Lecture #11

Vertical Field in a Tokamak

3.6 Large aspect-ratio

$$\nabla \cdot \frac{1}{R^2}\nabla\psi = -R^2 p'(\psi) - f(\psi)f'(\psi)$$

$\psi_0(R_0)$

Tokamak equilibria take a comparatively simple form for low-β, large aspect-ratio plasmas of circular cross-section. The ordering of quantities in terms of the inverse aspect-ratio, $\varepsilon = a/R$, is

$$B_\phi = B_{\phi 0}(R_0/R)(1+0(\varepsilon^2)) \qquad B_\theta \sim \varepsilon B_{\phi 0}$$
$$j_\phi \sim \varepsilon B_{\phi 0}/\mu_0 a \qquad j_\theta \sim \varepsilon^2 B_{\phi 0}/\mu_0 a$$
$$p \sim \varepsilon^2 B_{\phi 0}^2/\mu_0,\ (\beta \sim \varepsilon^2) \qquad \beta_p \sim 1$$

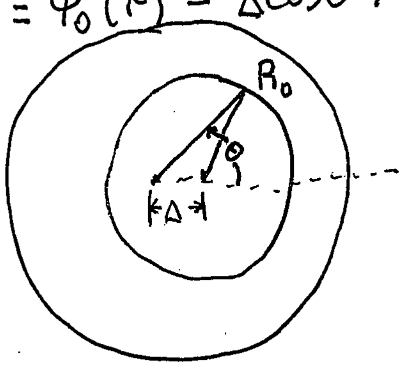
$$R_0^2 = (r\cos\theta - \Delta)^2 + r^2\sin^2\theta = r^2 - 2\Delta\cos\theta + \Delta^2$$
$$R_0 \simeq r - \Delta\cos\theta + \mathcal{O}(\Delta^2)$$
$$\psi = \psi_0(r - \Delta\cos\theta) = \psi_0(r) - \Delta\cos\theta\,\psi'(r)$$

where $B_{\phi 0}$ is the vacuum toroidal magnetic field at the major radius of the plasma R_0. The basic pressure balance equation is that of a cylinder,

$$\frac{dp}{dr} = j_\phi B_\theta - j_\theta B_{\phi 0}, \qquad 3.6.1$$

the equilibrium being specified by $j_\phi(r)$ and $p(r)$ with $p(a) = 0$. The azimuthal field is given by Ampère's equation

$$\mu_0 j_\phi = -\frac{1}{r}\frac{d}{dr}(rB_\theta)$$



and j_θ is then determined by eqn 3.6.1.

When toroidal effects are included, the flux surfaces form non-concentric circles as illustrated in Fig. 3.6.1(a). Using the co-ordinate system shown in Fig. 3.6.1(b) the Grad–Shafranov equilibrium eqn 3.3.9 may be written

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\psi - \frac{1}{R_0 + r\cos\theta}\left(\cos\theta\frac{\partial}{\partial r} - \sin\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right)\psi$$
$$= -\mu_0(R_0 + r\cos\theta)^2 p'(\psi) - \mu_0^2 f(\psi)f'(\psi). \qquad 3.6.2$$

Expanding ψ in ε,

$$\psi = \psi_0(r) + \psi_1(r,\theta).$$

$$p'(\psi_0+\psi_1) = p'(\psi_0) + \psi_1 p''(\psi_0)$$

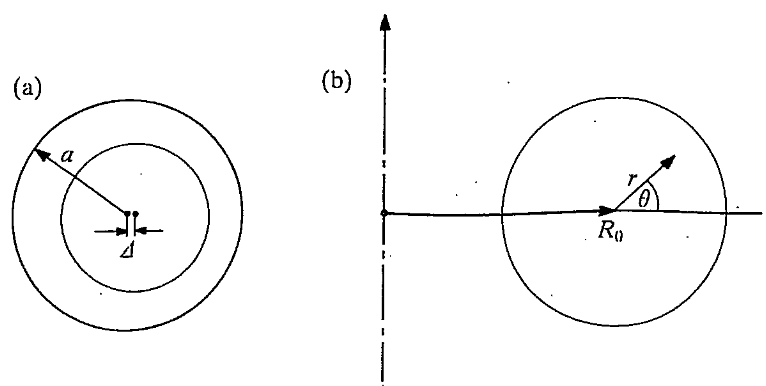


Fig. 3.6.1 (a) Showing circular flux surface displaced by a distance Δ with respect to outer flux surface whose centre is at a distance R_0 from the major axis. (b) Co-ordinate system (r, θ) with centre at major radius R_0.

$$f(\psi)f'(\psi) = f(\psi_0)f'(\psi_0) + \psi_1\left(f(\psi_0)f'(\psi_0)\right)'$$

ψ_0 is given by the leading order part of eqn 3.6.2, corresponding to eqn 3.6.1, that is

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\psi_0}{\mathrm{d}r}\right)=-\mu_0R_0^2p'(\psi_0)-\mu_0^2f(\psi_0)f'(\psi_0) \qquad 3.6.3$$

and ψ_1 satisfies the first-order part of eqn 3.6.2

$$\begin{aligned}\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\psi_1-\frac{\cos\theta}{R_0}\frac{\mathrm{d}\psi_0}{\mathrm{d}r}
&=-\mu_0R_0^2p''(\psi_0)\psi_1-\mu_0^2(f(\psi_0)f'(\psi_0))'\psi_1\\
&\qquad-2\mu_0R_0r\cos\theta p'(\psi_0)\\
&=-\frac{\mathrm{d}}{\mathrm{d}r}(\mu_0R_0^2p'(\psi_0)+\mu_0^2f(\psi_0)f'(\psi_0))\frac{\mathrm{d}r}{\mathrm{d}\psi_0}\psi_1\\
&\qquad-2\mu_0R_0r\cos\theta p'(\psi_0).\end{aligned} \qquad 3.6.4$$

$$A'(\psi_0)=\frac{dA}{dr}\frac{dr}{d\psi_0}=\frac{dA}{dr}\frac{1}{B_{\theta_0}}$$

If the flux surface ψ is displaced a distance $\Delta(\psi_0(r))$, ψ may be written

$$\begin{aligned}\psi&=\psi_0+\psi_1\\
&=\psi_0-\Delta(r)\frac{\partial\psi_0}{\partial R}\\
&=\psi_0-\Delta(r)\cos\theta\frac{\mathrm{d}\psi_0}{\mathrm{d}r}.\end{aligned} \qquad 3.6.5$$

Substituting the form for ψ_1 given by eqn 3.6.5 into eqn 3.6.4 leads to

$$\begin{aligned}&-\Delta\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\psi_0}{\mathrm{d}r}\right)\right)-\frac{1}{r}\left(\frac{\mathrm{d}r}{\mathrm{d}\psi_0}\right)\frac{\mathrm{d}}{\mathrm{d}r}\left(r\left(\frac{\mathrm{d}\psi_0}{\mathrm{d}r}\right)^2\frac{\mathrm{d}\Delta}{\mathrm{d}r}\right)-\frac{1}{R_0}\frac{\mathrm{d}\psi_0}{\mathrm{d}r}\\
&=\Delta\frac{\mathrm{d}}{\mathrm{d}r}(\mu_0R_0^2p'(\psi_0)+\mu_0^2f(\psi_0)f'(\psi_0))-2\mu_0R_0r\frac{\mathrm{d}p_0}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}\psi_0}\end{aligned} \qquad 3.6.6$$

and, from eqn 3.6.3, the first terms on the two sides of eqn 3.6.6 cancel, leaving

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(rB_{\theta0}^2\frac{\mathrm{d}\Delta}{\mathrm{d}r}\right)=\frac{r}{R_0}\left(2\mu_0r\frac{\mathrm{d}p_0}{\mathrm{d}r}-B_{\theta0}^2\right) \qquad 3.6.7$$

where the definition of the flux function given by eqn 3.2.2 has been used to replace $(\mathrm{d}\psi_0/\mathrm{d}r)/R_0$ by $B_{\theta0}$.

The solution of the differential equation 3.6.7 with $\mathrm{d}\Delta/\mathrm{d}r=0$ at $r=0$ and $\Delta(a)=0$ gives the displacement $\Delta(r)$ of the flux surfaces for a zero-order pressure $p_0(r)$ and azimuthal magnetic field $B_{\theta0}(r)$. Together with eqn 3.6.5 this then provides the solution $\psi(r,\theta)$.

Vertical Field

(shift the magnetic axis)

$$B_\theta = \frac{1}{R}\frac{\partial \psi}{\partial r} = \frac{1}{R_0 + a\cos\theta}\frac{\partial}{\partial r}\left(\psi_0 - \Delta(r)\frac{d\psi_0}{dr}\cos\theta\right)$$

use $\Delta(a) = 0$

$$\frac{d}{dr}\left(rB_{\theta 0}^2 \frac{d\Delta}{dr}\right) = \frac{r}{R_0}\left(2r\frac{dp_0}{dr} - B_{\theta 0}^2\right)$$

The calculation described in Section 3.6 leads to an approximate solution for the plasma equilibrium. This solution gives the variation of the poloidal magnetic field around the plasma surface at $r = a$. This in turn determines the vacuum magnetic field and prescribes the externally produced field necessary to maintain the equilibrium.

It is first necessary to determine $B_\theta(a)$ from the plasma equilibrium using the large aspect-ratio expansion introduced in Section 3.6. B_θ is given by

$$B_\theta = \frac{1}{R}\frac{\partial \psi}{\partial r} = \frac{1}{R_0 + r\cos\theta}\frac{\partial \psi}{\partial r} \qquad 3.7.1$$

with

$$\psi = \psi_0 - \Delta(r)\frac{\mathrm{d}\psi_0}{\mathrm{d}r}\cos\theta \qquad 3.7.2$$

and $\Delta(r)$ determined by the solution of eqn 3.6.7. Thus, using $\Delta(a) = 0$, eqns 3.7.1 and 3.7.2 give the poloidal field at $r = a$ as

$$B_\theta(a) = B_{\theta 0}(a)\left[1 - \left(\frac{a}{R_0} + \left(\frac{\mathrm{d}\Delta}{\mathrm{d}r}\right)_a\right)\cos\theta\right]. \qquad 3.7.3$$

The quantity $\mathrm{d}\Delta/\mathrm{d}r$ must now be calculated. Carrying out an integration of eqn 3.6.7 and integrating by parts on the right-hand side leads to

$$\frac{\mathrm{d}\Delta}{\mathrm{d}r} = \frac{2\mu_0}{rR_0B_{\theta 0}^2}\left\{r^2p_0 - \int_0^r\left(2p_0 + \frac{B_{\theta 0}^2}{2\mu_0}\right)r\,\mathrm{d}r\right\}. \qquad 3.7.4$$

Now, defining the poloidal beta and the internal inductance of the plasma by

$$\beta_\mathrm{p} = \frac{\int_0^a p_0 2\pi r\,\mathrm{d}r}{(B_{\theta 0}^2(a)/2\mu_0)\pi a^2}, \qquad l_i = \frac{\int_0^a (B_{\theta 0}^2/2\mu_0)2\pi r\,\mathrm{d}r}{(B_{\theta 0}^2(a)/2\mu_0)\pi a^2},$$

and taking $p_0(a) = 0$, eqn 3.7.4 gives

$$\left(\frac{\mathrm{d}\Delta}{\mathrm{d}r}\right)_a = -\frac{a}{R_0}\left(\beta_\mathrm{p} + \frac{l_\mathrm{i}}{2}\right). \qquad 3.7.5$$

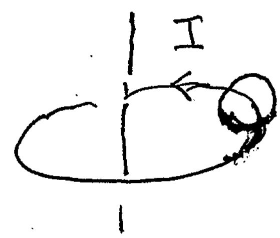
Substitution of eqn 3.7.5 into eqn 3.7.3 then leads to

$$B_\theta(a) = B_{\theta 0}(a)\left(1 + \frac{a}{R_0}\Lambda\cos\theta\right) \qquad 3.7.6$$

where

$$\Lambda = \beta_\mathrm{p} + \frac{l_\mathrm{i}}{2} - 1.$$

The vacuum magnetic field must now be matched to this solution for $B_\theta(a)$. The vacuum field is given by the solution of the equation $(\nabla \times \mathbf{B})_\phi = 0$. Using eqns 3.2.2, and the present co-ordinates, $(\nabla \times \mathbf{B})_\phi$ takes the form of the left hand side of eqn 3.6.2. In the large aspect-ratio approximation the solution for $r \ll R_0$ is



$$\psi = \underbrace{\frac{\mu_0 I}{2\pi} R_0\left(\ln\frac{8R_0}{r} - 2\right) + \frac{\mu_0 I}{4\pi}\left(r\left(\ln\frac{8R_0}{r} - 1\right)\right.}_{\text{solution for a ring of current}} \left. + \frac{c_1}{r} + c_2 r\right)\cos\theta, \qquad 3.7.7$$

where $I\ (= -2\pi a B_{\theta 0}(a)/\mu_0)$ is the plasma current. The values of the constants c_1 and c_2 are determined from the requirements that $B_\theta(a)$ matches the plasma solution given by eqn 3.7.6 and that $B_r(a) = 0$.

Substitution of eqn 3.7.7 into the expanded form of eqn 3.7.1 provides the vacuum solution for $B_\theta(a)$. Matching this to eqn 3.7.6 then gives

$$\frac{1}{a^2}c_1 - c_2 = \ln\frac{8R_0}{a} + 2\Lambda. \qquad 3.7.8$$

Since $B_r = -(1/R_0 r)\,\partial\psi/\partial\theta$, the requirement $B_r(a) = 0$ implies that the coefficient of $\cos\theta$ in eqn 3.7.7 is zero at $r = a$. Thus using eqn 3.7.8 the values of c_1 and c_2 are found to be

$$c_1 = a^2(\Lambda + \tfrac{1}{2}), \qquad c_2 = -\left(\ln\frac{8R_0}{a} + \Lambda - \tfrac{1}{2}\right),$$

and substitution into eqn 3.7.7 gives

$$\psi = \frac{\mu_0 I}{2\pi} R_0\left(\ln\frac{8R_0}{r} - 2\right) - \frac{\mu_0 I}{4\pi} r\left(\ln\frac{r}{a} + (\Lambda + \tfrac{1}{2})\left(1 - \frac{a^2}{r^2}\right)\right)\cos\theta.$$

The factor $(\ln(8R_0/r) - 1)$ in eqn 3.7.7 is an approximation, valid for $r \ll R_0$, to a function which is zero for $r \to \infty$. Thus at large r, ψ takes the form $(\mu_0 I/4\pi)c_2 r\cos\theta$ corresponding to a vertical field $(\mu_0 I/4\pi)c_2/R_0$, that is

$$B_v = -\frac{\mu_0 I}{4\pi R_0}\left(\ln\frac{8R_0}{a} + \Lambda - \tfrac{1}{2}\right).$$

This is the vertical magnetic field necessary to maintain the plasma in equilibrium, its effect being to provide an inward force to balance the outward hoop force of the plasma current.

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla P + \mathbf{j} \times \mathbf{B}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mathbf{j}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0$$

$$\mathbf{v} = \frac{d\boldsymbol{\xi}}{dt}$$

$$-\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E} = -\nabla \times (\mathbf{v} \times \mathbf{B})$$

$$\therefore \; +\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\frac{d\boldsymbol{\xi}}{dt} \times \mathbf{B} \right)$$

If we have an equilibrium without a flow $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times \left(\frac{\partial \boldsymbol{\xi}}{\partial t} \times \mathbf{B} \right)$$

$$\mathbf{B}_1 = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0) = (\mathbf{B}_0 \cdot \nabla)\boldsymbol{\xi} - \mathbf{B}_0 \nabla \cdot \boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \nabla)\mathbf{B}_0$$

$$\rho \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = -\nabla P + \mathbf{j} \times \mathbf{B}$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \left(\rho \frac{d\boldsymbol{\xi}}{dt} \right) ; \quad \rho_1 = -\boldsymbol{\xi} \cdot \nabla \rho_0 - \rho_0 \nabla \cdot \boldsymbol{\xi}$$

$$\frac{d}{dt}\left(P/\rho^{\gamma}\right) = 0 \qquad \text{ideal gas law.}$$

(in Lagrangian Frame, i.e moving with fluid)

$$\frac{1}{\rho^{\gamma}}\frac{dP}{dt} - \frac{\gamma P}{\rho^{\gamma+1}}\frac{d\rho}{dt} = 0$$

$$\frac{dP}{dt} = \frac{\gamma P}{\rho}\frac{d\rho}{dt} = -\gamma P \, \mathrm{div} \cdot \underset{\sim}{v} = -\gamma P \, \mathrm{div} \cdot \frac{d\underset{\sim}{\xi}}{dt}$$

Now linearizing,

$$P = P_0(\underset{\sim}{r} - \underset{\sim}{\xi}) - \gamma P_0 \, \mathrm{div} \cdot \underset{\sim}{\xi}$$

$$\parallel$$

$$P_0 + P_1 = P_0(r) - \underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} P_0 - \gamma P_0 \underset{\sim}{\nabla} \cdot \underset{\sim}{\xi}$$

$$P_1 = -\underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} P_0 - \gamma P_0 \underset{\sim}{\nabla} \cdot \underset{\sim}{\xi}$$