THE ASTROPHYSICAL JOURNAL, 567:L81–L83, 2002 March 1

DYNAMO ACTION IN HALL MAGNETOHYDRODYNAMICS

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Received 2001 August 27; accepted 2002 January 25; published 2002 February 11

ABSTRACT

We show that the generally neglected Hall term in the equations for two-fluid magnetohydrodynamics may have a profound effect on α-dynamo action. The new calculation, in addition to subsuming the standard results from the mean field approach, contains a contribution to the α-coefficient entirely due to the Hall current in the microscale.

Subject headings: galaxies: magnetic fields — magnetic fields — MHD — stars: magnetic fields

1. INTRODUCTION

One-fluid magnetohydrodynamics (MHD), the standard framework for describing dynamo activity in astrophysical environments, cannot be expected to properly describe collisionless plasmas because it fails to distinguish the relative motions between different species. It is nonetheless interesting that MHD often turns out to be a reasonable description of the large-scale bulk dynamics of the fluid, as long as the fluid does not support a significant electric field in its own frame of reference.

A first step toward creating a more appropriate theory for collisionless dynamics might be to include the dominant two-fluid effects through a generalized Ohm's law: $\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = -\boldsymbol{j} \times \boldsymbol{B}/ne$, which includes the Hall effect, considered to be the most important in a wide range of cases of interest (n is the particle density, e the electron charge). In order for the Hall term to be important, the characteristic length scale (L) of the system should be on the order of the Hall scale, $L_{\rm H} = cv_{\rm A}/V_0\omega_{\rm ip}$, where c is the speed of light, $v_{\rm A}$ the Alfvén speed, V_0 a characteristic speed, and $\omega_{\rm ip}$ the ion plasma frequency. Inclusion of the Hall term leads to the Hall-MHD equations that display, among other properties, the freezing of the magnetic field to the electron flow rather than to the bulk velocity field. Consequently, the Hall term is likely to exert a major influence on the generation of the magnetic field through dynamo activity.

The astrophysical systems where the Hall effect might be important are relatively well known. Some examples are as follows:

The interstellar medium.—The characteristic Hall length scale is about 1000 km, while typical length scales for turbulence in this medium range from 100 to 10^5 km (Spangler 1999). While these length scales are still very small for helical effects to be relevant, Hall effects can be important in large Prandtl number dynamos (Kinney et al. 1999), which take place in scales below the kinetic energy dissipation scale.

Turbulence in the early universe.—The Hall effect can affect the inverse cascade of magnetic helicity, which is believed to be responsible for the generation of large-scale magnetic fields (Tajima et al. 1992).

White dwarfs and neutron stars.—Evidence of strong Hall effect exists for the relatively wide range of magnetic fields detected in these objects (Urpin & Yakovlev 1980; Shalybkov & Urpin 1997).

In the present work, we go beyond the standard MHD treatment and calculate the contribution of the Hall term to the α-effect. We also avoid using the first-order smoothing approximation (FOSA), the standard closure scheme of the traditional mean field dynamo theories, since it requires the magnetic Reynolds number or the Strouhal number to be much smaller than unity. Such conditions are not satisfied in a number of relevant cases, such as the cases of the Sun and the galaxies. An alternative closure has recently been proposed (Blackman & Field 1999) that partially solves some of these problems. However, limitations in the derivation of the mean field still remain. In § 2 we derive an expression for α that includes the Hall effect, and in § 3 we obtain quantitative results for a set of Hall-MHD equilibria. In § 4 we summarize our conclusions.

2. THE HALL-MHD EQUATIONS

Ideal and incompressible Hall-MHD is described by the modified induction and the Euler equation:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times [(\boldsymbol{U} - \nabla \times \boldsymbol{B}) \times \boldsymbol{B}],$$

$$\frac{\partial \boldsymbol{U}}{\partial t} = \boldsymbol{U} \times (\nabla \times \boldsymbol{U}) + (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} - \nabla\left(P + \frac{U^2}{2}\right), \tag{1}$$

where the velocity and magnetic fields are expressed in units of a characteristic speed and lengths are in units of the Hall length. We are interested in finding out whether this system is able to generate a macroscale magnetic field from an initial microscale configuration, consisting of a small seed magnetic field along with a substantial velocity field.

We assume the initial state $\boldsymbol{u}_0$, $\boldsymbol{b}_0$ to be a solution of equation (1) in the absence of a long-scale field. We now perturb the system about this microscale solution, with $\boldsymbol{B} = \overline{\boldsymbol{B}} + \boldsymbol{b} + \boldsymbol{b}_0$ and $\boldsymbol{U} = \overline{\boldsymbol{U}} + \boldsymbol{u} + \boldsymbol{u}_0$, where the overbar denotes spatially or statistically averaged long-scale fields while $\boldsymbol{u}$ and $\boldsymbol{b}$ are small-scale perturbations. Note that while $\boldsymbol{b}_0$ is the short-scale magnetic field in the absence of $\overline{\boldsymbol{B}}$, $\boldsymbol{b}$ is the perturbation

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when $\overline{\boldsymbol{B}}$ is present and need not be isotropic. All small-scale fields have zero averages, while their products in general do not. Substituting into equation (1) using the equation for $\boldsymbol{b}_0$ and taking averages, we find an equation for the evolution of the large-scale averaged magnetic field:

$$\frac{\partial \overline{\boldsymbol{B}}}{\partial t} = \nabla \times [(\overline{\boldsymbol{U}} - \nabla \times \overline{\boldsymbol{B}}) \times \overline{\boldsymbol{B}} + \langle \nabla \times (\boldsymbol{u}_0^e \times \boldsymbol{b} + \boldsymbol{u}^e \times \boldsymbol{b}_0) \rangle, \tag{2}$$

where $\boldsymbol{u}^e \equiv \boldsymbol{u} - \nabla \times \boldsymbol{b}$ is the electron flow velocity. Quadratic terms in $\boldsymbol{b}$ and $\boldsymbol{u}$ were dropped, as is usually done in the mean field theory. Note that although it is a common assumption, it is not clear that these terms will remain negligible once the mean field grows to finite amplitudes. However, our assumption is less restrictive than the standard FOSA used in mean field derivations and is more akin to that of Blackman & Field (1999).

We can also derive an equation for the small-scale perturbed magnetic field $\boldsymbol{b}$. Here we drop terms involving spatial derivatives of the mean fields because the variations of the long-scale fields are negligible on the microscale. Finding corrections to the α-coefficient as our current focus, we also ignore the averaged terms, as they will not contribute to the equation for $\overline{\boldsymbol{B}}$. We obtain

$$\frac{\partial \boldsymbol{b}}{\partial t} = \nabla \times (\boldsymbol{u}_0^e \times \overline{\boldsymbol{B}} - \boldsymbol{b}_0 \times \overline{\boldsymbol{U}}). \tag{3}$$

In a similar manner, we can write the perturbed Euler equation,

$$\frac{\partial \boldsymbol{u}}{\partial t} = (\nabla \times \boldsymbol{b}_0) \times \overline{\boldsymbol{B}} - (\overline{\boldsymbol{U}} \cdot \nabla)\, \boldsymbol{u}_0 - \nabla p. \tag{4}$$

From the divergence of equation (4), we obtain $p = -\boldsymbol{b}_0 \cdot \overline{\boldsymbol{B}}$ for the small-scale pressure perturbation, which when substituted in equation (4) yields

$$\frac{\partial \boldsymbol{u}}{\partial t} = (\overline{\boldsymbol{B}} \cdot \nabla)\, \boldsymbol{b}_0 - (\overline{\boldsymbol{U}} \cdot \nabla)\, \boldsymbol{u}_0. \tag{5}$$

To obtain an expression for α, we close equation (2) by approximating the time derivatives in equations (3) and (5) by multiplications by an inverse correlation time. Note that this step, which amounts to the assumption that the correlation time τ is finite, is common in mean field theory, even though there is at present no evidence, experimental or numerical, that it is valid in general. We are working on an extension of the present model that does not rely on this assumption.

Assuming a weakly anisotropic turbulence and eliminating terms involving $\overline{\boldsymbol{U}}$ in the short-scale equations in the proper reference frame, the evolution equation for the mean field $\overline{\boldsymbol{B}}$ becomes

$$\frac{\partial \overline{\boldsymbol{B}}}{\partial t} = \nabla \times [(\overline{\boldsymbol{U}} - \nabla \times \overline{\boldsymbol{B}}) \times \overline{\boldsymbol{B}} + \alpha \overline{\boldsymbol{B}}], \tag{6}$$

where $\alpha \overline{\boldsymbol{B}}$ denotes the microscale contribution to the magnetic field generation with

$$\alpha = \frac{\tau}{3}(-\langle \boldsymbol{u}_0^e \cdot \nabla \times \boldsymbol{u}_0^e \rangle + \langle \boldsymbol{b}_0 \cdot \nabla \times \boldsymbol{b}_0 \rangle - \langle \boldsymbol{b}_0 \cdot \nabla \times \nabla \times \boldsymbol{u}_0^e \rangle). \tag{7}$$

The standard results from the mean field theory can be extracted from equation (7) by neglecting the terms originating from the Hall effect. By replacing $\boldsymbol{u}_0^e$ with $\boldsymbol{u}_0$ and dropping the last term, equation (7) reduces to $\alpha = \tau(-\langle \boldsymbol{u}_0 \cdot \nabla \times \boldsymbol{u}_0 \rangle + \langle \boldsymbol{b}_0 \cdot \nabla \times \boldsymbol{b}_0 \rangle)/3$. The kinetic term (Krause & Rädler 1980) is modified by a magnetic correction derived by Pouquet, Frisch, & Leorat (1976). Our more general expression (eq. [7]) differs from the classic result in two ways: it replaces the kinetic helicity (of the bulk motion) by the helicity of the electron flow, and it contains an extra term due to the Hall current in the microscale. A nontrivial consequence of the latter is that while the expression by Pouquet et al. (1976) is zero in a pure Alfvénic state, $\boldsymbol{u} = \pm \boldsymbol{b}$ (Gruzinov & Diamond 1994), equation (7) is not. In § 3 we compute the expression for α for double-Beltrami states, which were shown to be the dynamically selected states in Hall MHD (Mahajan & Yoshida 1998).

3. THE DOUBLE-BELTRAMI EQUILIBRIA

To assess the importance of the Hall term on α, let us assume the initial state $\boldsymbol{u}_0$, $\boldsymbol{b}_0$ to be a double-Beltrami state (Mahajan & Yoshida 1998). The two Beltrami conditions,

$$\boldsymbol{u}_0 - \nabla \times \boldsymbol{b}_0 = \frac{\boldsymbol{b}_0}{a}, \qquad \boldsymbol{b}_0 + \nabla \times \boldsymbol{u}_0 = d\boldsymbol{u}_0, \tag{8}$$

express rather basic physical laws, namely, that the inertialess electrons follow the field line and the ions follow the field lines modified by their vorticity. The parameters a and d measure the magnetic and generalized helicity, and for slowly evolving systems, they are constants of motion labeling the state (Mahajan et al. 2001; Ohsaki et al. 2001). These equilibria do not require any exact symmetry (such as Grad-Shafranov) or negligible $\boldsymbol{U}$ and ∇P (such as Taylor states). In addition, the velocity and magnetic fields are treated on an equal footing in the double-Beltrami state, making it an equilibrium of choice for the dynamo problem, where the velocity fields are so fundamental.

Double-Beltrami conditions (eq. [8]) are always accompanied by the Bernoulli condition, $\nabla(p_0 + u_0^2/2) = 0$, where p_0 is the equilibrium pressure. Double-Beltrami conditions allow a general solution in terms of two single-Beltrami fields ($\nabla \times \boldsymbol{b}_0 = \lambda \boldsymbol{b}_0$), with the inverse length scales determined by $\lambda_\pm = -r/2 \pm (r^2/4 - s)^{1/2}$ (where $r = 1/a - d$ and $s = 1 - d/a$). For dynamo applications, we are interested in the situation where the two scales are widely separated, requiring $r^2 \gg 4s$. The long scale will be associated with the characteristic scale of the system, while the shorter scale is associated with the turbulence. To reflect vanishingly small long-scale velocity and magnetic fields, the initial fields are assumed to be purely short scale (λ^{-1} is the short scale), given by $\nabla \times \boldsymbol{b}_0 = \lambda \boldsymbol{b}_0$ and $\boldsymbol{u}_0 = (\lambda + 1/a)\, \boldsymbol{b}_0$.

One of the fundamental consequences of the Hall term is to effectively replace the bulk velocity by the electron flow in the expression for α. Considering the Beltrami conditions and the initial fields, this ratio is

$$\frac{u_0^e}{u_0} = \frac{1}{\lambda a + 1}, \tag{9}$$

which can vary over a rather wide range. In the initial stages of the dynamo, this ratio is of great significance in determining the difference in α caused by the Hall term from its standard value. In Figure 1a we display this ratio as a function of the equilibrium parameters a and d. Note that for large $|a|$ and $|d|$, $u_0^e \approx 0$, implying that the α-effect is virtually suppressed. On the other hand, when $|a|$ is small, u_0^e is strongly enhanced, while the bulk motion in the microscopic scale remains small.

This equilibrium condition allows the compact form $\alpha = \tau F(a, d) \langle u_0 \cdot \nabla \times u_0 \rangle /3$. In Figure 1$b$ we display the dependence of the function $F(a, d)$ on the parameters a and d $[F(a, d) = [1 - (1/a + \lambda)/a]/(1/a + \lambda)^2]$. Figure 1$c$ shows the regions where the separation of scales exists. Note that we have an almost vanishing α-effect if $|a|$ and $|d|$ are large, since for this case the Hall cancellation makes the electron fluid motion much slower than the bulk motion. On the other hand, when $|a|$ is small, there is again a serious disparity between the electron fluid and the bulk motion, but with u_0^e much bigger than the bulk velocity in the microscale. As a result, the α-dynamo becomes much more effective. In these two asymptotic limits, the Hall term plays a dominant role.

4. CONCLUSIONS

This Letter contains a first derivation of the α-dynamo coefficient when Hall MHD is used to describe the plasma. A more complete two-fluid self-consistent approach will be presented in a later work.

The changes induced by the Hall term are twofold: The first is to cause the replacement of the bulk kinetic helicity by the helicity of the electron flow in the formula for the α-dynamo effect. Since these two-fluid motions can eventually become quite different, we find large suppression or enhancement of dynamo action (as compared to the standard MHD), depending on the state of the system. The second is the introduction of an additional term in the α-coefficient (due to Hall currents in the microscale), which survives the standard cancellation of the kinetic and magnetic contributions for Alfvénic perturbations.

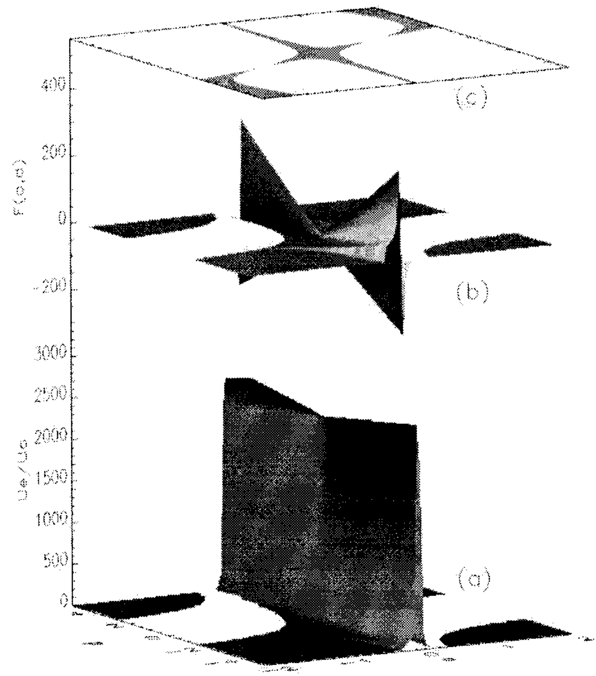

FIG. 1.—(a) Ratio of u_e/u_0. Holes correspond to regions where the length scales are complex. (b) Amplitude of F. (c) Region where separation of scales is fulfilled.

We would like to thank the anonymous referee for very fruitful comments that helped improve the present manuscript. P. D. M. is a fellow of CONICET, and D. O. G. is a member of the Carrera del Investigador del CONICET. P. D. M. and D. O. G. acknowledge support from the University of Buenos Aires through grant X209/00, and S. M. M. acknowledges support from US Department of Energy contract DE-FG03-96ER-54366.

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