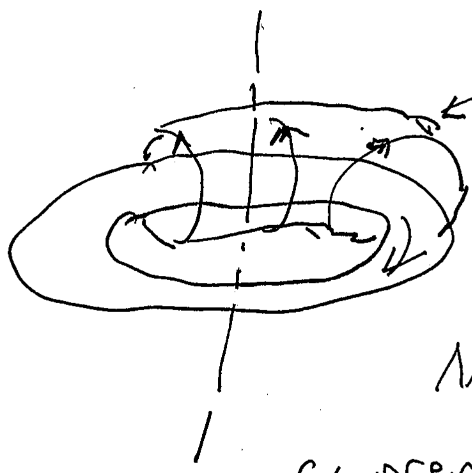
Lecture #10

Toroidal Grad-Shafranov Equation

3-D confinement, with symmetry



← external coils for toroidal field

Note that a toroidal current exhibits a "hoop" stress

$\circ \quad |\overset{R}{—} \ast$ opposite currents repel

$$\text{Force} \simeq I B L \quad , \quad L = 2\pi R$$

$$B \sim I/R$$

$$\text{Force} \simeq I^2$$ cause expansion

A vertical field, B_V, with a counter acting force needs to be supplied:

$$\underset{\sim}{B}_V \simeq \frac{a}{R} B_p \simeq \frac{I}{R}$$

We represent the magnetic field as:

$$\underset{\sim}{B} = \underset{\sim}{\nabla} \rho \times \underset{\sim}{\nabla} \psi - I \underset{\sim}{\nabla} \rho$$

(mixed representation

Poloidal Field $\equiv$ contra variant vector

Toroidal Field $\equiv$ co-variant vector

Pressure $\equiv$ only a function of ~~of~~ ψ

$$\underset{\sim}{\nabla} P = \underset{\sim}{J} \times \underset{\sim}{B}$$

$$\therefore \quad \underset{\sim}{B} \cdot \underset{\sim}{\nabla} P = 0$$

As we move along field line, which generates a nested flux tube, pressure cannot vary. Hence

$P = P(\psi)$, $\quad \underset{\sim}{\nabla} P = \underset{\sim}{\nabla} \psi \dfrac{\partial P}{\partial \psi}$

From $\underset{\sim}{\nabla} \cdot \underset{\sim}{J} = 0$, and toroidal symmetry, and $\underset{\sim}{J} \cdot \underset{\sim}{\nabla} \psi = 0$ (as $\nabla \psi \frac{\partial p}{\partial \psi} = J \times \underset{\sim}{B}$) it follows that the poloidal current is parallel to the poloidal magnetic field.

$$\underset{\sim}{J_p} = \alpha \, \underset{\sim}{\nabla} \varphi \times \underset{\sim}{\nabla} \psi \quad \left(\text{only way } \underset{\sim}{J_p} \perp \text{ to both } \underset{\sim}{\nabla} \varphi \text{ and } \underset{\sim}{\nabla} \psi \right)$$

Further, from ampere's law we take $\int \underset{\sim}{B} \cdot d\underset{\sim}{l} = -\int \frac{I}{R} R \, d\varphi = -I 2\pi$ in toroidal symmetry direction, and it must be equal to the total poloidal current, $\mathcal{I}(\psi)$ through the ribbon area ~~shown~~ generated by the element

$$d\underset{\sim}{A} = \frac{\underset{\sim}{\nabla} \theta \, d\varphi \, d\psi}{\mathcal{J}} \Rightarrow 2\pi \frac{\underset{\sim}{\nabla} \theta \, d\psi}{\mathcal{J}}$$

$$\therefore \; -2\pi I = \mathcal{I}(\psi) = \int \underset{\sim}{J_p} \cdot d\underset{\sim}{A} = 2\pi \int \alpha \frac{\underset{\sim}{\nabla} \theta \cdot \nabla \varphi \times \nabla \psi}{\mathcal{J}} \, d\psi = 2\pi \int_0^{\psi} \alpha \, d\psi'$$

$$\therefore \; \alpha = \frac{-\partial I(\psi)}{\partial \psi}$$

The current $I(\psi)$ includes the poloidal current in the plasma as well as the poloidal current in the magnets forming the vacuum toroidal field.

Now let us consider force balance

$$\nabla\psi \frac{\partial p}{\partial \psi} = (J_p + J_T) \times (B_T + B_p)$$

$$B_p = \nabla\phi \times \nabla\psi \quad ; \qquad B_T = -I(\psi)\nabla\phi$$

$$J_p = \frac{\partial I}{\partial \psi} \nabla\phi \times \nabla\psi \quad ; \qquad J_T = \nabla\phi\, J\cdot\nabla\phi \; R^2$$

Now $\nabla\phi \cdot J = \nabla\phi \cdot \nabla \times (\nabla\phi \times \nabla\psi)$ (using $\nabla\times\nabla\phi = 0$)

$$= -\nabla\cdot\left(\nabla\phi \times (\nabla\phi \times \nabla\psi)\right)$$

$$= -\nabla\cdot\Big[\nabla\phi \underbrace{\nabla\phi\cdot\nabla\psi}_{0} - \underbrace{\nabla\phi\cdot\nabla\phi}_{1/R^2} \nabla\psi$$

$$= \nabla\cdot\left(\frac{\nabla\psi}{R^2}\right)$$

$\therefore$

$$\nabla\psi \frac{\partial p}{\partial \psi} = J_p \times B_T + J_T \times B_p$$

$$= -\frac{\partial I}{\partial \psi}(\nabla\phi \times \nabla\psi) \times I(\psi)\nabla\phi + \nabla\phi \times (\nabla\phi \times \nabla\psi)\, \nabla\cdot\left(\frac{\nabla\psi}{R^2}\right)$$

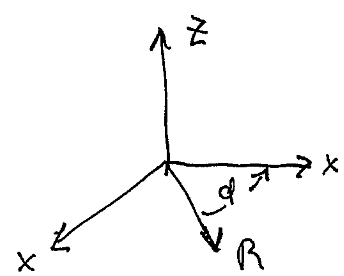
$$= -\frac{\partial I}{\partial \psi} I(\psi) \frac{\nabla\psi}{R^2} - \frac{\nabla\psi}{R^2} \nabla\cdot\left(\frac{\nabla\psi}{R^2}\right)$$

Thus Equating both sides gives the Grad - Shafranov Equation

$$\boxed{\nabla \cdot \left(\frac{\nabla \psi}{R^2} \right) = R^2 \frac{\partial p}{\partial \psi} + I(\psi) \frac{\partial I(\psi)}{\partial \psi}}$$

An example of a solution is the Solov'ev solution, when $I(\psi) = 0$. In this case there is no external magnets supplying toroidal fields. He also chooses $\frac{\partial p}{\partial \psi} = \frac{\partial p}{\partial \psi}\Big|_0 \equiv$ constant

In cylindrical geometry



(R, ϕ, Z)

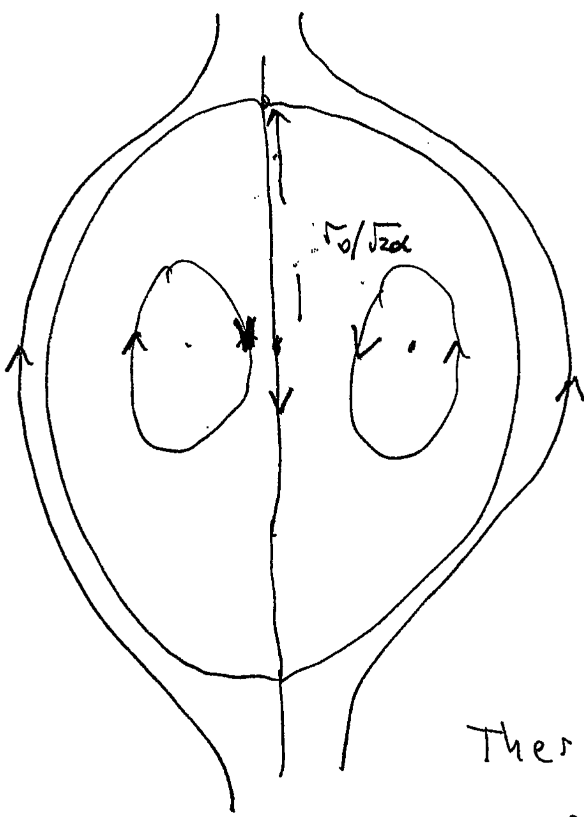
the G-S equation takes the form

$$R \frac{\partial}{\partial R} \frac{1}{R} \frac{\partial \psi}{\partial R} + \frac{\partial^2 \psi}{\partial Z^2} = - \frac{\partial p}{\partial \psi}\Big|_0$$

~~The~~ A solution is of the form:

$$\psi = \psi_0 \frac{r^2}{r_0^4}\left(2r_0^2 - r^2 - 4\alpha^2 z^2\right)$$

(determine how these parameters depend on $\left.\frac{\partial p}{\partial \psi}\right|_0$)



This solution describes plasma on closed field lines ($\psi > 0$) and open field lines ($\psi < 0$)

There ~~is~~ are magnetic nulls at $r=0$, $z = \pm r_0/\alpha\sqrt{2}$. Also at O-point, ($z=0$, $r=r_0$)

Typically we are interested in case where $p=0$ on separatrix ($\psi=0$ in this case) and $p=0$ on open field lines

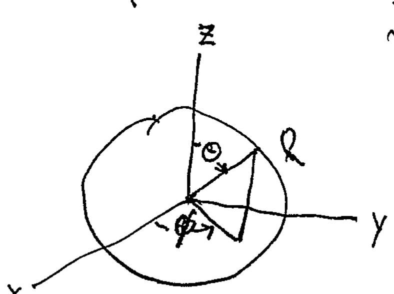
For this case the solution of the problem is incomplete, in that the required vertical needs to be determined. In general this is a hard problem.

The one straight-forward case is when the separatrix is a sphere ($\alpha = 1/2$). The a matching analytic solution can be found. It is easiest to show in spherical coordinates ($\rho^2 = r^2 + z^2$) where G-S equation is

$$\frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho^2 \sin\theta}\frac{\partial}{\partial \theta}\sin\theta\frac{\partial \psi}{\partial \theta} = -\rho^2 \sin^2\theta \frac{\partial P}{\partial \psi} - I(\psi)\frac{\partial I(\psi)}{\partial \psi}$$

If rhs vanishes;

$$\psi = \left(\frac{A}{\rho} + B\rho^2\right)\cos\theta$$



for $\rho > \sqrt{2}\, r_0$

This solution can be matched to the interior solution, (an exercise you need to do for your homework). Note $B\rho^2\cos\theta$ is the flux function for a uniform vertical field.