# Homework #2
# Solutions

Solution HW#1

(1) Solev'ev equilibrium (spherical boundary)

$$\psi = \psi_0 \frac{r^2}{r_0^4}\left(2r_0^2 - r^2 - z^2\right)$$

Solves equation

$$\nabla \cdot \frac{1}{r^2}\nabla\psi = - \frac{\partial p(\psi)}{\partial \psi} = \text{Const}; \quad \psi < \psi_0$$

$$\parallel$$

$$\frac{1}{r^2}\frac{\partial^2\psi}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}\psi = -\frac{\partial p}{\partial \psi}$$

$$\frac{1}{r}\frac{\partial \psi}{\partial r} = \psi_0\left[\frac{2}{r_0^4}\left(2r_0^2 - r^2 - z^2\right) - \frac{2r^2}{r_0^4}\right] +$$

$$-8\psi_0/r_0^4 \quad -2\psi_0/r_0^4 \quad = \psi_0\left[\frac{4r_0^2 - 4r^2 - 2z^2}{r_0^4}\right]$$

$$\frac{\partial^2\psi}{\partial z^2}\frac{1}{r^2} + \frac{\partial}{r\,\partial r}\frac{1}{r}\frac{\partial\psi}{\partial r} = -\frac{10\psi_0}{r_0^4} = -\frac{\partial p}{\partial \psi}$$

$$\therefore \quad \psi_0 = \frac{1}{10}\frac{\partial p}{\partial \psi} r_0^4, \quad \Delta p = 10\psi_0^2/r_0^4 = P(|\psi = \psi_{max}|)$$

(Note $\psi_0 < 0$) to have $P(\psi = |\psi_{max}|) > 0$ 
$P(\psi = 0) = 0$

~~$A\psi$~~

Now to match:

In spherical coordinates GS equation

$$\frac{\partial^2\psi}{\partial \rho^2} + \frac{\sin\theta}{\rho^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\psi = 0 \quad (\text{in vaccuum}) \quad (1)$$

$$\rho^2 = r^2 + z^2; \quad \rho\sin\theta = r$$

Solution in vacuum is

$$\psi = \sin^2\theta\left(A\rho^2 + \frac{B}{\rho}\right) \qquad \rho \geq \sqrt{2}r_0$$

~~(check this)~~

Solution from ~~vacuum~~ plasma at interface with vacuum is

$$\psi = \frac{\psi_0 r^2}{r_0^4}\left[2r_0^2 - \rho^2\right]$$

$$= \psi_0 \frac{\rho^2}{r_0^4}\sin^2\theta\left[2r_0^2 - \rho^2\right]$$

~~We n.~~ At $\rho = \sqrt{2}r_0$ $\psi(\rho,\theta) = 0$ at plasma. Equate $\psi$ from vacuum (this is equivalent to requiring $\underline{B}\cdot\hat{\underline{\rho}} \neq 0$ be continuous)

$$A\,2r_0^2 + B/\sqrt{2}r_0 = 0, \quad \frac{A}{B} = \frac{-1}{2\sqrt{2}\,r_0^3}$$

Now we need $\frac{\partial\psi}{\partial\rho}$ continuous (i.e $B_\theta$ contin

$$\left.\frac{\partial\psi_v}{\partial\rho}\right|_{\rho=\sqrt{2}r_0} = \sin^2\theta\left[2A\sqrt{2}r_0 - \frac{B}{2r_0^2}\right] = A\sin^2\theta\left[2\sqrt{2}r_0 + \sqrt{2}r_0\right] = 3A\sqrt{2}\,r_0$$

$$\left.\frac{\partial\psi_p}{\partial\rho}\right|_{\rho=\sqrt{2}r_0} = \frac{\sin^2\theta\,\psi_0}{r_0^4}\left[\cancel{\phantom{xx}} - 2\rho^3\big|_{\rho=\sqrt{2}r_0}\right] = \left(-4\sqrt{2}\frac{r_0^3}{r_0^4}\psi_0\right)\sin^2\theta = -\frac{4\sqrt{2}\psi_0}{r_0}\sin^2\theta$$

$$\therefore A = -4\psi_0/3r_0^2$$

$$\psi_r = -\frac{4\psi_0}{3r_0^2}\sin^2\theta\left(\rho^2 - \frac{2\sqrt{2}\,r_0^3}{\rho}\right)$$

A vertical field, $\underset{\sim}{B}_{ver}$, is given by

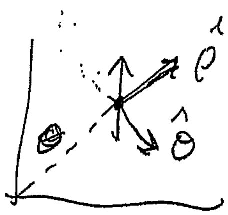
$$\underset{\sim}{B}_{ver} = \underset{\sim}{\nabla}\phi \times \underset{\sim}{\nabla}\psi_{vr}$$

with $\psi_{vr} = \frac{\mp B_{vr}\rho^2 \sin^2\theta}{2}$

as
$$\underset{\sim}{B}_{ver} = \frac{1}{\rho\sin\theta}\left(\hat{\phi}\times\hat{r}\frac{\partial\psi_v}{\partial\rho} + \hat{\phi}\times\frac{\hat{\theta}}{\rho}\frac{\partial\psi}{\partial\theta}\right)$$

$$= \frac{B_{vr}}{\rho\sin\theta}\left[-\hat{\theta}\,\rho\sin^2\theta + \hat{\rho}\rho\sin\theta\cos\theta\right]$$

$$= B_{vr}\left[\hat{\rho}\cos\theta - \hat{\theta}\sin\theta\right]$$




expression for vertical field

$$\therefore \quad \frac{B_v}{2} = -\frac{\cancel{4}}{3}\frac{\psi_0}{r_0^2} = \frac{\sqrt{P_0}\,r_0^2}{r_0^2\sqrt{\cancel{10}}}\frac{\cancel{4}}{3}$$

where we used $P_0 = 10\psi_0^2/r_0^4$ obtained earlier and that $\psi_0 < 0$

$$\beta_v = \frac{2P_0}{B_v^2} = \frac{180}{64} = \frac{90}{32} = \frac{45}{16}$$ (quite a large number

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(#2) Assuming that the poloidal current is

$$I(\psi) = \lambda \psi$$

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and that the pressure is

$$p = 0$$

Since we have a force free current, the Grad-Shafranov equation is given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial \psi}{\partial r}\right)\right] = -\lambda^2 \psi$$

$$\Rightarrow r^2 \frac{\partial^2 \psi}{\partial r^2} + r\frac{\partial \psi}{\partial r} + \lambda^2 r^2 \psi = 0$$

If we let $x = \lambda r$, then

$$x^2 \frac{\partial^2 \psi}{\partial x^2} + x\frac{\partial \psi}{\partial x} + x^2 \psi = 0 ,$$

which is Bessel's equation with $n = 0$.

$$\Rightarrow \psi = \psi_0 J_0(x) = \psi_0 J_0(\lambda r)$$

Since we require that $\psi(a) = 0$, $J_0(\lambda a) = 0$, so that $\boxed{\lambda = x_0/a}$

where $x_0 \cong 2.4048$ is the first root of $J_0$.

Then, the magnetic fields are given by

$$\begin{cases} B_z = I(\psi) = \lambda \psi(r) = \lambda \psi_0 J_0(\lambda r) \\ B_\theta = \dfrac{\partial \psi}{\partial r} = \psi_0 \dfrac{\partial J_0(\lambda r)}{\partial r} = -\lambda \psi_0 J_1(\lambda r) \end{cases}$$

Thus,

$$\frac{B_\theta(a)}{B_z(0)} = \frac{-J_1(\lambda a)}{J_0(0)} = -J_1(x_0) \cong -0.5191$$