Lecture 21

Ballooning Modes

&

TAE modes.

Mercier's sufficient condition for instability

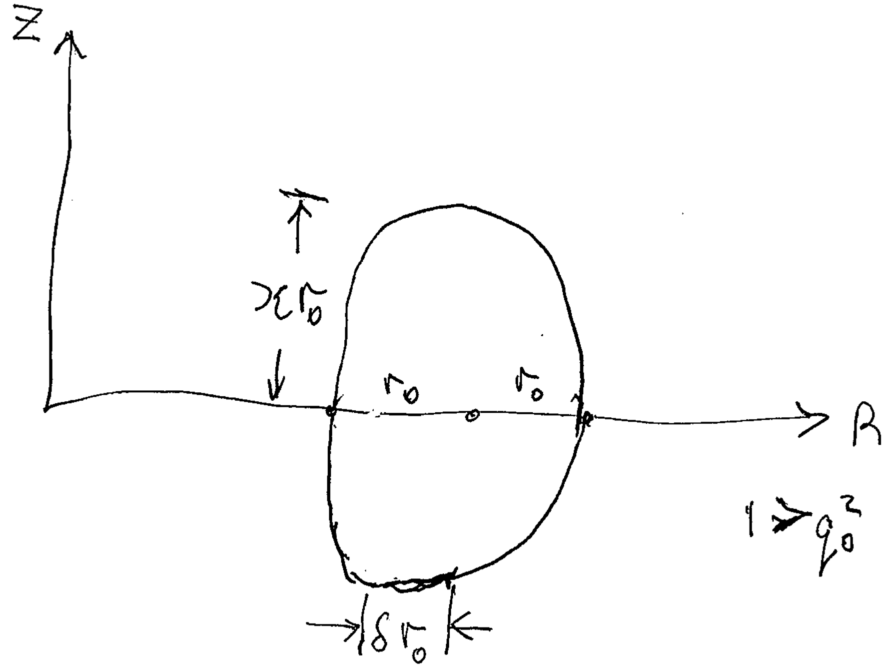
$$-\beta' r \geq \frac{s^2}{4(1-q^2)} \quad ; \quad \beta' = \frac{2dp}{B^2 dr}$$

$$s = \frac{r\,dq}{q\,dr}$$

This applies to equilibria that are shifted circles

Shaping conditions effects stability

elongation, triangularity



For Instability condition on axis $1 \geq q_0^2$ is replaced by:

$$1 \geq q_0^2 \left\{ 1 - \frac{4}{1+3\kappa^2}\left[\frac{3(\kappa^2-1)}{4(\kappa^2+1)}\left(\kappa^2 - \frac{2\delta}{\epsilon}\right) + \frac{(\kappa-1)^2 \beta_{p0}}{\kappa(\kappa+1)} \right] \right\} \quad (1)$$

Note: (1) There is an improved condition when $\kappa > 1$, for stability if $\frac{2\delta}{\epsilon} > \kappa^2$

(2) β dependence of stability only arises when equilibrium is elongated, not for shift circular flux surfaces

Stability arises because flux surface averaged field lines ~~av~~ weigh more to the inside of the plasma.

2% beta limit from shifted circles can be raised to ~10% with elongation & triangularity

Ballooning Modes

As beta builds up modes can localize in the bad curvature region, and curvature - β drive, can beat out line bending stabilization

Recall model balloon mode calculation: $\xi = \frac{\underset{\sim}{b}\times\underset{\sim}{\nabla}\phi}{B}$, $\underset{\sim}{B}_\perp = -\underset{\sim}{b}\times\underset{\sim}{\nabla}A_{\parallel}$

$A_{\parallel} = b\,\partial\phi/\partial\ell$

$$\frac{\partial}{\partial\theta}\left(1+\Lambda^2\right)\frac{\partial\phi}{\partial\theta} + \alpha\left(\Lambda\sin\theta + \cos\theta\right)\phi = 0$$

$$\Lambda(\theta) = s(\theta-\theta_0) - \alpha\sin\theta\ , \quad s = rq'/q$$

$$\alpha = -\frac{\beta' r^2}{R_0 B_\theta^2} = -\beta' R q^2$$

This comes from

$$\delta W \propto \int d\theta\left[(1+\Lambda)^2\left(\frac{\partial\phi}{\partial\theta}\right)^2 - \alpha\phi^2\left(\Lambda\sin\theta + \cos\theta\right)\right]$$

If $\frac{\partial\phi}{\phi\,\partial\theta} \gtrsim 1$, $\alpha > 1$ can beat out bending

Therefore: $\beta' R q^2 \gtrsim 1$ expected to be stability criterion

Note: strong shear ($s > 1$) enhances stability from line bending.

The surprise of the solution is that is that there is a second regime of stability as shown in the figure.

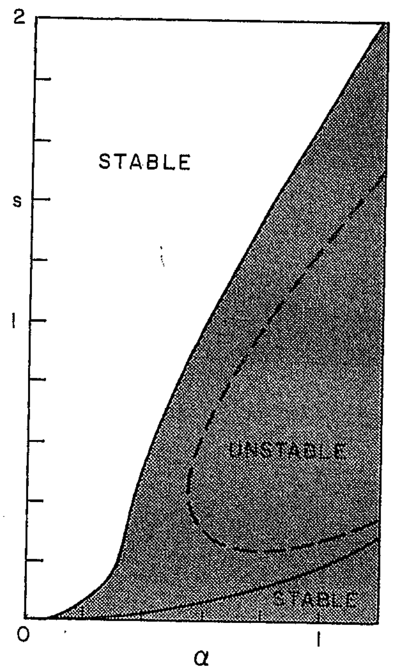


Figure 10.11. s vs. α marginal stability diagram for the analytic ballooning mode model. The region in the upper left corresponds to the first region of stability. The region in the lower right corresponds to the second region of stability. Courtesy J. Connor and M. Turner, Culham Laboratory, Great Britain.

Dashed curve is estimate obtain from trial function

$$\phi = \begin{cases} 1+\cos\theta & -\pi < \theta < \pi \\ 0 & |\theta| > \pi \end{cases}$$

Leads to an energy given by

$W = 1.39 \ s^2 - 2.17 s\alpha + \alpha^2 - \alpha + .5$

For a given alpha, marginal stability occurs at two s-values

$$s = 0.78\alpha \pm \left(.72\alpha - 0.36 - 0.11\alpha^2 \right)^{1/2}$$

illustrated by dashed curve.

Note: numerical solution more pessimistic (i.e larger region of instability) than this prediction

Also note that with negative s, there is no instability predicted (leads to transport barrier observed in late 1990's).

It is the pressure driven modulation of the local shear that allows second stability region.

$$s \to s - \alpha \cos\theta$$

To get growth rates of unstable modes, or to investigate frequencies of stable modes we need to add the kinetic energy to the system

$$\xi = -\frac{\underset{\sim}{b} \times \nabla \phi}{B}$$

$$\dot{\xi} = -\frac{b \nabla}{B} \frac{d\phi}{dt}$$

$$KE = \int d^3r \, \frac{\dot{\xi}^2 \rho}{2} \qquad \rho = \text{mass density}$$

If $\xi \rightarrow e^{-i\omega t}$

$$KE = \int d^3r \, \underset{\sim}{\xi} \cdot \underset{\sim}{\xi}^* \frac{\omega^2 \rho}{2}$$

$$= \int d^3r \, \nabla_\perp \phi \cdot \frac{\nabla_\perp \phi}{2} \frac{\omega^2}{V_A^2} \, , \quad V_A^2 = \frac{B^2}{\rho}$$

Including both the kinetic energy, the bending energy and the drive from pressure + curvature, leads to the equation

Remember $\nabla_\perp^2 \to \left(\frac{\partial S}{\partial \beta}\right)^2 \nabla\beta \cdot \nabla\beta$

$$\beta = \zeta - q(\psi)\,\theta$$

$$q^2 R^2 (1+\Lambda^2)\frac{\omega^2}{V_A^2}\phi + \frac{\partial}{\partial\theta}(1+\Lambda^2)\frac{\partial\phi}{\partial\theta}$$
$$+\alpha(\Lambda\sin\theta + \cos\theta)\phi = 0$$

$$\Lambda = s(\theta-\theta_0) - \alpha\sin\theta$$

If $\alpha \ll 1$, we will be stable, and we will be describing shear Alfven excitations.

Note, when we have instability $\omega^2 < 0$, growth rate is $\gamma = -i\omega$.

Let us look at the possibility of Alfven wave propagating along field lines

We neglect α

$$\Omega^2 \equiv \frac{\omega^2 q^2 R^2}{V_A^2}$$

Remember $\Lambda = 1 + s^2\theta^2$

However, even if $\alpha = 0$, there will be a modulation of the shear due to variation of $|\nabla\psi|^2$ and B^2 along a field line.

Hence, $s^2\theta^2$ should be replaced with $\Lambda^2 \equiv s^2\theta^2(1+\epsilon\cos\theta)$ where $\epsilon \simeq r/R$

Thus in $\alpha = 0$ limit, we consider model equation

$$\Omega^2(1+\Lambda^2)\phi + \frac{\partial}{\partial\theta}(1+\Lambda^2)\frac{\partial\phi}{\partial\theta} = 0$$

or

$$\left(\Omega^2\phi + \frac{\partial^2\phi}{\partial\theta^2}\right) = \frac{\epsilon s^2\theta^2\sin\theta - 2s^2\theta\cos\theta}{1+\Lambda^2}\phi$$

$$= \frac{\epsilon s^2\theta^2\sin\theta - 2s^2\theta\cos\theta}{1+s^2\theta^2(1+\epsilon\cos\theta)}\phi$$

We need to know the form of solution for large θ

$$\Omega^2\phi + \frac{\partial^2\phi}{\partial\theta^2} \simeq \epsilon \sin\theta\, \phi$$

If we did not have any modulation, the form of the solution would be at large θ

$$\phi = e^{\pm i\theta\Omega}$$

These are travelling Alfven waves, and the constitute the continuous spectrum of a sheared cylinder

Waves only on one (extended) field line. If a mode (strictly a quasi-mode) on from an antenna is localized in some region, it would radiate $e^{i\Omega\theta}$ to the right and $e^{-i\Omega\theta}$ to the left and cause radiation (or what is called <u>continuum</u>) damping.

If we now consider the outer solution with finite modulation $\epsilon \sin\theta\, \phi$, it would not normally change the structure of the continuum spectrum much. However, an exception occurs when $\Omega \simeq 1/2$. In this case a "gap" in the continuum results

Let $\Omega = \frac{1}{2} + \delta$, and for large θ, let $\phi(\theta) = \exp(i\frac{\theta}{2})\, \Phi^{+}(\theta) + e^{-i\frac{\theta}{2}}\, \Phi^{-}(\theta)$

where we assume $\frac{d\Phi}{\Phi\, d\theta} \ll 1$.

Then we ~~consider~~ substitute into our equation

$$\left(\frac{1}{4}+\delta\right)\left[\Phi^{+}e^{i\theta/2}+\Phi^{-}e^{-i\theta/2}\right] = -\frac{\partial^2}{\partial\theta^2}\phi - \epsilon\sin\theta\,\phi$$

$$= \frac{1}{4}\left[\Phi^{+}e^{i\theta/2}+\Phi^{-}e^{-i\theta/2}\right]$$

$$- i\left[\frac{\partial\Phi^{+}}{\partial\theta}(\theta)e^{i\theta/2} - \frac{\partial\Phi^{-}}{\partial\theta}e^{-i\frac{\theta}{2}}\right]$$

$$- \frac{\epsilon}{2i}\left[\Phi^{-}e^{i\theta/2} - \Phi^{+}(\theta)e^{-i\theta/2}\right]$$

$+$ higher order terms and ~~terms~~ terms $\propto e^{\pm 3i\frac{\theta}{2}}$

collecting the coefficients of $e^{i\theta/2}$ are $e^{-i\theta/2}$

$$e^{i\theta/2}\left[\left(\delta + i\frac{\partial}{\partial\theta}\right)\Phi^{+}(\theta) + i\frac{\epsilon}{2}\Phi^{-}(\theta)\right] = 0$$

$$e^{-i\theta/2}\left[\left(\delta - i\frac{\partial}{\partial\theta}\right)\Phi^{-}(\theta) + i\frac{\epsilon}{2}\Phi^{+}(\theta)\right] = 0$$

Coupled Equations

$$\left(\delta + i\frac{\partial}{\partial\theta}\right)\Phi^+ \quad - \frac{i\epsilon}{2}\Phi^- = 0$$

$$+\frac{i\epsilon}{2}\Phi^+ \quad \left(\delta - i\frac{\partial}{\partial\theta}\right)\Phi^- = 0$$

Let $\Phi^\pm = \hat{\Phi}^\pm e^{\pm\Lambda\theta}$

$$\begin{vmatrix} (\delta \pm i\Lambda) & -\frac{i\epsilon}{2} \\ +\frac{i\epsilon}{2} & (\delta \mp i\Lambda) \end{vmatrix} = 0$$

$$\delta^2 - \Lambda^2 - \epsilon^2/4 = 0$$

$$\Lambda^2 = \delta^2 - \epsilon^2/4$$

$$\Lambda = \pm\left[\delta^2 - \epsilon^2/4\right]^{1/2}$$

For large θ, localized mode fall off as $e^{-|\Lambda|\theta}$ with $\Lambda = [\delta^2 - \epsilon^2/4]^{1/2}$

Thus, normalized frequencies

$$\Omega = \frac{1}{2} \pm \frac{\epsilon}{2}$$

$$\frac{1-\epsilon}{2} < \Omega < \frac{1+\epsilon}{2}$$

do not propagate away, but evanesce.

This is ~~the~~ the Toroidal Alfven Wave Gap range of frequencies

Eigenmodes (TAE's) may arise in these gaps,
They do not radiate away, but evanesce, and give real eigen modes.

These eigenmodes can be excited by energetic particles (like alphas) and are a concern for confinement of alphas

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