Hamiltonian Description of Vlasov Dynamics: Action-Angle Variables for the Continuous Spectrum

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Abstract

The linear Vlasov-Poisson system for homogeneous, stable equilibria is solved by means of a novel invertible integral transform that is a generalization of the Hilbert transform. The integral transform provides a means for describing the dynamics of the continuous spectrum that is well-known to occur in this system. The results are interpreted in the context of Hamiltonian systems theory, where it is shown that the integral transform defines a canonical transformation to action-angle variables. A means for attaching Krein signature to a continuum eigenmode is given.

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1 Introduction

1.1 Motivation and overview

There are two main points of this paper: (1) to describe an integral transform for solving the Vlasov-Poisson equation linearized about a homogeneous, stable equilibrium and (2) to interpret this transform method of solution in the context of infinite degree-of-freedom Hamiltonian dynamics.

In Sec. 2 we describe the integral transform, a generalization of the Hilbert transform, that is specifically designed to unravel the mathematics of the continuous spectrum, which is well-known to occur in the linearized Vlasov-Poisson system. This transform was introduced elsewhere (Refs. [1] and [2]), but the level of rigor given here is much higher. If a method provides a rigorous solution, then one may question the necessity of the Hamiltonian formalism, which we present in Sec. 3. We address this question by describing below a sort of Hamiltonian philosophy.

In the past couple of hundred years a sizable body of lore has accumulated about finite degree-of-freedom Hamiltonian systems, and indeed remarkable progress has been made in the past 50 years. For example, it is now recognized that systems with one degree of freedom are integrable and that such systems are exceptional. It is known that systems with two degrees of freedom are generically nonintegrable, invariant tori are broken, and 'chaos' is the norm. In systems with three or more degrees of freedom invariant tori no longer separate phase space, and thus no longer provide barriers to transport. The philosophy in which one uses results and procedures from finite systems as a guide for investigation of infinite systems is one to which we subscribe. Relatively speaking, not nearly so much is known about infinite systems. Although all phenomena of finite systems occur in infinite systems, there are new phenomena that can occur only in infinite systems. One example of this is the continuous spectrum, and interpreting this in the Hamiltonian context is one of our main points.

Perhaps the most compelling argument for the Hamiltonian point of view resides in its generality. The most important equations of physics are Hamiltonian, and when one solves one problem one solves a whole class of important problems. For example, it has been proven that all stable finite degree-of-freedom Hamiltonian systems can be transformed so that

the Hamiltonian obtains the following normal form:

$$H = \sum_i^N \frac{\omega_i}{2} \left(p_i^2 + q_i^2 \right) = i \sum_i^N \omega_i \, Q_i \, P_i = \sum_i^N \omega_i \, J_i \tag{1}$$

which we have written three ways. The first way demonstrates that the dynamics is merely that of a collection of uncoupled oscillators or normal modes, while the last way corresponds to the action-angle variable description in which the angle variable is ignorable. The middle way is sometimes convenient and we record it for latter use.

So, is there a similar normal form for infinite degree-of-freedom Hamiltonian systems with a continuous spectrum? The answer to this question is yes, and as one might expect the normal form is given by something like the following:

$$H = \int \omega(u) \, J(u) \, du \, . \tag{2}$$

In Sec. 3 we demonstrate this for the Vlasov-Poisson system, but it is also true for the Vlasov-Maxwell system (Ref. [3]), the two-dimensional Euler fluid equations (Ref. [4]), and others. It is now clear that something general is going on: the continuous spectrum is resolved by a class of transformations that is a generalization of the Hilbert transform.

A feature of the normal form (1) that carries over to infinite dimensions is the notion of *signature*. The first form of (1) is a quadratic form, which need not be positive definite: in general the frequencies satisfy $\omega_i = \sigma_i |\omega_i|$ with $\sigma_i \in \{-1, 1\}$. Even though some of the oscillators may have negative frequencies, they are still stable oscillators. For the obvious reason they are called *negative energy modes*. It is known by Sylvester's theorem that the signature, defined to be the difference between the number of positive and negative modes, is invariant under real similarity transformations, and so to the extent that energy is defined, signature has physical meaning. Perhaps the most important role of signature is the one it plays in governing bifurcations. A theorem of Krein (independently proved by Moser) states that a necessary condition for the bifurcation from stability to instability is that colliding modes possess opposite sign. At the end of Sec. 3 we discuss signature in the context of the continuous spectrum of the Vlasov-Poisson system.

We conclude the present section by describing the Vlasov-Poisson system and collecting together some notions that will be used in the remainder of the paper.

1.2 The Vlasov-Poisson system

The Vlasov-Poisson system has a single dynamical variable, the phase space density $f(x, v, t)$, and can be thought of as a 1 + 1 + 1 field theory; i.e. a theory with one space, one velocity (or momentum), and one time variable. We have $f\colon X \times \mathbb{R}^2 \to \mathbb{R}$, where X is the spatial part and typically X is either S^1 or $\mathbb{R}$. In the former case we have periodic boundary conditions while in the latter case we have a some kind of decay condition at infinity. Physical initial conditions satisfy $f(x, v, t) \geq 0$.

The Vlasov-Poisson system is composed of the equation for the conservation of phase space density,

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}\frac{\partial \phi[x,t;f]}{\partial x}\frac{\partial f}{\partial v} = 0 \tag{3}$$

and Poisson's equation,

$$\phi_{xx} = -4\pi\left[e\int_{\mathbb{R}} f(x,v,t)\,dv + \rho_B\right], \tag{4}$$

where ρ_B is a background charge density that is chosen so that the total charge vanishes. Equations (3) and (4) formally conserve the following energy functional:

$$H = \frac{m}{2}\int_X\int_{\mathbb{R}} v^2 f\,dx\,dv + \frac{1}{8\pi}\int_X (\phi_x)^2\,dx\,, \tag{5}$$

which is composed of the sum of kinetic energy plus electrostatic energy pieces. This energy is actually the Hamiltonian for the infinite degree-of-freedom Hamiltonian description that we consider in Sec. 3.

In this work we are interested in perhaps the simplest plasma problem, that of linearization about a homogeneous, stable equilibrium. Thus we set $f = f_0(v) + \delta f(x, v, t)$, insert this in (3) and (4), and obtain

$$\frac{\partial \delta f}{\partial t} + v\frac{\partial \delta f}{\partial x} - \frac{e}{m}\frac{\partial \delta\phi[x,t;\delta f]}{\partial x}\frac{\partial f_0}{\partial v} = 0 \tag{6}$$

and

$$\delta\phi_{xx} = -4\pi e\int_{\mathbb{R}} \delta f(x,v,t)\,dv\,. \tag{7}$$

This linear system formally conserves the following energy functional:

$$H_L[\delta f] = -\frac{m}{2}\int_X\int_{\mathbb{R}} \frac{v}{f_0'}(\delta f)^2\,dx\,dv + \frac{1}{8\pi}\int_X (\delta\phi_x)^2\,dx \tag{8}$$

which will be seen in Sec. 3 to be the Hamiltonian for this system.

It is well-known that the solution to the full Vlasov-Poisson system can be written as a rearrangement: $f = \mathring{f} \circ Z$, where Z represents the solution to the characteristic equations run backwards in time. Equivalently, this can be written out as $f(x, v, t) = \mathring{f}(\mathring{x}(x, v, t), \mathring{v}(x, v, t))$. It is also well-known that not all rearrangements are allowed: only those that preserve the area measure $dxdv$. Thus it is natural to restrict initial perturbations to be rearrangements of the equilibrium state. In Refs. [5] and [1] we called such restricted variations *dynamically accessible* variations and obtained the following formula for them:

$$\delta f = [h, f] \,, \tag{9}$$

where

$$[A, B] := \frac{1}{m} \left(\frac{\partial A}{\partial x} \frac{\partial B}{\partial v} - \frac{\partial B}{\partial x} \frac{\partial A}{\partial v} \right) \tag{10}$$

is the usual Poisson bracket, f is any phase space density, and h is an arbitrary (sufficiently well-behaved) phase space function. Of interest here are variations about a homogeneous equilibrium state that are to be initial conditions for (6). Thus we have $\delta \mathring{f} = [h, f_0] = h_x f_0'/m$ where $f_0' := \partial f_0/\partial v$. This implies, for reasonable h's, that $\delta \mathring{f}$ has the same extrema as f_0. Also, it is easy to show that if a function is initially dynamically accessible, then under the dynamics of (6) it will be so for all time; i.e. δf at any time can be written in the form of $[h, f_0]$ for some function h.

In closing this section we point out that the idea of dynamical accessibility has recurred many times in the literature in different contexts, many of which are pointed out in Ref. [6], which along with Refs. [5], [1], and [7] is a source of more detailed information.

2 Linear Vlasov solution by integral transform

Now consider the integral transform solution. Because the continuous spectrum, our main object of concern, is associated with velocity dependence, we remove the spatial dependence by Fourier decomposition,

$$\delta f = \sum_k f_k(v, t) e^{ikx} \,, \qquad \delta \phi = \sum_k \phi_k(t) e^{ikx} \tag{11}$$

and write the linearized Vlasov-Poisson system as

$$\frac{\partial f_k}{\partial t} \quad + \quad ikv f_k - ik\phi_k \frac{e}{m} \frac{\partial f_0}{\partial v} = 0$$

$$k^2\phi_k \quad = \quad 4\pi e \int_{\mathbb{R}} f_k(v,t)\, dv\,. \tag{12}$$

These are the equations we solve for a single $k \neq 0$. Note, the assumption $k \neq 0$ is consistent with dynamical accessibility, because $h_x \sim ikh_k$. Hence, a dynamically accessible initial condition vanishes for $k = 0$ and the $k = 0$ term in the Fourier series for δf will remain zero for all time.

Equations (12) have been well-studied by essentially two methods: the Laplace transform method that originates with with Landau in 1946 and the normal mdes approach that originates with Van Kampen in 1955. Since these original papers there have been many works of varying degrees of rigor and generality. (See e.g. Refs. [8], [9], and [10].) The integral transform method treated here, which originates in Refs. [1] and [2], amounts to a coordinate change that makes the time integration trivial. This method is closest to Van Kampen's approach, but the spirit is very Hamiltonian, an interpretation that is deferred to Sec. 3.

The integral transform method is very simple once the transform is known and understood. We introdue the G-transform [given by Eq. (15)], which we prove has an inverse $\widehat{G}$ that transforms (12) into

$$\frac{\partial g_k}{\partial t} + iku\, g_k = 0\,. \tag{13}$$

The solution to (13) is obviously $g_k(u,t) = \mathring{g}_k \exp(-ikut)$, where $\mathring{g}_k(u) := g_k(u, t = 0)$. Using $\mathring{g}_k = \widehat{G}[\mathring{f}_k]$, where $\mathring{f}_k(v) := f_k(v, t = 0)$, we obtain the solution upon transforming back with the G-transform:

$$\begin{aligned} f_k(v,t) \quad &= \quad G[g_k(u,t)] \\ &= \quad G\left[\mathring{g}_k(u)e^{-ikut}\right] = G\left[\widehat{G}[\mathring{f}_k]e^{-ikut}\right]\,. \end{aligned} \tag{14}$$

In the remainder of this section we make this rigorous. In 2.1 we describe classes of equilibria and initial conditions that our solution method accommodates, and we remark on their physical significance. In 2.2 we review some properties of the Hilbert transform, which we use in 2.3 to prove the main theorems about the G-transform. Lastly in 2.4 we state precisely the nature of the solution.

2.1 Equilibria and initial conditions

Definition (VP1) *A function $f_0(v)$ is a* <u>*good*</u> <u>*equilibrium*</u> *if $f_0'(v)$ satisfies*

(i) $f_0' \in L_q(\mathbb{R}) \cap H_\alpha(\mathbb{R})$, *for some* q *such that* $1 < q < \infty$,

(ii) $\exists\, v* > 0$ *such that* $|f_0'(v)| < A|v|^{-\mu}, \forall\, |v| > v*$, *where* $A > 0$ *and* $\mu > 0$, *and*

(iii) $f_0'/v < 0$, $\forall\, v \in \mathbb{R}$, *or* f_0 *has the* <u>*mystery*</u> <u>*property*</u> *described below. (We assume* $f_0'(0) = 0$*.)*

Remarks:

1. H_α is the space of Hölder functions that satisfy $|f(x) - f(y)| < K|x - y|^\alpha$, where $0 < \alpha < 1$. For our purposes local Hölder is sufficient.

2. Since (12) only depends on f_0', conditions on f_0' are sufficient. However, the physics requires other conditions, such as $f_0(v) \geq 0$ and $\int_{\mathbb{R}} |v|^n f_0(v)\, dv < \infty$, for $n = 0, 1, 2$.

3. Item (iii) assures that the spectrum is entirely continuous, i.e. does not possess a discrete component. The mystery property relaxes the monotonicity condition and allows for negative energy modes, which were mentioned in Sec. 1 and are described in Sec. 3.

4. Maxwellian equilibria are included in our class of good equilibria, along with bump on tail equilibria with sufficiently small and slow bumps, and other interesting equilibria as well. However, we are excluding cold beams that are distributions, $f_0 \sim \delta(v - v_0)$, and waterbag equilibria and others of compact support. The solution method is more powerful than what is stated here and can be generalized to e.g. piecewise Hölder functions, and probably functions of BMO.

Definition (VP2) *A function,* $\mathring{f}_k(v)$*, is a* <u>*good*</u> <u>*initial*</u> <u>*condition*</u> *if it satisfies*

(i) $\mathring{f}_k(v), v\mathring{f}_k(v) \in L_p(\mathbb{R})$,

(ii) $\int_{\mathbb{R}} \mathring{f}_k(v)\, dv < \infty$.

Remarks:

1. The above requirements assure that all the terms of (12) exist initially. Item (ii) corresponds to the initial charge.

2. The physics suggests $\int_{\mathbb{R}} |v|^n \mathring{f}_k(v)\, dv < \infty$, for $n = 1, 2$, i.e. in addition to the charge, momentum and energy-like integrals should exist. These should even exist locally. Also $\int_{\mathbb{R}} v|\mathring{f}_k|^{\,2}/f_0'\, dv < \infty$, since this is part of H_L, the Hamiltonian functional.

2.2 Hilbert transform review

The Hilbert transform is defined by

$$H[g](x) := \frac{P}{\pi} \int_{\mathbb{R}} \frac{g(t)}{t-x}\, dt\,,$$

where P denotes the Cauchy principal value. There are many theorems about Hilbert transforms in the spaces L_p and H_α. (See e.g. Ref. [11].) We state some of them without proof.

Theorem (H1)

(ii) $H\colon L_p(\mathbb{R}) \to L_p(\mathbb{R})$, *for* $1 < p < \infty$, *is a bounded linear operator:*

$$\|H[g]\|_p \le A_p\, \|g\|_p\,,$$

where A_p *depends only on* p,

(ii) H *has an almost everywhere inverse on* $L_p(\mathbb{R})$, *given by*

$$H[H[g]] = -g\,,$$

and

(iii) $H\colon L_p(\mathbb{R}) \cap H_\alpha(\mathbb{R}) \to L_p(\mathbb{R}) \cap H_\alpha(\mathbb{R})$.

The proof of (H1) is given in the classical works of Plemelj, M. Riesz, Zygmund, Titchmarsh and others. (See e.g. [12] and [13].) It can also be an be extracted from the more general Calderón-Zygmund theory. (See e.g. [17] and [15].)

Theorem (H2) *If* $g_1 \in L_p(\mathbb{R})$ *and* $g_2 \in L_q(\mathbb{R})$ *with* $\frac{1}{p} + \frac{1}{q} < 1$, *then*

$$H[g_1 H[g_2] + g_2 H[g_1]] = H[g_1]H[g_2] - g_1 g_2\,.$$

A proof based on the Hardy-Poincaré-Bertrand theorem was given by Tricomi and can be found in [16].

Lemma (H3) *If $vg \in L_p(\mathbb{R})$, then*

$$H[vg](u) = u\, H[g](u) + \frac{1}{\pi} \int_{\mathbb{R}} g\, dv\,.$$

Proof : $\frac{v}{v-u} = \frac{u+v-u}{v-u} = \frac{u}{v-u} + 1$ □

2.3 The G-transform

Definition (G1) *The G-transform is defined by*

$$\begin{aligned} f(v) &= G[g](v) \\ &:= \epsilon_R(v)\, g(v) + \epsilon_I(v)\, H[g](v)\,, \end{aligned} \tag{15}$$

where

$$\epsilon_I(v) = -\pi \frac{\omega_p^2}{k^2} \frac{\partial f_0(v)}{\partial v}\,, \qquad \epsilon_R(v) = 1 + H[\epsilon_I](v)\,.$$

Remarks:

1. We suppress the dependence of ϵ and some other quantities upon k. Note, $\omega_p^2 := 4\pi n_0 e^2/m$ is the plasma frequency corresponding to an equilibrium of number density n_0.

2. $\epsilon = \epsilon_R + i\epsilon_I$ (when extended into the complex plane) is the plasma dispersion relation whose vanishing implies discrete normal eigenmodes. When ϵ does not vanish the system has only a continuous component to the spectrum.

3. Note that because $\epsilon_I \propto f_0' \in L_q(\mathbb{R}) \cap H_\alpha(\mathbb{R}) \Longrightarrow \epsilon_R - 1 \in L_q(\mathbb{R}) \cap H_\alpha(\mathbb{R})$, and because $\lim_{|v|\to\infty} \epsilon_I = 0$ and $\lim_{|v|\to\infty} \epsilon_R = 1$, both $\epsilon_R\,, \epsilon_I \in L_\infty(\mathbb{R})$.

Theorem (G2) *$G\colon L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$, is a bounded linear operator:*

$$\|G[g]\|_p \le B_p\, \|g\|_p\,,$$

where B_p depends only on p.
Proof : Since ϵ_I and ϵ_R are Hölder and bounded (Remark 3), $g \in L_p(\mathbb{R})$, $H[g] \in L_p(\mathbb{R}) \Longrightarrow G[g] \in L_p(\mathbb{R})$. By the triangle inequality, Hölder's inequality, and the boundedness of $H[g]$,

$$\begin{aligned} \|G[g]\|_p &\leq \|\epsilon_R\, g\|_p + \|\epsilon_I\, H[g]\|_p \\ &\leq \|\epsilon_R\|_\infty \|g\|_p + \|\epsilon_I\|_\infty \|H[g]\|_p \\ &\leq B_p \|g\|_p \,. \end{aligned}$$

□

Theorem (G3) *If f_0 is a good equilibrium, then $G[g]$ has an almost everywhere inverse,*

$$\widehat{G}\colon L_p(\mathbb{R}) \to L_p(\mathbb{R})\,,$$

for $1/p + 1/q < 1$, given by

$$\begin{aligned} g(u) &= \widehat{G}[f](u) \\ &:= \frac{\epsilon_R(u)}{|\epsilon(u)|^2}\, f(u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2}\, H[f](u)\,. \end{aligned}$$

where $|\epsilon|^2 := \epsilon_R^2 + \epsilon_I^2$.
Proof : First we show $g \in L_p(\mathbb{R})$ and then $g = \widehat{G}[G[g]$.

If $\epsilon_R(u)/|\epsilon(u)|^2$ and $\epsilon_I(u)/|\epsilon(u)|^2$ are bounded, then clearly $g \in L_p(\mathbb{R})$. For good equilibria the numerators are bounded and everything is Hölder, so it is only necessary to show that $|\epsilon|$ is bounded away from zero. We now show that either of the conditions of (VP1)(iii) assures this.

If the first condition is satisfied then $|f_0'| > 0$ for $v \neq 0$ and $f_0'(0) = 0$. Therefore we need only look at $v = 0$ and $v = \infty$ to assure $\epsilon \neq 0$. At $v = 0$

$$\epsilon_R(0) = 1 - \frac{\omega_P^2}{k^2} \int_{\mathbb{R}} \frac{f_0'}{v} dv \; > 1 > 0\,,$$

while as $v \to \infty$, $\epsilon_R \to 1$.

The second condition, the mystery property, is equivalent to the Penrose criterion (see e.g. [9]), a criterion that uses the argument principle of complex analysis to show $|\epsilon| \neq 0$. It states that there are no discrete modes if for all minima, v_m, of f_0

$$\int_{\mathbb{R}} \frac{f_0(v) - f_0(v_m)}{(v - v_m)^2}\, dv < 0\,,$$

or better if

$$\int_{\mathbb{R}} \frac{f_0'(v)}{v - v_m}\, dv < 0\,.$$

Note this criterion is independent of k, but allows inversion of the G-transform $\forall\, k$.

That $\widehat{G}$ is the inverse follows directly upon inserting $G[g]$ of (G1) into $g = \widehat{G}[G[g]]$, and using (H2) and $\epsilon_R(v) = 1 + H[\epsilon_I]$. We write out the steps:

$$\begin{aligned}
g(u) &= \widehat{G}[f](u) = \frac{\epsilon_R(u)}{|\epsilon(u)|^2}\, f(u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2}\, H[f](u) \\
&= \frac{\epsilon_R(u)}{|\epsilon(u)|^2}\, [\epsilon_R(u)\, g(u) + \epsilon_I(u)\, H[g](u)] \\
&\qquad - \frac{\epsilon_I(u)}{|\epsilon(u)|^2}\, H\left[\epsilon_R(u')\, g(u') + \epsilon_I(u')\, H[g](u')\right](u) \\
&= \frac{\epsilon_R^2(u)}{|\epsilon(u)|^2}\, g(u) + \frac{\epsilon_R(u)\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) \\
&\qquad - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H\left[H[\epsilon_I]\, g + \epsilon_I\, H[g]\right](u) \\
&= \frac{\epsilon_R^2(u)}{|\epsilon(u)|^2}\, g(u) + \frac{\epsilon_R(u)\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) \\
&\qquad - \frac{\epsilon_I(u)}{|\epsilon(u)|^2}\, [H[\epsilon_I](u) H[g](u) - g(u)\, \epsilon_I(u)] \\
&= g(u) + \frac{\epsilon_R(u)\epsilon_I(u)}{|\epsilon(u)|^2} H[g] - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[g] - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[\epsilon_I] H[g] \\
&= g(u) + \frac{\epsilon_R(u)\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u)\, [1 + H[\epsilon_I](u)] \\
&= g(u) + \frac{\epsilon_R(u)\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) \epsilon_R(u) = g(u)
\end{aligned}$$

□

Lemma (G4) *If ϵ_I and ϵ_R are as above, then*

(i) for $vf \in L_p(\mathbb{R})$,
$\widehat{G}[vf](u) = u\, \widehat{G}[f](u) - \frac{\epsilon_I}{|\epsilon|^2} \frac{1}{\pi} \int_{\mathbb{R}} f\, dv\,,$

(ii) $\widehat{G}[\epsilon_I](u) = \frac{\epsilon_I(u)}{|\epsilon|^2(u)}$

(iii) and if $f(u,t)$ and $g(v,t)$ are strongly differentiable in t; i. e. the mapping $t \mapsto f(t) = f(t,\cdot) \in L_p(\mathbb{R})$ is differentiable (the usual difference quotient converging in the L_p sense) then

a) $\widehat{G}\left[\frac{\partial f}{\partial t}\right] = \frac{\partial \widehat{G}[f]}{\partial t} = \frac{\partial g}{\partial t}$,

b) $G\left[\frac{\partial g}{\partial t}\right] = \frac{\partial G[g]}{\partial t} = \frac{\partial f}{\partial t}$.

Proof : (i) goes through like (H3), (ii) follows from $\epsilon_R = 1 + H[\epsilon_I]$, and (iii) follows because G is bounded and linear. □

2.4 The solution

Now we are in a position to justify the solution (14) that was described at the beginning of this section.

Theorem (S1) *For good initital conditions and equilibria,*

$$f_k(v,t) = G\left[\widehat{G}[\mathring{f}_k]e^{-ikut}\right]$$

is an $L_p(\mathbb{R})$ solution of (12).

Remarks:

1. The solution above can be compared to that for a sum over eigenvectors, G_j: $z^i(t) = a^j\, G^i_j\; e^{i\,\omega_j t}$, which when projected onto an initial condition $\mathring{z}$ gives $a^j = (G^k_j)^{-1}\,\mathring{z}^{\,k}$. Whence $z^i(t) = G^i_j\,(G^k_j)^{-1}\,\mathring{z}^{\,k}\,e^{i\,\omega_j t}$.

2. The analogue of G^i is the Van Kampen singular eigenfunction

$$\mathcal{G}(u,v) = \epsilon_I(v)\,\frac{\mathcal{P}}{\pi}\frac{1}{u-v} + \epsilon_R(v)\,\delta(v-u)\,,$$

   which is the eigenfunction for the continuous spectrum, indexed by $u \in \mathbb{R}$.

3 Hamiltonian Description

Now we interpret the results of Sec. 2 in the Hamiltonian context. In 3.1 we briefly review the generalization of Hamiltonian theory that is possessed by the equations that describe the dynamics of classical media in terms of

Eulerian variables. In 3.2 we do two main things: we change variables so that the Hamiltonian description of (20) achieves canonical form in terms of canonically conjugate variables, (q_k, p_k), and then we use the results of Sec. 2 to construct a canonical transformation to new canonical variables, (Q_k, P_k), in terms of which the Hamiltonian H_L achieves diagonal form. This latter form is essentially action-angle form, which we describe. The section is concluded with a brief discussion of signature and Krein's theorem.

3.1 Hamiltonian structure for Vlasov-Poisson

There is a standard and general Hamiltonian structure possessed by physical systems that describe media in terms of Eulerian variables. For example, ideal fluid equations, including magnetohydrodynamics, and various kinetic theories, including the Vlasov-Poisson equations, possess this struture. (See Ref. [6] and many references therein.) It is a structure that dates back to the nineteenth century and the work of Sophus Lie, but its importance for describing fluid and plasma equations was not realized in generality until around 1980. Since then a great deal of research, much of it with a definite modern geometric flavor, has been done.

Ordinarily Hamilton's equations are written as follows:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}\,, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad i = 1, 2, \ldots . \tag{16}$$

However, if these equations are written in terms of arbitrary noncanonical variables or coordinates, they obtain the form

$$\dot{z}^i = J^{ij}\frac{\partial H}{\partial z^j} = [z^i, H]\,, \tag{17}$$

where the Poisson bracket becomes

$$[A, B] := \frac{\partial A}{\partial z^i} J^{ij}(z) \frac{\partial B}{\partial z^j}\,.$$

The essence of being Hamiltonian lies in the fact that $[A, B]$ is a Lie bracket, i.e. it is bilinear and satisfies

$$[A, B] = -[B, A]$$

and

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0\,,$$

for all functions A, B and C. The latter condition, known as the Jacobi identity, is the crux of the matter: by a nineteenth century result credited to Darboux, if $\det J \neq 0$, then at least locally there exists a canonical set of coordinates in which Eqs. (17) have the form of (16). Also, in the nineteenth century Sophus Lie studied the case where $\det J = 0$ and concluded that a system of coordinates exist in which Eqs. 17 have the form of a smaller canonical system plus some coordinates, called Casimir invariants, that are constants of motion.

Lie studied the bracket

$$J^{ij} = c^{ij}_k z^k \quad \Longleftrightarrow \quad [A, B] = c^{ij}_k z^k \frac{\partial A}{\partial z^i} \frac{\partial B}{\partial z^j} \,,$$

where c^{ij}_k are the structure constants (or operators) of some Lie algebra. But it was unbeknownst to Lie that this kind of Poisson bracket, now called a noncanonical Poisson bracket of Lie-Poisson type, describes continuous media. The quantity J, called the Poisson tensor or cosymplectic form, is linear in the dynamical variables, unlike that for conventional (canonical) field theories such as the Klein-Gordon equation.

It is apparent that the Vlasov-Poisson system cannot possess canonical form. It energy, as given by (5), is quadratic in f and such Hamiltonians give rise to linear theories like the simple harmonic oscillator or the wave equation. Also, f is the only dynamical variable, and so a conjugate is missing. A simple dimensional analysis reveals that if this system is to be noncanonically Hamiltonian, then the Poisson bracket must be linear in f. The Vlasov-Poisson system possesses an infinite-dimensional generalization of the structure studied by Lie. The table shows the correspondence between quantities in a finite-dimensional theory and those for the Vlasov-Poisson system.

Finite	$\longleftrightarrow$	*VP*
z	$\longleftrightarrow$	f
i	$\longleftrightarrow$	(x, v)
$A(z)$	$\longleftrightarrow$	$F[f]$
$\delta A = \nabla A \cdot \delta z$	$\longleftrightarrow$	$\delta F = \int_X \int_{\mathbb{R}} \frac{\delta F}{\delta f} \, \delta f \, dx \, dv$
$J^{ij} = c^{ij}_k z^k$	$\longleftrightarrow$	$\mathcal{J} = [f, \ \cdot \ ]$

The first row compares dynamical variables; the second, indices, with i being discrete and $(x, v) \in X \times \mathbb{R}$; the third, phase space funtion(als); the

fourth compares first variations and serves to define the functional derivative $\delta F/\delta f$, which is evidently an infinite-dimensional gradient; and the last row compares cosymplectic forms.

The noncanonical Hamiltonian for the Vlasov-Poisson system was first given in [18]. The Poisson bracket, with the cosymplectic form of the table, is the following:

$$\{F,G\} = \int_X \int_{\mathbb{R}} f \left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right] dx\, dv\,, \tag{18}$$

where F and G are functionals of f. In terms of (18) the Vlasov-Poisson system can be written as

$$\frac{\partial f}{\partial t} = \{f, H\} = [f, \mathcal{E}].$$

where H is the energy functional of (5), $\mathcal{E} = mv^2/2 + e\phi$, and $[f, \mathcal{E}]$ is defined by (10).

Letting $f = f_0(v) + \delta f$, as in Sec. 1.2, and linearizing gives

$$\{F,G\}_L = \int_X \int_{\mathbb{R}} f_0 \left[\frac{\delta F}{\delta\delta f}, \frac{\delta G}{\delta\delta f}\right] dx\, dv\,, \tag{19}$$

where F and G are now functionals of δf and $\delta F/\delta\delta f$ is an unfortunate notation for the functional derivative of F with respect to δf. With the Hamiltonian (energy) of (8), (19) gives the linearized Vlasov-Poisson equations in the noncanonical Hamiltonian form

$$\frac{\partial \delta f}{\partial t} = \{\delta f, H_L\}_L\,. \tag{20}$$

3.2 Canonization and diagonalization

Upon expanding δf in a Fourier series as in (11), the Poisson bracket (19) can be written as

$$\{F,G\}_L = \sum_{k=1}^{\infty} \frac{ik}{m} \int_{\mathbb{R}} f_0' \left(\frac{\delta F}{\delta f_k}\frac{\delta G}{\delta f_{-k}} - \frac{\delta G}{\delta f_k}\frac{\delta F}{\delta f_{-k}}\right) dv\,, \tag{21}$$

and the Hamiltonian (8) becomes

$$\begin{aligned} H_L &= -\frac{m}{2}\sum_k \int_{\mathbb{R}} \frac{v}{f_0'}\,|f_k|^2\, dv + \frac{1}{8\pi}\sum_k k^2|\phi_k|^2 \\ &=: \sum_{k,k'} \int_{\mathbb{R}}\int_{\mathbb{R}} f_k(v)\,\mathcal{O}_{k,k'}(v|v')\, f_{k'}(v')\, dv\, dv'\,. \end{aligned} \tag{22}$$

In terms of (21) and (22), (12) can be written in noncanonical Hamiltonian form as

$$\frac{\partial f_k}{\partial t} = \{f_k, H_L\}_L \,, \tag{23}$$

which is merely the projection of (20) onto Fourier modes.

This description can now be canonized, i.e. written in terms of canonical variables, by the simple scaling transformation,

$$q_k(v,t) = \frac{m}{ikf_0'} f_k(v,t) \,, \qquad p_k(v,t) = f_{-k}(v,t) \,,$$

which gives

$$\{F, G\}_L = \sum_{k=1}^{\infty} \int_{\mathbb{R}} \left(\frac{\delta F}{\delta q_k} \frac{\delta G}{\delta p_k} - \frac{\delta G}{\delta q_k} \frac{\delta F}{\delta p_k} \right) dv \,.$$

Note that a possible singularity occurs in this transformation at points where $k = 0$ or $f_0' = 0$; however, both of these are ruled out by restricting initial conditions to be dynamically accessible.

Diagonalization is achieved using the following mixed variable generating functional to generate a canonical transformation from (q, p) to (Q, P) :

$$\begin{aligned} \mathcal{F}[q,P] &= \sum_{k=1}^{\infty} \int_{\mathbb{R}} q_k(v)\, G[P_k](v)\, dv \\ &= \sum_{k=1}^{\infty} \Bigg(\int_{\mathbb{R}} \epsilon_R(v)\, q_k(v)\, P_k(v)\, dv \\ &\qquad\qquad + \frac{P}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\epsilon_I(v)}{u-v}\, q_k(v)\, P_k(u)\, dv\, du \Bigg) \,. \end{aligned} \tag{24}$$

As is usual for generating functions of this type, the transformation is given by

$$p_k(v) = \frac{\delta \mathcal{F}[q,P]}{\delta q_k(v)} = G[P_k](v) \,, \qquad Q_k(u) = \frac{\delta \mathcal{F}[q,P]}{\delta P_k(u)} = G^{\dagger}[q_k](u) \,. \tag{25}$$

The new Hamiltonian in terms of these variables takes the form

$$H_L = \sum_{k=1}^{\infty} \int_{\mathbb{R}} i\, \omega_k(u)\, Q_k(u)\, P_k(u)\, du \,, \tag{26}$$

where $\omega_k(u) = ku$. Thus we have transformed to an infinite-dimensional generalization of the normal form of (1). This is diagonalization.

Above we have introduced complex variables in our transformations. However, ultimately the dynamical variables of Vlasov theory correspond to real physical quantities and are thus real variables. Everything we have done could have been done entirely in terms of real variables. This would have been awkward, but, as for finite degree-of-freedom systems, it has the advantage of keeping track of the invariant signature of the Hamiltonian. The signature can be recovered by the following transformation to action-angle variables, (J_k, θ_k),

$$\begin{aligned} Q_k(u,t) &= \sqrt{J_k(u,t)}\; e^{i\sigma_k\theta_k(u,t)}\,, \\ P_k(u,t) &= -i\sqrt{J_k(u,t)}\; e^{-i\sigma_k\theta_k(u,t)}\,, \end{aligned} \tag{27}$$

where the signature is given by $\sigma_k(u) := \mathrm{sgn}(ku\epsilon_I)$. In terms of these variables the Hamiltonian becomes

$$H_L = \sum_{k=1}^{\infty} \int_{\mathbb{R}} \sigma_k(u)\,\omega_k(u) J_k(u,t)\,du\,, \tag{28}$$

where $\omega_k(u) := |ku|$, and the Poisson bracket becomes

$$\{F,G\}_L = \sum_{k=1}^{\infty} \int_{\mathbb{R}} \left(\frac{\delta F}{\delta\theta_k}\frac{\delta G}{\delta J_k} - \frac{\delta G}{\delta\theta_k}\frac{\delta F}{\delta J_k} \right) du\,. \tag{29}$$

Thus we have achieved the normal form of (2).

The quantity $\sigma_k(u)$ assigns a signature to a continuum eigenmode, analogous to that for finite systems. In the study of finite systems, one assumes the Hamiltonian depends upon a parameter and then investigates bifurcations as this parameter is varied. As mentioned above, when discrete eigenmodes collide Krein's theorem states that a necessary condition for the bifurcation to instability is that the colliding modes have opposite signature. The Penrose criterion can be used to analyze bifurcations in the Vlasov-Poisson system, and indeed it can be shown that unstable modes emerge from the continuum at places where positive and negative continuum eigenmodes meet, a result that suggests an infinite dimensional version of Krein's theorem. However, unlike the finite case, care must be taken in defining the 'distance' from the bifurcation point. Because any point of the continuum is point-wise arbitrarily close to either signature, a Sobolev type norm is required. A general discussion of Krein's theorem for infinite dimensions will be given in future work, along with several items described in the next section.

4 Summary and future work

We have described an integral transform, or equivalently a coordinate change, that enabled us to obtain an L_p solution of the Vlasov-Poisson system linearized about good equilibria with good initial conditions. The equilibria considered possess a continuous spectrum with no discrete component. The Hamiltonian transformation to action-angle variables was given. Signature for the continuous spectrum was defined and comments about Krein's theorem for infinite systems were made.

It is clear that there are many other problems suggested by the Hamiltonian philosophy described in the Introduction. We list a few of them:

1. Defining the general class of linear Hamiltonian systems with continuous spectra that can be solved by a general class of integral transforms similar to those presented here.

2. Inclusion of discrete spectra in the class above.

3. Investigation of bifurcation theory in the presence of the continuous spectrum and the role of negative energy modes.

4. The generalization of the theory of adiabatic invariants.

5. Nonlinear Hamiltonian perturbation theory with the continuous spectrum.

These will be the subjects of future papers.

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