Lecture # 12

MHD Energy Principle

Linearization of MHD equilibria

$$\rho \frac{d^2 \underset{\sim}{\xi}}{dt^2} = -\underset{\sim}{\nabla} p_1 + \underset{\sim}{j_1} \times \underset{\sim}{B_0} + \underset{\sim}{j_0} \times \underset{\sim}{B_1} \equiv \underset{\sim}{F}(\xi)$$

$$\underset{\sim}{B_1} = \underset{\sim}{\nabla} \times (\underset{\sim}{\xi} \times B_0) \equiv \underset{\sim}{Q}$$

$$p_1 = -\xi \cdot \nabla p_0 - \gamma p_0 \underset{\sim}{\nabla} \cdot \underset{\sim}{\xi}$$

$$j_1 = \underset{\sim}{\nabla} \times \underset{\sim}{B_1}$$

If we are looking for a normal mode

$$\xi = e^{-i\omega t} \xi(\underset{\sim}{r})$$

$$-\omega^2 \rho \underset{\sim}{\xi} = \underset{\sim}{F}(\underset{\sim}{\xi})$$

It turns out that $F(\xi)$ is a self-adjoint linear operator (as we will exhibit shortly)

If we multiply a (take inner product) by an adjoint function $\underset{\sim}{\eta}$

$$-\omega^2 \rho \underset{\sim}{\xi} \cdot \underset{\sim}{\eta} = \underset{\sim}{\eta} \cdot \underset{\sim}{F}(\underset{\sim}{\xi})$$

Integrate over all space (assume appropriate boundary condition regularity)

$$\omega^2 = -\frac{\int d^3r\, \underset{\sim}{\eta} \cdot \underset{\sim}{F}(\xi)}{\int d^3r\, \rho\, \underset{\sim}{\eta} \cdot \underset{\sim}{\xi}} \equiv -\frac{\int d^3r\, \underset{\sim}{\eta} \cdot \underset{\approx}{F} \cdot \underset{\sim}{\xi}}{\int d^3r\, \rho\, \underset{\sim}{\eta} \cdot \underset{\sim}{\xi}}$$

It will turn out that

$$\int d^3r\, \underset{\sim}{\eta} \cdot \underset{\approx}{F} \cdot \underset{\sim}{\xi} = \int d^3r\, \underset{\sim}{\xi} \cdot \underset{\sim}{F} \cdot \underset{\sim}{\eta}$$

where $\underset{\sim}{F}$ is real. This is the condition for self-adjointness, which guarantees the an eigenvalue ω^2, is real.

If $\omega^2 > 0$, modes oscillate in ~~space~~ time, and the mode is stable

If $\omega^2 < 0$, modes grow as $e^{\gamma t}$, where $\gamma^2 = -\omega^2$. These are unstable modes, and cause a great deal of difficulty for confinement.

When we have self-adjointness, there is a very physical interpretation to the terms.

$\frac{\omega^2}{2}\int \rho \; \underset{\sim}{\xi} \cdot \underset{\sim}{\xi}^* \, d^3r$ $\equiv$ kinetic energy of a mode $\equiv KE$

$-\frac{1}{2}\int d^3r \; \underset{\sim}{\xi}^* \cdot \underset{\approx}{F} \cdot \underset{\sim}{\xi}$ $\equiv$ potential energy of a mode $\equiv PE$

$KE + PE \equiv$ constant (as in any conservative system)

However if $\omega^2 > 0$, energy must be imputted into the system to have an excitation

But if $\omega^2 < 0$, no extra energy is needed, but instead system continuously deforms, growing larger in time, until either catastrophe, or limited excursion with nonlinear stabilization arising

If one can show that

$$\delta W = \frac{1}{2}\int d^3x\, \underset{\sim}{\xi}^* \cdot \underset{\approx}{F} \cdot \underset{\sim}{\xi} > 0$$

for any allowable perturbation (e.g. compatible with the ideal Ohm's law) then the system is stable

Indeed as in QM, by finding the minimum $\xi(\underset{\sim}{r})$ that produces the potential energy, we obtain ~~the~~ the eigenfunction of the system. However, ~~from~~ stability considerations, showing $\delta W > 0$, is frequently enough (but not all the time)

The first exercise we are confronted with is to show that $\underset{\approx}{F}$ is self-adjoint.

In general it is not a pretty calculation as it involves considerable algebra. Here we limit outselves to a plasma in contact with a conducting wall, so that $\underset{\sim}{\xi} \cdot \underset{\sim}{\hat{n}} = 0$ ($\underset{\sim}{\hat{n}} \equiv$ normal to the wall)

From Freidberg
"Ideal Magnetohydrodynamics"

$F(\underset{\sim}{\xi}) = -\underset{\sim}{\nabla} P_1 + \underset{\sim}{J}_0 \times \underset{\sim}{B}_1 + \underset{\sim}{J}_1 \times B_0$

$\boxed{P_1 = -\underset{\sim}{\xi} \cdot \nabla p - \gamma \underset{\sim}{\nabla} \cdot \xi}$; $\underset{\sim}{B}_1 \equiv \underset{\sim}{Q} = \underset{\sim}{\nabla} \times (\underset{\sim}{\xi} \times \underset{\sim}{B})$

APPENDIX A. SELF-ADJOINTNESS OF THE FORCE OPERATOR F

The goal of Appendix A is to show that the force operator $\mathbf{F}$ is self-adjoint; that is

$$\int \boldsymbol{\eta} \cdot \mathbf{F}(\boldsymbol{\xi})\, d\mathbf{r} = \int \boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\eta})\, d\mathbf{r} \tag{A.1}$$

where $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are two arbitrary vectors satisfying the boundary condition $\mathbf{n} \cdot \boldsymbol{\xi} = \mathbf{n} \cdot \boldsymbol{\eta} = 0$ on the surface. This corresponds to the perfectly conducting wall boundary condition.

The integrand can be written as

$$\boldsymbol{\eta} \cdot \mathbf{F}(\boldsymbol{\xi}) = \boldsymbol{\eta} \cdot \left[\frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{Q} + \frac{1}{\mu_0}(\nabla \times \mathbf{Q}) \times \mathbf{B} + \nabla(\boldsymbol{\xi} \cdot \nabla p + \gamma p \nabla \cdot \boldsymbol{\xi})\right] \tag{A.2}$$

with $\mathbf{Q} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$. The last term is integrated by parts yielding

$$\boldsymbol{\eta} \cdot \mathbf{F}(\boldsymbol{\xi}) = \boldsymbol{\eta} \cdot \left[\frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{Q} + \frac{1}{\mu_0}(\nabla \times \mathbf{Q}) \times \mathbf{B} + \nabla(\boldsymbol{\xi} \cdot \nabla p)\right] - \gamma p(\nabla \cdot \boldsymbol{\xi})(\nabla \cdot \boldsymbol{\eta}) \checkmark \tag{A.3}$$

One now writes $\boldsymbol{\xi} = \boldsymbol{\xi}_\perp + \xi_\parallel \mathbf{b}$, $\boldsymbol{\eta} = \boldsymbol{\eta}_\perp + \eta_\parallel \mathbf{b}$. The term in the square bracket in Eq. (A.3) has no parallel component; specifically,

a. $$\frac{1}{\mu_0}\mathbf{B} \cdot (\nabla \times \mathbf{B}) \times \mathbf{Q} = -\mathbf{Q} \cdot \mathbf{J} \times \mathbf{B} = -\mathbf{Q} \cdot \nabla p = \nabla \cdot [\nabla p \times (\boldsymbol{\xi} \times \mathbf{B})] = -\nabla \cdot [(\boldsymbol{\xi} \cdot \nabla p)\mathbf{B}] \tag{A.4}$$

└ using $B \cdot \nabla p = 0$

$Q \cdot \nabla P = (\underset{\sim}{\nabla} \times (\underset{\sim}{\xi}_\perp \times \underset{\sim}{B})) \cdot \nabla P = -\underset{\sim}{\nabla} \cdot (\underset{\sim}{\nabla} P \times (\underset{\sim}{\xi}_\perp \times B))$

b. $$\mathbf{B} \cdot \nabla(\boldsymbol{\xi} \cdot \nabla p) = \nabla \cdot [(\boldsymbol{\xi} \cdot \nabla p)\mathbf{B}]$$

Clearly, the parallel component cancels. Consequently one finds

$$\boldsymbol{\eta} \cdot \mathbf{F}(\boldsymbol{\xi}) = -\gamma p(\nabla \cdot \boldsymbol{\xi})(\nabla \cdot \boldsymbol{\eta}) + I$$

where I is a function only of the perpendicular components of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$:

$$I(\boldsymbol{\xi}_\perp, \boldsymbol{\eta}_\perp) = \boldsymbol{\eta}_\perp \cdot \left[\frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{Q} + \frac{1}{\mu_0}(\nabla \times \mathbf{Q}) \times \mathbf{B} + \nabla(\boldsymbol{\xi}_\perp \cdot \nabla p)\right] \tag{A.5}$$

$$I = \eta_\perp \cdot \left[(\nabla \times B) \times Q + (\nabla \times Q) \times B + \nabla(\xi_\perp \cdot \nabla p) \right]$$

The last term is now integrated by parts and the first two terms rewritten using standard vector identities:

$$I = \frac{1}{\mu_0}\boldsymbol{\eta}_\perp \cdot [\mathbf{Q}\cdot\nabla\mathbf{B} + \mathbf{B}\cdot\nabla\mathbf{Q} - \nabla(\mathbf{B}\cdot\mathbf{Q})] - (\boldsymbol{\xi}_\perp\cdot\nabla p)\nabla\cdot\boldsymbol{\eta}_\perp \quad \text{(A.6)}$$

The three terms inside the square brackets in Eq. (A.6) are expanded as follows:

$$\begin{aligned}
\boldsymbol{\eta}_\perp\cdot(\mathbf{Q}\cdot\nabla\mathbf{B}) &= \boldsymbol{\eta}_\perp\cdot[\overset{a}{(\mathbf{B}\cdot\nabla\boldsymbol{\xi}_\perp)\cdot\nabla\mathbf{B}} - \overset{b}{(\boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B})\cdot\nabla\mathbf{B}}] - B^2(\boldsymbol{\eta}_\perp\cdot\boldsymbol{\kappa})\nabla\cdot\boldsymbol{\xi}_\perp \\
\boldsymbol{\eta}_\perp\cdot(\mathbf{B}\cdot\nabla\mathbf{Q}) &= \mathbf{B}\cdot\nabla(\boldsymbol{\eta}_\perp\cdot\mathbf{Q}) - \mathbf{Q}\cdot(\mathbf{B}\cdot\nabla\boldsymbol{\eta}_\perp) \\
&= \nabla\cdot[(\boldsymbol{\eta}_\perp\cdot\mathbf{Q})\mathbf{B}] - (\mathbf{B}\cdot\nabla\boldsymbol{\xi}_\perp - \boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B} \\
&\quad - \mathbf{B}\nabla\cdot\boldsymbol{\xi}_\perp)\cdot(\mathbf{B}\cdot\nabla\boldsymbol{\eta}_\perp) \qquad \text{(A.7)} \\
&= -(\mathbf{B}\cdot\nabla\boldsymbol{\xi}_\perp)\cdot(\mathbf{B}\cdot\nabla\boldsymbol{\eta}_\perp) + \underset{c}{(\boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B})\cdot(\mathbf{B}\cdot\nabla\boldsymbol{\eta}_\perp)} \\
&\quad - B^2(\boldsymbol{\eta}_\perp\cdot\boldsymbol{\kappa})\nabla\cdot\boldsymbol{\xi}_\perp \\
-\boldsymbol{\eta}_\perp\cdot\nabla(\mathbf{B}\cdot\mathbf{Q}) &= -\nabla\cdot[(\mathbf{B}\cdot\mathbf{Q})\boldsymbol{\eta}_\perp] + (\mathbf{B}\cdot\mathbf{Q})\nabla\cdot\boldsymbol{\eta}_\perp \\
&= -B^2(\nabla\cdot\boldsymbol{\xi}_\perp)(\nabla\cdot\boldsymbol{\eta}_\perp) - [\boldsymbol{\xi}_\perp\cdot\nabla B^2/2 + B^2(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa})]\nabla\cdot\boldsymbol{\eta}_\perp
\end{aligned}$$

In the second and third terms, full divergence contributions have been dropped since they integrate to zero. Combining terms one finds

$$\begin{aligned}
\boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi}) = &-\frac{B^2}{\mu_0}(\nabla\cdot\boldsymbol{\xi}_\perp)(\nabla\cdot\boldsymbol{\eta}_\perp) - \frac{1}{\mu_0}(\mathbf{B}\cdot\nabla\boldsymbol{\xi}_\perp)\cdot(\mathbf{B}\cdot\nabla\boldsymbol{\eta}_\perp) \\
&- \gamma p(\nabla\cdot\boldsymbol{\xi})(\nabla\cdot\boldsymbol{\eta}) \\
&- \left[\boldsymbol{\xi}_\perp\cdot\nabla\left(p + \frac{B^2}{2\mu_0}\right) + \frac{B^2}{\mu_0}\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}\right]\nabla\cdot\boldsymbol{\eta}_\perp \\
&- 2\frac{B^2}{\mu_0}(\boldsymbol{\eta}_\perp\cdot\boldsymbol{\kappa})\nabla\cdot\boldsymbol{\xi}_\perp \\
&+ R \qquad \text{(A.8)}
\end{aligned}$$

$$2B^2(\xi_\perp\cdot\kappa)\nabla\cdot\eta_\perp$$

where

$$\mu_0 R = \boldsymbol{\eta}_\perp\cdot[\underset{a}{(\mathbf{B}\cdot\nabla\boldsymbol{\xi}_\perp)\cdot\nabla\mathbf{B}} - \underset{b}{(\boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B})\cdot\nabla\mathbf{B}}] + \underset{c}{(\boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B})\cdot(\mathbf{B}\cdot\nabla\boldsymbol{\eta}_\perp)} \quad \text{(A.9)}$$

The middle line is simplified by noting that $\boldsymbol{\xi}_\perp\cdot\nabla(p + B^2/2\mu_0) = (B^2/\mu_0)(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa})$. The quantity R can be rewritten by using the two identities

$$\begin{aligned}
\nabla\cdot\{[\boldsymbol{\eta}_\perp\cdot(\boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B})]\mathbf{B}\} &= \underset{(c)}{(\mathbf{B}\cdot\nabla\boldsymbol{\eta}_\perp)\cdot(\boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B})} \\
&\quad + \boldsymbol{\eta}_\perp\cdot\underset{(a)}{(\mathbf{B}\cdot\nabla\boldsymbol{\xi}_\perp)}\cdot\nabla\mathbf{B} + \boldsymbol{\eta}_\perp\cdot(\mathbf{B}\boldsymbol{\xi}_\perp:\nabla\nabla)\mathbf{B} \\
\underset{(b)}{\boldsymbol{\eta}_\perp\cdot(\boldsymbol{\xi}_\perp\cdot\nabla)(\mathbf{B}\cdot\nabla\mathbf{B})} &= \underset{(d)}{\boldsymbol{\eta}_\perp\cdot(\boldsymbol{\xi}_\perp\cdot\nabla\mathbf{B})\cdot\nabla\mathbf{B}} + \boldsymbol{\eta}_\perp\cdot(\mathbf{B}\boldsymbol{\xi}_\perp:\nabla\nabla\mathbf{B}) \quad \text{(A.10)}
\end{aligned}$$

To within a dive

$R = -\frac{1}{\mu}$

The final result

$\int \boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi})\,d\mathbf{r}$

$$\begin{aligned}
Q &= \nabla\times(\xi_\perp\times B) \\
&= B\cdot\nabla\xi_\perp - \xi_\perp\cdot\nabla B \\
&\quad - B\nabla\cdot\xi_\perp
\end{aligned}$$

which is clearly

$$\boxed{\begin{aligned}
&B\nabla\cdot\xi\cdot(B\cdot\nabla\eta_\perp) \\
&= B^2\nabla\cdot\xi\,(b\cdot\nabla)(\eta_\perp\cdot b) \\
&\quad - B^2\nabla\cdot\xi\;\eta_\perp\cdot\kappa
\end{aligned}}$$

To within a divergence term which integrates to zero, R is given by

$$R = -\frac{1}{\mu_0}\boldsymbol{\eta}_\perp \cdot (\boldsymbol{\xi}_\perp \cdot \nabla)(\mathbf{B} \cdot \nabla \mathbf{B}) = -(\boldsymbol{\eta}_\perp \boldsymbol{\xi}_\perp : \nabla\nabla)\left(p + \frac{B^2}{2\mu_0}\right) \quad \text{(A.11)}$$

$\sim (d)$

The final result is

$$\int \boldsymbol{\eta} \cdot \mathbf{F}(\boldsymbol{\xi})\, d\mathbf{r} = -\int d\mathbf{r}\Big[\frac{1}{\mu_0}(\mathbf{B} \cdot \nabla \boldsymbol{\xi}_\perp) \cdot (\mathbf{B} \cdot \nabla \boldsymbol{\eta}_\perp) + \gamma p (\nabla \cdot \boldsymbol{\xi})(\nabla \cdot \boldsymbol{\eta})$$
$$+ \frac{B^2}{\mu_0}(\nabla \cdot \boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa})(\nabla \cdot \boldsymbol{\eta}_\perp + 2\boldsymbol{\eta}_\perp \cdot \boldsymbol{\kappa}) \quad \text{(A.12)}$$
$$- \frac{4B^2}{\mu_0}(\boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa})(\boldsymbol{\eta}_\perp \cdot \boldsymbol{\kappa}) + (\boldsymbol{\eta}_\perp \boldsymbol{\xi}_\perp : \nabla\nabla)\left(p + \frac{B^2}{2\mu_0}\right)\Big]$$

which is clearly a self-adjoint form by inspection.

$$(\underset{\sim}{B} \cdot \underset{\sim}{\nabla})\underset{\sim}{B} = (\underset{\sim}{B} \cdot \underset{\sim}{\nabla})(\underset{\sim}{b} B)$$
$$= \underset{\sim}{\kappa} B^2 + \underset{\sim}{b}(\underset{\sim}{b} \cdot \nabla)\frac{B^2}{2}$$
$$= \underset{\sim}{\nabla}\left(p + \frac{B^2}{2}\right)$$

$$\kappa B^2 = \nabla_\perp\left(p + \frac{B^2}{2}\right)$$
$$= \nabla\left(p + \frac{B^2}{2}\right)$$

(7)