Lecture # ~~24~~ 25

Poisson Equation with magnetic field

Vlasov Equation

Statement of conservation of probability of $f(\underset{\sim}{r}, \underset{\sim}{p}, t)\, d^3r\, d^3p$

$$\frac{\partial f(\underset{\sim}{r}, \underset{\sim}{p}, t)}{\partial t} + \frac{\partial}{\partial \underset{\sim}{r}}\left(\dot{\underset{\sim}{r}}\, f\right) + \frac{\partial}{\partial \underset{\sim}{p}}\left(\dot{\underset{\sim}{p}}\, f\right) = 0$$

Since $\mathcal{V}(\Delta \underset{\sim}{r}, \Delta \underset{\sim}{p})$ is conserved [i.e. 6-D (Jacobian)$^{-1}$] in motion of particle, what flows into a fixed phase space volume, must lead to ~~an~~ corresponding increase in particle number (like counting charge in 3-D to get continuity equation)

There is another physical meaning of Vlasov-equation. The value of $f(\underset{\sim}{r}, p, t)$ along the particle trajectory does not change.

$$\frac{d}{dt} f(\underset{\sim}{r}, \underset{\sim}{p}, t) = \left[\frac{\partial}{\partial t} + \dot{\underset{\sim}{r}} \cdot \frac{\partial}{\partial \underset{\sim}{r}} + \dot{\underset{\sim}{p}} \cdot \frac{\partial}{\partial \underset{\sim}{p}}\right] f(\underset{\sim}{r}, \underset{\sim}{p}, t) = 0$$

The two forms are identical ~~by~~ by virtue of the Hamiltonian form of dynamics

$$\dot{\mathbf{r}} = \frac{\partial H(\mathbf{r},\mathbf{p},t)}{\partial \mathbf{p}}$$

$$\dot{\mathbf{p}} = -\frac{\partial H(\mathbf{r},\mathbf{p},t)}{\partial \mathbf{r}}$$

Then conservation form from 6-D flux

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathbf{r}}\cdot\frac{\partial H(\mathbf{r},\mathbf{p},t)}{\partial \mathbf{p}} - \frac{\partial}{\partial \mathbf{p}}\cdot\frac{\partial H(\mathbf{r},\mathbf{p},t)}{\partial \mathbf{r}}\right] f(\mathbf{r},\mathbf{p},t) = 0$$

equivalent to constancy of $f(\mathbf{r},p,t)$ along trajectory

$$\left[\frac{\partial}{\partial t} + \frac{\partial H(r,p,t)}{\partial \mathbf{p}}\cdot\frac{\partial}{\partial \mathbf{r}} - \frac{\partial H(\mathbf{r},p,t)}{\partial \mathbf{r}}\frac{\partial}{\partial \mathbf{p}}\right] f(r,p,t)$$

$$+ f\left[\frac{\partial}{\partial \mathbf{r}}\frac{\partial}{\partial \mathbf{p}} - \frac{\partial}{\partial \mathbf{p}}\frac{\partial}{\partial \mathbf{r}}\right] H(r,p,t)$$

last ~~term~~ $= 0$

$= 0$ – this term vanishes

$$= \left[\frac{\partial}{\partial t} + \dot{\mathbf{r}}\cdot\frac{\partial}{\partial \mathbf{r}} + \dot{\mathbf{p}}\frac{\partial}{\partial \mathbf{p}}\right] f(\mathbf{r},p,t) \equiv \frac{d f(\mathbf{r},p,t)}{dt} = 0$$

Stationary Vlasov Equation

$$\dot{\underset{\sim}{r}} \cdot \frac{\partial f}{\partial \underset{\sim}{r}} + \dot{\underset{\sim}{p}} \frac{\partial f}{\partial \underset{\sim}{p}} = 0$$

$$= \frac{\partial H}{\partial \underset{\sim}{p}} \cdot \frac{\partial f}{\partial \underset{\sim}{r}} - \frac{\partial H}{\partial \underset{\sim}{r}} \cdot \frac{\partial f}{\partial \underset{\sim}{p}} = 0$$

$$\equiv [H, f] = 0$$

↖ Poisson Brackett

Fundamental theorem in Hamiltonian dynamics $C_i \equiv C_i(\underset{\sim}{r}, \underset{\sim}{p})$

$$[H, f(C_i)] = 0$$

where C_i are constants of motion of a particle ~~for~~ Hamilton H.

Thus, a distribution function that are functions of the constants of motion, satisfy stationary Vlasov equation

$$\left[\underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{r}} + \dot{\underset{\sim}{p}} \frac{\partial}{\partial \underset{\sim}{p}}\right] f(C_i(\underset{\sim}{r}, \underset{\sim}{p})) = 0$$

Let us consider electro-static perturbations of a particle in a uniform magnetic field

Let us take $\underset{\sim}{B} \parallel \hat{z}$ and plasma independent of y

$$\underset{\sim}{B} = B_0 \hat{\underset{\sim}{z}} \quad , \quad \underset{\sim}{A} = x B_0 \hat{\underset{\sim}{y}}$$

$$P_y = m v_y + \frac{e A_y}{c} \equiv \text{constant of motion}$$

$$= m\left(v_y + \frac{e B_0}{mc} X\right)$$

or equivalenty $\quad X_g = X + v_y/\omega_c$

(for uniform field $P_y = m\omega_c X_g$)

Other constants of motion are

(a) $\frac{E}{m} = \frac{1}{2}\left(v_\parallel^2 + v_\perp^2\right) \to E$ (energy/mass)

(b) $\frac{v_\perp^2}{2} \xrightarrow[\text{non-uniform field}]{} \mu \equiv \frac{v_\perp^2}{2B}$ (if B non-uniform in space)

(μ is only an adiabatic invariant; can lead to some modifications which may be treated later)

For now we treat $v_\perp^2$ as a constant of motion in an unperturbed magnetic field: Also $v_\parallel = \pm\{2E - v_\perp^2\}^{1/2}$ constant of motion in this case.

We will consider linear-Vlasov equation $f(\underset{\sim}{r}, \underset{\sim}{p}, t)$

$$\left[\frac{\partial}{\partial t} + \underset{\sim}{v}\cdot\frac{\partial}{\partial \underset{\sim}{r}} + \underbrace{\frac{e}{c}\underset{\sim}{v}\times\underset{\sim}{B}}_{=\dot{p}}\cdot\frac{\partial}{\partial \underset{\sim}{p}}\right]\delta f_i = e\,\underset{\sim}{\nabla}\phi\cdot\frac{\partial F_i(\underset{\sim}{r},p,t)}{\partial \underset{\sim}{p}}$$

$$\nabla^2\phi = -4\pi e\left[\int d^3p\, Z_i f_i - \int d^3p\, Z_e f_e\right]$$

$$F = F\left(x + \frac{v_y}{\omega_c}, v_\parallel, v_\perp^2\right) = F(x_g, v_\parallel, v_\perp^2)$$

First just consider single species (say electrons)

$$\left.\frac{\partial}{\partial \underset{\sim}{p}}\right|_r = \left.\frac{1}{m}\frac{\partial}{\partial \underset{\sim}{v}}\right|_r \qquad \left(\begin{array}{l}\text{as } \underset{\sim}{r} \text{ held fixed} \\ \text{as } \underset{\sim}{p} \text{ varies}\end{array}\right)$$

Notice:

$$\frac{\partial F}{\partial \underset{\sim}{v}} = \hat{y}\frac{\partial X_g}{\partial v_y}\frac{\partial F}{\partial X_g} + \underset{\sim}{\hat{z}}\frac{\partial F}{\partial v_\parallel} + 2\underset{\sim}{v_\perp}\frac{\partial F}{\partial v_\perp^2}\text{, and } \therefore$$

$$\underset{\sim}{\nabla}\phi\cdot\frac{\partial F}{\partial \underset{\sim}{v}} = \frac{\partial\phi}{\partial y}\frac{1}{\omega_c}\frac{\partial F}{\partial X_g} + \frac{\partial\phi}{\partial z}\frac{\partial F}{\partial v_\parallel} + 2\underset{\sim}{v_\perp}\cdot\underset{\sim}{\nabla_\perp}\phi\frac{\partial F}{\partial v_\perp^2}$$

spatial gradient came from v_y derivative

$$n\bar{\underset{\sim}{V}} = \int d^3p \, \underset{\sim}{v} \, F_0(\underset{\sim}{R}_g, v_\parallel, v_\perp^2)$$

$$= \int d^3p \, v \, F_0\left(\underset{\sim}{r} - \frac{b \times v}{\omega_c}, v_\parallel, v_\perp^2\right)$$

$$\simeq \int d^3p \, \underset{\sim}{v} \left(F_0(\underset{\sim}{r}, v_\parallel, v_\perp^2) - \frac{b \times \underset{\sim}{v}}{\omega_c} \cdot \nabla F(\underset{\sim}{r}, v_\parallel, v_\perp^2)\right)$$

$$\left(\text{If} \quad F_0(v_\parallel) = F_0(-v_\parallel)\right) \qquad \left(\frac{d^3p}{m} = d^3v\right)$$

$$\simeq - \int d^3p \, v \, \frac{\underset{\sim}{b} \times \underset{\sim}{v}}{\omega_c} \cdot \nabla \, F_0(\underset{\sim}{r}, \underset{\sim}{v}_\parallel, v_\perp^2)$$

$$d^3p = m^3 d^3v$$

$$d^3v = dv_\parallel \, dv_\perp v_\perp \, d\varphi \qquad \underset{\sim}{v} = v_\parallel \underset{\sim}{b} + v_\perp \cos\varphi \hat{x} + v_\perp \cos\varphi \hat{y}$$

$$2\pi \, \overline{\underset{\sim}{b} \times \underset{\sim}{v} \, \underset{\sim}{v} \cdot \underset{\sim}{\nabla}} = 2\pi \, b \times \left[\frac{v_\perp^2}{2} \underset{\approx}{I} + \left(v_\parallel^2 - \frac{v_\perp^2}{2}\right) \underset{\sim}{b}\underset{\sim}{b}\right] \cdot \underset{\sim}{\nabla}$$

$$= 2\pi \frac{v_\perp^2}{2} (\underset{\sim}{b} \times \underset{\sim}{\nabla})$$

$$\bar{V} = \frac{\underset{\sim}{b} \times \underset{\sim}{\nabla}}{2\omega_c} \int d^3p \, v_\perp^2 F = \frac{\underset{\sim}{b} \times \underset{\sim}{\nabla} P_\perp}{eB}$$

perpendicular pressure

$$\text{with} \quad P_\perp = m \int d^3p \, \frac{v_\perp^2}{2} F = m \int d^3p \left(\frac{v_x^2}{2} + \frac{v_y^2}{2}\right) F$$

$$= m \int d^3p \, v_x^2 F$$

$$n\underset{\sim}{\bar{V}} = \frac{c \underset{\sim}{b} \times \underset{\sim}{\nabla} P_\perp}{e_j B} \equiv \text{diamagnetic flow}$$

Diamagnetic Current

$$j = \sum n e_j \underset{\sim}{V} = \frac{c \underset{\sim}{b} \times \underset{\sim}{\nabla} P_{\perp total}}{B} \qquad \left(P_{\perp total} = \sum_j P_{\perp j} \right)$$

$j \equiv$ all species e.g. electrons + protons + ...

$\left(\text{Recall} \quad \frac{\underset{\sim}{j}}{c} \times \underset{\sim}{B} = \underset{\sim}{\nabla} p \leftarrow \text{pressure} \right.$ of MHD fluid theory

$$\left. \underset{\sim}{j} = \frac{c \underset{\sim}{b} \times \underset{\sim}{\nabla} p}{B} \leftarrow \text{pressure} \right)$$

Now let us examine perturbed Vlasov equation: $\left(V_\parallel \equiv V_z \text{ in this case} \right)$

$Df/Dt \equiv$

$$\left[\frac{\partial}{\partial t} + \underset{\sim}{v} \cdot \frac{\partial}{\partial \underset{\sim}{r}} + \omega_c \underset{\sim}{v} \times \underset{\sim}{b} \cdot \frac{\partial}{\partial \underset{\sim}{v}} \right] f$$

$$= \frac{e}{m} \left[\frac{1}{\omega_c} \frac{\partial F}{\partial X_g} \hat{y} + \frac{\partial F}{\partial V_\parallel} \hat{z} + 2 \frac{\partial F}{\partial V_\perp^2} \underset{\sim}{V_\perp} \right] \cdot \underset{\sim}{\nabla} \phi(\underset{\sim}{r}, t)$$

Introduce Trick to ease integration

note: $\underset{\sim}{V_\perp} \cdot \underset{\sim}{\nabla} \phi = \left[\frac{D}{Dt} - \frac{\partial}{\partial t} - V_\parallel \frac{\partial}{\partial z} \right] \phi$

$$\text{Let} \quad f = \frac{e}{m} \frac{\partial F}{\partial V_\perp^2/2} \phi(r, t) + g(\underset{\sim}{r}, \underset{\sim}{v})$$

$$\text{Take} \quad \phi = \phi(t) \exp\left[i \underset{\sim}{k_\perp} \cdot \underset{\sim}{r} + i k_z z \right] + \text{c.c.}$$

$$\underset{\sim}{k_\perp} = k_\perp \cos\theta \, \underset{\sim}{\hat{x}} + k_\perp \sin\theta \, \hat{y}$$

Then $\underset{\sim}{\nabla}_r \phi \to i\underset{\sim}{k}\,\phi_k$

$$\frac{D g(r,v,t)}{Dt} = \frac{ie\phi}{m}\underbrace{\left[\frac{k_y}{\omega_c}\frac{\partial F}{\partial X_g} + k_z\frac{\partial F}{\partial v_z} + (\omega - k_z v_z)\frac{\partial F}{\partial v_\perp^2/2}\right]}_{\text{functions of constants of motion in equilibrium orbits}}$$

$$F = F(X_g, v_z, v_\perp^2)$$

Integrate by method of characteristics:

$$g(r,v,t) = \frac{ie}{m}\left[\frac{k_y}{\omega_c}\frac{\partial F}{\partial X_g} + k_z\frac{\partial F}{\partial v_z} + (\omega - k_z v_z)\frac{\partial F}{\partial v_\perp^2/2}\right]$$

$$\int_{-\infty}^{t} dt'\, \phi(t', \underset{\sim}{r}(t'-t), \underset{\sim}{v}(t'-t))$$

where $\underset{\sim}{r}(0) = \underset{\sim}{r}$

$\underset{\sim}{v}(0) = \underset{\sim}{v}$, $\quad t' - t \equiv \tau$

$r(\tau) = R_g(\tau) + \frac{\underset{\sim}{b}\times \underset{\sim}{v}(t)}{\omega_c}$; $\quad R_g(\tau) = \underset{\sim}{R}(0) + v_\parallel \tau \hat{\underset{\sim}{z}}$; $\quad \underset{\sim}{R}(0) = \underset{\sim}{r} - \frac{\underset{\sim}{b}\times\underset{\sim}{v}}{\omega_c}$

$\therefore\ \underset{\sim}{r}(\tau) = \cancel{R_g(\tau)} + \frac{\underset{\sim}{b}\times\underset{\sim}{v}(\tau)}{\omega_c} - \underset{\sim}{b}\times\underset{\sim}{v}/\omega_c + \underset{\sim}{r} + v_\parallel \tau \hat{\underset{\sim}{z}}$

$\underset{\sim}{v}(\tau) = v_\perp \cos(\psi - \omega_c\tau)\,\hat{\underset{\sim}{x}} + v_\perp \sin(\psi - \omega_c \tau)\,\hat{\underset{\sim}{y}}$

$\underset{\sim}{v} = v_\perp \cos\psi\, \hat{\underset{\sim}{x}} + v_\perp \sin\psi\, \hat{\underset{\sim}{y}}$

$\cancel{R_g(\tau) = R_g(0) + v_\parallel \tau}$

For $\phi(t', r') = \phi_k \exp\left[-i\omega t' + i\underset{\sim}{k}\cdot\underset{\sim}{r}'\right]$

$\underset{\sim}{k} = k_\parallel \hat{z} + k_\perp \cos\theta\, \hat{x} + k_\perp \sin\theta\, \hat{y}$

$\underset{\sim}{k}\cdot\underset{\sim}{b}\times\underset{\sim}{v}(\tau) = k_\perp v_\perp \sin(\cancel{\psi} + \omega_c t + \theta)$

We have essentially assumed taken a WKB approximation in x

$$\phi(x) = \exp\left(i \int^{x} k(x')\,dx' \right) \hat{\phi}(x)$$

where $\dfrac{d\hat{\phi}(x)}{k(x)\,dx} << 1$, also $\dfrac{d^2 k}{k^2 dx} << 1$

Then

$$\phi(x(\tau)) = \hat{\phi}(x) \exp\left(i \int_0^{x} k(x')\,dx' \right) \exp\left[i k(x)(x(\tau) - x) \right]$$

Thus we have

$$\int_{-\infty}^{t} dt' \, \phi(t', \underset{\sim}{r}(t'-t), \underset{\sim}{v}(t'-t)) =$$

$$\phi_k e^{-i\omega t} \int_{-\infty}^{0} d\tau \, e^{-i\omega\tau + i k_{\parallel} v_{\parallel} \tau + i \frac{k_\perp v_\perp}{\omega_c} \left(\begin{array}{l} \sin(-\psi + \theta + \omega_c \tau) \\ - \sin(-\psi + \theta) \end{array} \right)}$$

$$= \phi_k e^{-i\omega t} \int_{-\infty}^{0} d\tau \, \exp[-i(\omega - k_{\parallel} v_{\parallel})\tau] \exp[+i \sin(\psi + \theta + \omega_c \tau)] \cdot \exp[-i \sin(\psi + \theta)]\}$$

note

$$\exp[i x \sin\theta] = \sum_n J_n(x) \exp(i\theta n)$$