial \underset{\sim}{v}} - \frac{m_i}{m_j} f_i(\underset{\sim}{v}) \frac{\partial f_j(v')}{\partial \underset{\sim}{v}'} \right]$$

$$\Gamma_i = \frac{4\pi e_i^4 \ln\Lambda}{m_i^2}, \quad \underset{\sim}{g} = \underset{\sim}{v} - \underset{\sim}{v}'$$

$$\frac{\partial^2 |g|}{\partial \underset{\sim}{v} \partial \underset{\sim}{v}} = \frac{1}{g}\left(\underset{\approx}{I} - \hat{\underset{\sim}{g}}\hat{\underset{\sim}{g}}\right);$$

Note it is easy to establish that the stationary solutions are:

~~Base unit for collision frequency is~~

~~$\nu_0^{i/j} = 4\pi e_j^2 e_i^2 \ln\Lambda n_j / m_i^2 v_{thi}^3$~~

~~species i colliding with species j.~~

$$F_{maxj}(\underset{\sim}{v}') \frac{\partial}{\partial \underset{\sim}{v}} F_{maxi}(\underset{\sim}{v}) - \frac{\partial F_{maxj}(\underset{\sim}{v}')}{\partial \underset{\sim}{v}'} F_{max}(\underset{\sim}{v}) \frac{m_i}{m_j}$$

$$= -\frac{(\underset{\sim}{v} - \underset{\sim}{v}') m_i F_{maxj}(v') F_{maxi}(\underset{\sim}{v})}{v_{thi}^2} \propto -\underset{\sim}{g}$$

The vanishing follows since

$$\underset{\sim}{g} \cdot \left(\underset{\approx}{I} g^2 - \underset{\sim}{g}\,\underset{\sim}{g} \right) = 0$$

In the NRL formulary there are expressions the relaxation rate for a test particle

$$\frac{d\bar{\underset{\sim}{V}}_\alpha}{dt} = -\nu_s^{\alpha/\beta}\, \underset{\sim}{V}_\alpha \qquad \text{drag, or slowing down}$$

$$\frac{d}{dt}\left\langle (\underset{\sim}{V}_\alpha - \underset{\sim}{V}_0)_\perp^2 \right\rangle = \nu_\perp^{\alpha/\beta}\, \underset{\sim}{V}_0^2 \qquad \text{transverse diffusion}$$

$$\frac{d}{dt}\left\langle (V_\alpha - \underset{\sim}{V}_0)_\parallel^2 \right\rangle = \nu_\parallel^{\alpha/\beta}\, V_\alpha^2 \qquad \text{parallel diffusion}$$

$$\frac{d}{dt}\left\langle V_\alpha^2 \right\rangle = -\nu_\epsilon\, V_\alpha^2 \qquad \text{energy loss}$$

How can we calculate these values from the Fokker Planck collision operator?

Start with

Simple Fokker Planck Equation in 1-D

$$\frac{\partial f}{\partial t} = \nu \frac{\partial}{\partial v} \cdot \left(\underset{\sim}{v} + v_{th}^2 \frac{\partial}{\partial v} \right) f$$

Note that the stationary solution is $f = \dfrac{\exp\left(-\dfrac{v^2}{2 v_{th}^2}\right) n_0}{(2\pi v_{th}^2)^{1/2}}$

Any initial condition relaxes to this Maxwellian

For example let the initial distribution be

$$f(0) = \delta(v - v_0)\, n_0$$

Let us solve the evolution equation.

We note that the solution of the diffusion equation

$$0 = \frac{\partial f}{\partial t} + V_{th}^2 \nu \frac{\partial f}{\partial v^2}$$

is, if when $t=0$ $f = \delta(v-v_0)$,

$$f(v) = \frac{1}{(4t\nu V_{th}^2)^{1/2}} \exp\left(-\frac{(v-v_0)^2}{4\nu V_{th}^2 t}\right)$$

Easy to show that

$$f(v) \xrightarrow[t\to 0]{} \delta(v-v_0)$$

But Fokker-Planck Equation is of form

$$\frac{\partial f}{\partial t} - \nu\left[f + v\frac{\partial f}{\partial v} + V_{th}^2 \frac{\partial^2 f}{\partial v^2}\right] = 0$$

To rid ourselves of the $-\nu f$ we define $f(v) = g(v)e^{\nu t}$, then

$$\frac{\partial g(v)}{\partial t} - \nu\left[v\frac{\partial g}{\partial v} + V_{th}^2 \frac{\partial^2 g}{\partial v^2}\right] = 0$$

Now as a characteristic of linear term satisfies $\frac{dv}{dt} = -\nu v$, with solution

$v = u e^{-\nu t}$, or $u = v e^{\nu t}$

Then with new variables,

$$u = v e^{\nu t}$$

$$t' = t$$

The F.P. equation becomes

$$\frac{\partial g}{\partial t} - \nu V_{th}^2 \frac{\partial^2 g}{\partial u^2} e^{-2\nu t} = 0$$

Thus we have the form we desire, except we need to introduce a time variable,

$$dT = dt\, e^{-2\nu t}$$

$$T = \frac{1}{2\nu}\left(1 - e^{-2\nu t}\right) \qquad T \xrightarrow[t\to 0]{} t$$

$$T \xrightarrow[t\to\infty]{} \frac{1}{2\nu},$$

Then

$$\frac{\partial g}{\partial T} - \nu V_{th}^2 \frac{\partial^2 g}{\partial u^2} = 0$$

$$\therefore \quad g = \frac{1}{(4\pi T V_{th}^2)^{1/2}} \exp\left(-\frac{(u-u_0)^2}{4\nu V_{th}^2 T}\right)$$

Substituting the ~~old~~ variables gives

$$f(v) = \frac{\exp\left[- \left(\frac{(v - v_0 e^{-\nu t})^2}{2 v_{th}^2 (1 - \exp(-2\nu t))} \right) \right]}{\left[2 v_{th}^2 \pi \, [1 - \exp(-2\nu t)] \right]^{1/2}}$$

$$\xrightarrow[\nu t \ll 1]{} \exp\left[-\frac{(v - v_0)^2}{4 v_{th}^2 \nu t} \right] \Big/ \left[4 v_{th}^2 \nu t \right]^{1/2}$$

$$\xrightarrow[\nu t \gg 1]{} \frac{\exp\left(- v^2 / 2 v_{th}^2 \right)}{\left[2\pi v_{th}^2 \right]^{1/2}}$$

Note that the initial rate of spreading is given by

$$\int_{-\infty}^{\infty} f(v) (v - v_0)^2 dv = \int_{-\infty}^{\infty} dv \, \frac{\exp\left[-\frac{(v - v_0)^2}{4 v_{th}^2 t} \right] (v - v_0)^2}{(4 v_{th}^2 t)^{1/2}}$$

$$= 2 v_{th}^2 t \int_{-\infty}^{\infty} dv \, \frac{d}{dv} \exp\left[-\frac{(v - v_0)^2}{4 v_{th}^2 t} \right] \frac{(v - v_0)}{(4 v_{th}^2 t)^{1/2}}$$

$$= 2 v_{th}^2 \nu t = \langle (v - v_0)^2 \rangle$$

Therefore

$$\frac{d\langle (v-v_0)^2\rangle}{dt} = 2\nu\, V_{th}^2$$

This result can also be obtained perturbatively without solving FP Equation

Initially $f(v) = \delta(v-v_0)$

Now

$$\frac{\partial f}{\partial t} = \nu \frac{\partial}{\partial v}\left(v - V_{th}^2 \frac{\partial}{\partial v}\right) f$$

$\therefore$

$$\int \frac{\partial}{\partial t} f\, (v-v_0)^2\, dv \equiv \frac{\partial}{\partial t}\langle v - v_0\rangle^2$$

$$= \nu \int dv \left(\frac{\partial}{\partial v}\left(v - V_{th}^2 \frac{\partial}{\partial v}\right) f\right)(v-v_0)^2$$

on rhs $f = \delta(v-v_0)$ accurate when

$\nu t \ll 1$

$$\therefore \quad \frac{\partial}{\partial t}\langle (v-v_0)^2\rangle = \nu \int_{-\infty}^{\infty} dv\, (v-v_0)^2 \frac{\partial}{\partial v}\left(v - V_{th}^2\frac{\partial}{\partial v}\right)\delta(v-v_0)$$

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$$\frac{\partial}{\partial t}\langle (v-v_0)^2\rangle = -2\nu \int_{-\infty}^{\infty} dv\,(v-v_0)\left(v - v_{th}^2\frac{\partial}{\partial v}\right)\delta(v-v_0)$$

$$= -2\nu \int_{-\infty}^{\infty} dv \left[(v-v_0)\, v\,\delta(v_0-v) - v_{th}^2\,\delta(v-v_0) \right]$$

$$= +2\nu v_{th}^2 \quad \checkmark \quad \text{as stated}$$

This ~~initial~~ rate of spreading, can be calculated this way with the Coulomb, Fokker-Planck equation ($\underset{\sim}{g} = \underset{\sim}{v} - \underset{\sim}{v}'$)

$$\frac{\partial f_i}{\partial t} = \frac{\Gamma_i}{2}\sum_j \left(\frac{e_j}{e_i}\right)^2 \frac{\partial}{\partial \underset{\sim}{v}}\int d^3v'\,\frac{\partial^2 |g|}{\partial \underset{\sim}{v}'\partial \underset{\sim}{v}'}$$

$$\cdot\left[f_j(\underset{\sim}{v}')\frac{\partial f_i(\underset{\sim}{v})}{\partial \underset{\sim}{v}} - \frac{m_i}{m_j} f_i(v)\frac{\partial f_j(\underset{\sim}{v}')}{\partial \underset{\sim}{v}'}\right]$$

Now let $f_i = \delta(v-v_0)$ at $t=0$

consider just one species for simplicity,

where $f_j(\underset{\sim}{v}) = \dfrac{n_0}{(2\pi v_{thj}^2)^{3/2}}\exp\left(-\dfrac{v^2}{2v_{thj}^2}\right)$

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Substitute collision integral & integrate by parts

$$\frac{\partial}{\partial t}\langle(\underset{\sim}{v}-\underset{\sim}{v}_0)^2\rangle = \int d^3v \frac{\partial f}{\partial t}(\underset{\sim}{v}-\underset{\sim}{v}_0)^2$$

$$= -2\Gamma_{ij} n_j \int d^3v' d^3v\, (\underset{\sim}{v}-\underset{\sim}{v}_0)\cdot \frac{(\underset{\sim}{I}-\hat{\underset{\sim}{g}}\hat{\underset{\sim}{g}})}{g} \frac{\exp\left(-\frac{v'^2}{2v_{thj}^2}\right)}{(2\pi v_{thj}^2)^{3/2}} \frac{\partial}{\partial \underset{\sim}{v}}\delta(\underset{\sim}{v}-\underset{\sim}{v}_0)$$

we need to annihilate $\underset{\sim}{v}-\underset{\sim}{v}_0$ term (only non-vanishing term)

$$= 2n_0\Gamma_{ij}\int d^3v' d^3v\, \underset{\approx}{I} : \frac{\exp\left(-\frac{v'^2}{2v_{thj}^2}\right)\delta(\underset{\sim}{v}-\underset{\sim}{v}_0)(\underset{\approx}{I}g^2-\underset{\sim}{g}\underset{\sim}{g})}{(2\pi v_{thj}^2)^{3/2} g^3}$$

$$I:I=3$$

$$I:\underset{\sim}{g}\underset{\sim}{g}=g^2$$

$$= 4n\Gamma_{ij}\int d^3v' \frac{\exp\left(\frac{-v'^2}{2v_{thj}^2}\right)}{(2\pi v_{thj}^2)^{3/2}\,[v'^2+v_0^2-2v'v_0\cos\theta]^{1/2}}$$

$$d^3v' = 2\pi v'^2 dv'\, d\theta$$

$$= \frac{4\pi n_0\Gamma_{ij}}{(2\pi v_{thj}^2)^{3/2}}\int_0^\infty dv'\, v'^2 \exp\left(-\frac{v'^2}{2v_{thj}^2}\right)\int_0^\pi \frac{d\theta \sin\theta}{[v'^2+v_0^2-2v'v_0\cos\theta]^{1/2}}$$

$$= \frac{4\pi n\Gamma_{ij}}{(2\pi v_{thj}^2)^{3/2}}\int_0^\infty \frac{dv'\, v'^2 \exp\left(-\frac{v'^2}{2v_{thj}^2}\right)}{(2v'v_0)}\left[(\underset{\sim}{v}_0+\underset{\sim}{v}') - |v'-v_0|\right]$$

$$= \frac{8\pi n \Gamma_{ij}}{(2\pi v_{thj}^2)^{3/2}} \left[\int_{v_0}^{\infty} dv\, v \exp\left(-v^2/2v_{thj}^2\right) + \int_0^{v_0} \frac{dv\, v^2}{v_0} \exp\left(-\frac{v^2}{2v_{thj}^2}\right) \right]$$

$$= \frac{8\pi n \Gamma_{ij}}{(2\pi v_{thj}^2)^{3/2}} \left[v_{thj}^2 \exp\left(-\frac{v_0^2}{2v_{thj}^2}\right) + \frac{v_{thj}^3}{v_0} \int_0^{v/v_{thj}} dx\, e^{-x^2/2} - v_{thj}^2 e^{-\frac{v_0^2}{2v_{th}^2}} \right]$$

$$\frac{d}{dt} \langle (\underset{\sim}{v} - \underset{\sim}{v_0}) \cdot (\underset{\sim}{v} - \underset{\sim}{v_0}) \rangle = \left(\frac{2}{\pi}\right)^{1/2} \frac{n_0 e_i^2 e_j^2 \ln\Lambda}{v_{thj} m_i^2} \left[\frac{v_{thj}}{v_0} \int_0^{v_0/v_{thj}} dx\, e^{-x^2/2} \right]$$

$$\longrightarrow \left(\frac{2}{\pi}\right)^{1/2} \frac{n_0 e_i^2 e_j^2 \ln\Lambda}{m_i} \begin{cases} \left(\dfrac{1}{v_{thj}}\right) & v_0 \ll v_{thj} \\ \left(\dfrac{1}{v_0}\right)\left(\dfrac{\pi}{2}\right)^{1/2}, & v_0 \gg v_{thj} \end{cases}$$