Normal Modes and Continuous Spectra

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Abstract

We consider stability problems arising in fluids, plasmas and stellar systems that contain singularities resulting from wave-mean flow or wave-particle resonances. Such resonances lead to singularities in the differential equations determining the normal modes at the so-called critical points or layers. The locations of the singularities are determined by the eigenvalue of the problem, and as a result, the spectrum of eigenvalues forms a continuum. We outline a method to construct the singular eigenfunctions comprising the continuum for a variety of problems.

I. INTRODUCTION

In theory of fluids, plasmas and stellar systems, we frequently encounter the question of the stability of equilibria. The answer is provided in part on determining the evolution of an infinitesimal disturbance away from equilibrium, an approach that usually goes by way of a normal mode expansion. This approach can at times be very powerful, and amounts to solving an eigenvalue problem. It can, however, run into difficulty in circumstances for which that eigenvalue problem is, in some sense, irregular.

What we might call regular eigenvalue problems involve the solution of a set of ordinary differential equations with regular coefficients on a domain of finite size. As illustrated by the classical Sturm-Liouville problem, the eigenvalue spectrum turns out to be composed of an infinite number of distinct points. Like the characteristic frequencies of a vibrating string, these correspond to the distinct, normal modes. One might say that the set of irregular problems consists of everything that doesn't fall into this category. For many examples, the eigenvalue spectrum retains a simple form, but in general it is not the case, and the spectrum may consist of only a finite number of discrete modes or continuous intervals.

Here we are concerned with situations for which the eigenvalue problem is irregular and the resulting spectrum is at least partly continuous. This kind of a spectrum can arise as a result of solving the problem on an infinite domain, in which case there is simply no quantization condition. Of more interest are problems in which the set of ordinary differential equations is not autonomous and contains coefficients that become singular at points within the domain.

In physical situations, singularities in the equations governing the evolution of an infinitesimal disturbance can result from a variety of effects, and they do not always affect the form of the eigenspectrum. An important class of problems for which the singularity

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has direct repercussions on the eigenspectrum occurs in fluids, plasmas and stellar systems. These are ideal problems in which there are wave-mean flow or wave-particle resonances which result in the creation of a continuous eigenvalue spectrum. In these circumstances, coefficients in the differential problem are formally singular at the point at which resonance occurs. Moreover, *that point is determined by the speed of a wave-like perturbation or, equivalently, the eigenvalue.*

The existence of a continuous spectrum for an inviscid, shearing fluid was known to Rayleigh,[1] although he was not directly interested in it. In this context, an explicit solution for the spectrum was given by Fjørtøft and Høiland in the 1940s for the special case of incompressible Couette flow (see Ref. 2). The complications associated with finding the eigenvalues surround the presence of the singularity in the equations, which occurs where the advection of the perturbation exactly cancels its natural speed; a layer in the channel associated with such a singularity is commonly referred to as a critical layer.

In plasma theory we have an analogous situation at the points in phase space for which the equilibrium particle velocity matches the phase speed of the disturbance. This led to a classical problem in plasma theory that was eventually solved by Landau, leading to the celebrated phenomenon of Landau damping. That solution went by way of Laplace transforms which is naturally tailored to the initial-value problem. The parallel procedure using a continuum variety of normal modes was proposed by Van Kampen,[3] and considered in fluid contexts by Case.[4] In this paper we follow the directions indicated by Van Kampen for more general problems than the relatively simple plasma and fluid equilibria considered by Van Kampen and Case.

In what follows, we first describe the general method (which is discussed in greater detail and applied to parallel shear flow by Balmforth and Morrison[5]). Then, in the general context, the problem of plasma oscillations is reviewed. The remaining sections on parallel shear flow, shear flow in shallow water theory, incompressible circular vortices, and differentially

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rotating disks, are the bulk of the paper. We conclude with a discussion of the uses of singular eigenfunctions.

II. METHOD

An important feature of the solutions that compose the continuous spectrum is that they are not regular functions; they can contain kinks, discontinuities or singularities at the critical layers. Finding the solutions with standard numerical techniques for regular ordinary differential equations is then problematic. Here we describe an alternative method to construct the singular eigenfunctions. Related procedures have been used in neutron transport theory,[6] scattering theory,[7,8] and plasma physics.[9]

Most informally we can speak of a system governed by an equation of the form,

$$(x - x_*)\mathcal{L}_x\phi = \mathcal{M}_x\phi, \tag{1}$$

for some eigenfunction ϕ, and differential operators $\mathcal{L}_x$ and $\mathcal{M}_x$. The point x_* is contained within the domain, $\mathcal{D}$, and is really the eigenvalue. The operator $\mathcal{L}_x$ contains the leading derivatives in the problem, and consequently the equation is formally singular at the critical point $x = x_*$.

Our method follows Van Kampen's treatment of plasma oscillations in the Vlasov-Poisson equation (we give his solution in the next section). We first divide through by the coefficient $x - x_*$. Such an operation is not mathematically defined, however; the resulting equation has a right-hand side which is not a well-behaved function of position. We attach meaning to the expression by interpreting it in a distributional sense, and we use the Cauchy principal value, $\mathcal{P}$, to handle the singular term. Then,

$$\mathcal{L}_x\phi = \mathcal{P}\,\frac{\mathcal{M}_x\phi}{x - x_*} + \mathcal{C}(x_*)\delta(x - x_*), \tag{2}$$

where $\mathcal{C}$ is an arbitrary amplitude and $\delta(x)$ is the delta function.

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The solution of a differential equation like (2) with a delta-function inhomogeneous term is most easily found by converting that equation to an integral equation. In order to achieve this result, we introduce the Green function of the operator $\mathcal{L}_x$ which we denote by $\mathcal{K}(x, x')$. Then we can write (2) in the form,

$$\phi(x) = \mathcal{P} \int_{\mathcal{D}} \mathcal{K}(x, x') \frac{\mathcal{M}_{x'}\phi(x')}{x' - x_*} dx' + \mathcal{C}(x_*)\mathcal{K}(x, x_*). \tag{3}$$

Equation (3) is an inhomogeneous integral equation. Its kernel, $(x' - x_*)^{-1}\mathcal{K}(x, x')\mathcal{M}_{x'}$, is singular at the critical point, and we could use the methods of singular integral equation theory[10] to solve it. However, as yet, this is no clear simplification of the problem, but we have not specified $\mathcal{C}$. At our disposal is a normalization condition. If we fix the normalization of the eigenfunction, we determine $\mathcal{C}$. Certain normalizations lead to simplifications in our problem. In particular, if we require that

$$\int_{\mathcal{D}} \mathcal{L}_x \phi \, dx =: \Lambda, \tag{4}$$

we observe that

$$\mathcal{C} = \Lambda - \mathcal{P} \int_{\mathcal{D}} \frac{\mathcal{M}_x \phi}{x - x_*} \, dx. \tag{5}$$

If we substitute this relation into our integral problem (3), we see that

$$\phi = \Lambda \mathcal{K}(x, x_*) + \int_{\mathcal{D}} \mathcal{F}_{x_*}(x, x')\phi(x')dx', \tag{6}$$

where

$$\mathcal{F}_{x_*}(x, x') = \frac{\mathcal{K}(x, x') - \mathcal{K}(x, x_*)}{x' - x_*} \mathcal{M}_{x'} \tag{7}$$

is a kernel with a parametric dependence on x_*. This is another integral equation, but, whereas (3) was singular, (6) is not. In other words, our normalizing operation (4) has regularized the integral problem. In fact equation (6) is a standard Fredholm equation.[11]

Fredholm theory tells us that equation (6) has two kinds of solutions. If there are homogeneous solutions that satisfy

$$\tilde{\phi} = \lambda \int_{\mathcal{D}} \mathcal{F}_{x_*}(x, x')\tilde{\phi}(x')dx', \tag{8}$$

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for certain values of λ, and if there are no values of x_* for which $\lambda = 1$, then there are no homogeneous solutions to (6). Fredholm theory then demonstrates that there is a unique particular solution. If homogeneous solutions do exist with $\lambda = 1$ for certain values of x_*, then a solution only exists if the inhomogeneous term satisfies an additional relation (the so-called Fredholm Alternative), and it is not unique.

Provided, we have no homogeneous solutions, then, the method allows us to construct singular eigenfunctions by solving a simpler, regular problem. Moreover, it would establish the existence of a unique solution of the kind we seek. Sometimes it can be verified directly that no such homogeneous solutions exist; also, numerical techniques can be used. Should homogeneous solutions exist, precautions must be taken to assure a unique and bounded solution to our original problem. One way to do this is by suitably scaling the amplitude of the singular mode, Λ. In particular, we can select $\Lambda = \mathcal{D}(x_*)\widehat{\Lambda}$, where the function $\mathcal{D}(x_*)$ vanishes at the eigenvalues for which there exists a homogeneous solution (it is the Fredholm determinant), and $\widehat{\Lambda}$ is bounded. This scaling forces the inhomogeneous term to vanish whenever a homogeneous solution appears, and so we always find a unique, bounded eigenfunction.

III. PLASMA OSCILLATIONS

We first apply the method to the one-dimensional, Vlasov-Poisson equation, which reproduces Van Kampen's original solution. In this problem we have an equilibrium described by a distribution function, $f_0(v)$, where v is the phase-space velocity coordinate. Infinitesimal perturbations of the distribution function, $f(x, v, t)$, satisfy the linearized Vlasov equation together with the Poisson equation for the electric field, $E(x, t)$. Because the equilibrium is independent of the spatial coordinate x, we can Fourier transform the equations, or, equivalently, look for solutions where the perturbations of the distribution function and the electric field are, respectively, of the forms $f(v)\exp[ik(x - ut)]$ and $E\exp[ik(x - ut)]$, where k is a

wavenumber and u is the wavespeed. The governing equations are then,

$$(u - v)f + \frac{eE}{m} \frac{df_0}{dv} = 0 \tag{9}$$

and

$$k^2 E = -4\pi e \int_0^\infty f(v)dv, \tag{10}$$

where e and m are the particles' charge and mass. If we take a solution of the form (2) for f, by dividing equation (9) by $(u - v)$, we obtain,

$$f = \frac{e}{m} \mathcal{P} \frac{E f_0'}{u - v} + \mathcal{C}(u)\delta(u - v). \tag{11}$$

If we integrate this expression over v, and use the normalization indicated by equation (4), we find that,

$$\mathcal{C} = \Lambda - \frac{4\pi e^2}{mk^2} \mathcal{P} \int_0^\infty \frac{f_0'}{u - v} dv. \tag{12}$$

In this problem, there is no dispersion relation; solutions exist for all eigenvalues, u. The associated eigenfunctions are given by (11) with (12). It is not necessary to solve a Fredholm problem in this case because the Poisson equation has the simple, "degenerate" kernel, $\mathcal{K} \equiv 1$. The kernel of the Fredholm equation therefore vanishes everywhere, and $\phi \equiv 1$.

IV.   INCOMPRESSIBLE SHEARS

A slightly more complicated example is the problem considered by Rayleigh.[12] He studied an inviscid fluid configuration consisting of a shear flow contained within a channel. If we denote x and y as the spatial coordinates along and across the channel, then a flow with velocity profile $U(y)$ within the domain $-\infty < x < \infty$ and $-1 < y < 1$ exists as an equilibrium of the two-dimensional Euler equations. Infinitesimal perturbations about this equilibrium can be taken to be of the form, $u(y) \exp ik(x - ct)$, $v(y) \exp ik(x - ct)$ and $p(y) \exp ik(x - ct)$ for the two velocity components and pressure fluctuation. The eigenvalue

is c, and there is a critical layer at $y = y_*$, at which point $U(y) = U(y_*) = c$. The perturbations satisfy the equations,

$$ik(U - c)u + U'v = -ikp, \tag{13}$$

$$ik(U - c)v = -p' \tag{14}$$

and

$$iku + v' = 0, \tag{15}$$

where the equilibrium density has been set to unity. By representing the perturbation's velocity field in terms of a stream function, $\psi(y)$, we can formally manipulate these expressions into Rayleigh's equation,

$$(U - c)(\psi'' - k^2\psi) = U''\psi. \tag{16}$$

Rayleigh's equation is a relatively well-studied equation.[2] Various integral relations can be derived from it. These indicate that there are no discrete eigenmodes unless there is an inflexion point, $U'' = 0$, somewhere within the flow. Such modes are either purely real, in which case their critical layers lie exactly at the inflexion point, or they are complex, indicating decaying/growing pairs. All other neutral modes must have critical layers that lie within the channel; they are intrinsically irregular and we expect them to comprise a continuum, *i.e.* the singular, continuous spectrum.

Rayleigh's equation is clearly of the form of equation (1), provided $U(y)$ is a monotonic function. If we assume this to be the case, then the generalization of the Van Kampen eigenfunction is,

$$\omega(y) = \mathcal{P}\frac{U''(y)\psi(y)}{U(y) - c} + \left[\Lambda - \mathcal{P}\int_{-1}^{1}\frac{U''(y')\psi(y')}{U(y') - c}\,dy'\right]\delta(y - y_*), \tag{17}$$

which is the vorticity fluctuation, and ψ satisfies the Fredholm equation (6) (but in the variable y), with

$$\mathcal{F}_{y_*}(y, y') = \frac{\mathcal{K}(y, y') - \mathcal{K}(y, y_*)}{U(y') - U(y_*)}U''(y'), \tag{18}$$

and $\mathcal{K}(y, y')$ being Green's function of the two-dimensional Laplace equation, i.e.

$$\mathcal{K}(y, y') = \begin{cases} -\sinh k(1-y)\sinh k(1+y')/k\sinh 2k & \text{for } y > y', \\ -\sinh k(1-y')\sinh k(1+y)/k\sinh 2k & \text{for } y \leq y'. \end{cases} \tag{19}$$

Some solutions to the Fredholm problem are shown in Fig. 1. These are computed for the flow profiles, $U(y) = y + y^3/10$ and $U(y) = y + y^3$. The continuity of fluid elements requires that ψ remains continuous across the channel, but it does have a discontinuity in slope. In these cases, the absence of homogeneous solutions to our Fredholm problem can be established numerically by constructing the Fredholm determinant.[11] Hence, we set Λ to unity.

It is not necessary to assume that the profile is monotonic. If $U(y)$ is multivalued in places, we have multiple critical layers for the corresponding wave speeds. This complicates the construction of singular eigenfunctions, but it can still be done, with some modification to the method.[5]

V. SHEARS IN SHALLOW WATER

A more complicated situation than Rayleigh's problem is when the shearing fluid is compressible. An example in which the two-dimensional character of the configuration is retained is for the flow of shallow water through a channel, a physical situation of interest in an oceanographical context.[13,14,15]

From a physical point of view, we expect a different spectrum for the stability eigenvalue problem, because compressibility introduces an additional degree of freedom into the dynamics of the fluid. In particular, in Rayleigh's problem, there are only vortical motions. For compressible fluid we also expect sound waves, or, in the shallow water system, surface gravity waves. (The similarity between the acoustical dispersion relation of a two-dimensional compressible fluid and that of the surface gravity waves of a shallow fluid system has led to some confusion in the past.[16])

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In addition to the singular modes, we therefore anticipate a new class of modes, and from the earlier studies these are expected to compose a discrete portion of the complete eigenspectrum.

The equations for perturbations to a shearing, shallow fluid of undisturbed, uniform depth and velocity profile $U(y)$ (using a coordinate system like above and assuming monotonic velocity profiles), are[13]

$$ik(U - c)u + U'v = -\frac{ik}{Fr^2}\, h, \tag{20}$$

$$ik(U - c)v = -\frac{1}{Fr^2}\, h' \tag{21}$$

and

$$ik(U - c)h + iku + v' = 0, \tag{22}$$

where the velocity components are again given by u and v, h is the y-dependent piece of the depth perturbation, and the dependence $\exp ik(x - ct)$ has again been introduced. These equations have been scaled to make them dimensionless; this introduces the Froude number, Fr, which is the ratio of the characteristic, mean flow speed to a typical surface gravity wavespeed (or a characteristic Mach number of a two-dimensional, compressible fluid).

From these expressions we can derive a second-order equation for h; namely

$$(U - c)\left\{h'' + k^2[Fr^2(U - c)^2 - 1]h\right\} = 2U'h'. \tag{23}$$

Another relation of interest comes from the vorticity and continuity equations,

$$(U - c)(v'' - k^2 v) - U''v = -ik[(U - c)^2 h]'. \tag{24}$$

The incompressible limit, in which we should recover Rayleigh's equation, is obtained by taking $Fr \to 0$ and $h \to 0$, but with the ratio h/Fr^2 finite. Accordingly, (24) reduces to Rayleigh's equation since $ik\psi = v$.

The next step is to divide through by a factor of $U - c$, take a principal part and add a delta-function. In our current example, we need to be a little careful about how we should

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accomplish this. In analogy with Rayleigh's equation, we can clearly divide the second relation (24) by $U - c$ and proceed along the lines outlined by the method. A similar procedure for (23) does not seem to work for the following reasons.

Equation (23) contains a singular point, namely $y = y_*$. About that point, we have Frobenius expansions of the form[13]

$$h \sim (y - y_*)^3 \sum_{n=0}^{\infty} a_n (y - y_*)^n$$

and

$$h \sim -\frac{k^2 U''(y_*)}{2U'(y_*)} (y - y_*)^3 \log(y - y_*) \sum_{n=0}^{\infty} a_n (y - y_*)^n + \sum_{n=0}^{\infty} b_n (y - y_*)^n. \qquad (25)$$

If, for the moment, we consider Couette flow, for which $U'' = 0$, then we observe that the singular point in the equation for h is entirely regular. In other words, it is a removable singularity (in fact the equation for u in this case contains no singular terms). Moreover, the solutions of (23) form a complete basis set of regular functions (the surface gravity modes). There does not seem to be any need, then, to include singular eigenfunctions. However, in order to determine the evolution of the fluid, we need to represent both an initial height and an initial velocity field. This requires two independent sets of basis functions, and the surface gravity modes alone are in general insufficient. The singular mode spectrum is still therefore needed in order to complete the problem.

Even though there is no principal-value singularity in equation (23), we could nevertheless add a delta function on dividing by $U - c$. This leads to a particular solution for h that might represent the singular eigenmode. Indeed, that solution generally has a discontinuity in its first derivative of h. However, such an eigensolution is ruled out if we use the physical requirement that the pressure gradient be continuous. Even were this objection not to preclude such solutions, we would then be forced to work with highly divergent vorticity fluctuations (in the sense that the singularity at the critical point is not just a simple pole). Moreover, these appear to be of no relation to the singular modes of Rayleigh's equation,

yet Rayleigh's solutions should be recovered in the incompressible limit.

The resolution of this difficulty lies in equation (24) and the fact that (23) was derived from the continuity equation (22). The continuity equation contains information only about the divergence of the velocity. In deriving equation (23) we therefore omit crucial, singular details of the vorticity field. That field evolves according to (24). Applying our procedure to this equation gives,

$$v'' - K^2 v + 2U' \frac{Fr^2 k^2}{K^2} \left[(U - c)v' - U'v \right] = \mathcal{P} \frac{U''v}{U - c} + \mathcal{C}\delta(y - y_*). \qquad (26)$$

where

$$K^2 = k^2 \left[1 - Fr^2(U - c)^2 \right]. \qquad (27)$$

This is the shallow water version of Rayleigh's equation. In writing this equation, we have introduced another singular term, namely the term with a denominator of K^2. That quantity vanishes at the points $y = y_t^{\pm}$, for which

$$U(y_t^{\pm}) = c \pm \frac{1}{Fr}. \qquad (28)$$

These singular terms have no counterpart in the equation for h, (23), reflecting how they are removable singularities (the Frobenius expansions for v about these singular points are both purely regular). Formally we can write the equation for v in the form of equation (2), thence solve it according to our method. This requires us to build a Green function for the operator on the left-hand side of (26), but then our Fredholm problem is straightforward to solve.

Buried in equation (26) are both the eigensolutions of the continuum, and the discrete modes which correspond to the surface gravity waves. In addition, should the flow profile violate Rayleigh's criterion, there may be discrete solutions related to the vortical instabilities of the incompressible problem. When the Froude number is very small, we expect that the two types of solutions are well separated on the spectral plane. For larger values of Fr, the distinction may not be so clear.

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This particular problem is interesting in that it provides an example where we have to be a little careful about simply writing down principal values and delta functions in order to find singular eigenfunctions. Applying the method to the equation for h produces ambiguous results; the equation for v seems to be the best way to go. None the less, there is a certain amount of freedom in choosing which equation to work with, or into which physical quantity we should introduce a principal-value singularity or delta function. At the end of the day, it is how well the resulting eigenfunctions behave as a unique, complete basis set that determines the optimal choice.

The problem also highlights another ambiguity. We decided not to treat the singularities occurring equation (26) at the points $y = y_t^{\pm}$ by the method since they were removable (of zero dividing zero form) and so no principal-value piece was necessary. However, for linear shear in both compressible and incompressible fluid, $U'' = 0$, and there is no principal-value singularity in the equation for v even at the critical layer. In Rayleigh's equation, the delta function piece must still be added into the equation in order to find a solution (this is Høiland and Fjørtøft's result[2]). Similarly, in the shallow water equation (26) we could also retain the delta function, but now there is no distinction between the importance of the critical point, and the other, removable, singular points $y = y_t^{\pm}$. In principle, then, we could add alternative terms, $\mathcal{E}\delta(y - y_t^{\pm})$, to the equation. This would lead to another two sets of singular eigenfunctions.

In practice it is unlikely that the new sets of solutions are as useful as the original one because we expect continuum eigenfunctions for every wave speed that matches the mean flow. From (28) we see that this would give solutions with singularities outside the channel. From a physical point of view, we interpret $y_t^{\pm}$ to be the turning points for the surface gravity waves (the points for which these waves are reflected). There is no obvious reason why we should allow the eigenfunctions to be irregular at these points.

VI. INCOMPRESSIBLE VORTICES

As a prelude to discussing an astrophysical application of our method we now discuss another simple situation. This is the incompressible, two-dimensional vortex. Kelvin[17] considered such configurations with piecewise continuous vorticity distributions. These equilibria support interfacial-type discrete modes which were of interest to Kelvin, but not directly relevant to the continuum modes we derive here. More recently, the stability of a two-dimensional, incompressible vortex has regained importance since it has become feasible to experimentally simulate the dynamics of such a configuration with an electron plasma.[18]

In polar coordinates, (r, θ), we have an equilibrium given by an angular velocity distribution, $V(r) = r\Omega(r)$. For the sake of simplicity we again take monotonic profiles for Ω. Perturbations to the vortical structure can be described by a streamfunction, $\psi(r) \exp im(\theta - \nu t)$, where m is the azimuthal quantum number, for which the perturbed velocity components are given by

$$u = -\frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad \text{and} \qquad v = \frac{\partial \psi}{\partial r}. \tag{29}$$

Then Rayleigh's equation for ψ can be written in the form,

$$r(\Omega - \nu)\left[\frac{1}{r}(r\psi')' - \frac{m^2}{r^2}\psi\right] = \zeta'\psi, \tag{30}$$

where the mean vorticity is given by

$$\zeta = \frac{1}{r}(r^2\Omega)'. \tag{31}$$

A straightforward application of our method yields the singular eigenfunctions,

$$\omega = -\mathcal{P}\frac{\zeta'\psi}{r(\Omega - \nu)} - \mathcal{C}\delta(r - r_*), \tag{32}$$

where r_* denotes the critical ring, or the co-rotation radius, for which $\Omega(r_*) = \nu$. The streamfunction ψ satisfies the Fredholm equation,

$$\psi(r) = \Lambda\mathcal{K}(r, r_*) + \int_0^\infty \frac{\mathcal{K}(r, r') - \mathcal{K}(r, r_*)}{\Omega(r) - \Omega(r_*)}\zeta'(r')\psi(r')dr', \tag{33}$$

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and the Green function of Laplace's equation in these coordinates is

$$\mathcal{K}(r, r') = -\int_0^\infty J_m(kr) J_m(kr') \frac{dk}{k}.$$ (34)

The stability criterion for the vortex is simply that ζ' not vanish. This also excludes any discrete modes in the spectrum.

VII.   DIFFERENTIALLY ROTATING FLUID DISKS

Compressible generalizations of Kelvin's vortices have lately prompted interest regarding noise-generation problems in aerodynamic contexts[19] and in disk theory in astrophysics.[20,21] In the latter situations, we consider slender configurations like the shallow-water shears considered above. These disks are essentially two-dimensional, being hydrostatically stratified in the vertical, and variations in thickness provide the most important effects of compressibility. Equilibria are determined by a surface density distribution, $\Sigma(r)$, in addition to the rotation rate, $\Omega(r)$. If we consider barotropic configurations which are not self-gravitating, but rotate about some central mass, then a disturbance can be represented (to leading order in thinness), in polar coordinates, by the velocity components $u(r) \exp im(\theta - \nu t)$ and $v(r) \exp im(\theta - \nu t)$, and by perturbation in the enthalpy, $h(r) \exp im(\theta - \nu t)$, in the midplane of the disk. The equations of motion can be written in the form,

$$im(\Omega - \nu)u - 2\Omega v = -h',$$ (35)

$$im(\Omega - \nu)v + \zeta u = -\frac{im}{r} h$$ (36)

and

$$im(\Omega - \nu)\sigma + \frac{1}{r}(r\Sigma u)' + \frac{im}{r}\Sigma v = 0,$$ (37)

where the surface density perturbation, σ, is related to the midplane enthalpy by

$$\sigma = \frac{\Sigma h}{c_s^2},$$ (38)

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with c_s a function of the local vertical structure of the disk, or the local, surface gravity wavespeed. The undisturbed vorticity has again been represented by $\zeta = (r^2\Omega)'/r$.

From these equations we can derive the relation,

$$(\Omega - \nu) \left[\frac{1}{r} \left(\frac{r\Sigma}{D} h' \right)' - \frac{m^2 \Sigma}{r^2 D} h - \frac{\Sigma}{c_s^2} h \right] + \frac{2}{r} \left(\frac{\Omega\Sigma}{D} \right)' h = 0, \qquad (39)$$

where

$$D = 2\Omega\zeta - m^2(\Omega - \nu)^2. \qquad (40)$$

This is the counterpart of equation (23) for the shallow-water shear. Like that equation, it is singular at the critical point, $r = r_*$ (the singularities for which $D = 0$, the so-called Lindblad resonances are removable; these are the analogues of the turning points (28) of the shallow-water problem discussed earlier), except for the case in which the potential vorticity,

$$Q = \zeta/\Sigma, \qquad (41)$$

is uniform. Then there are two regular solutions at the critical ring. This case corresponds to the Couette, shallow-water shear flow example. Like that example, there would therefore appear to be no need for singular eigenfunctions, and the regularity of the pressure derivatives in (35) and (36) precludes us from dividing through by $\Omega - \nu$ and adding a delta function. In other words, once again we cannot straightforwardly apply the method to equation (40).

In order to find the continuum modes we first consider an inelastic approximation to the equations. This is obtained by taking the limit $c_s \to \infty$. Then, the surface gravity waves are filtered out of the problem and the continuity equation becomes,

$$\frac{1}{r} (r\Sigma u)' + \frac{im}{r} \Sigma v = 0. \qquad (42)$$

We can introduce a streamfunction, ψ, to solve this equation. It is given by

$$u = \frac{im}{r\Sigma} \psi \qquad \text{and} \qquad v = -\frac{1}{\Sigma} \psi'. \qquad (43)$$

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In terms of this variable, we write the perturbed potential vorticity equation as

$$r(\Omega - \nu) \left[\frac{1}{r} \left(\frac{r}{\Sigma} \psi' \right)' - \frac{m^2}{r^2 \Sigma} \psi \right] = Q' \psi. \qquad (44)$$

This is a Rayleigh-like equation which generalizes the incompressible version, equation (30). Since the source of the singularity is now evident we can apply our method to (44) and derive singular eigenfunctions for this inelastic approximation. In particular, we have

$$q = \frac{1}{r} \left(\frac{r}{\Sigma} \psi' \right)' - \frac{m^2}{r^2 \Sigma} \psi = \mathcal{P} \frac{Q' \psi}{r(\Omega - \nu)} + \mathcal{C} \delta(r - r_*) \qquad (45)$$

for the potential vorticity fluctuation, with ψ determined from a suitable Fredholm equation.

To return to the full problem, we again write down the potential vorticity equation. Without approximation it is,

$$r(\Omega - \nu) \left[\frac{1}{r} \left(\frac{r}{\Sigma} \psi' \right)' - \frac{m^2}{r^2 \Sigma} \psi \right] = Q' \psi + (\Omega - \nu) \mathcal{S}, \qquad (46)$$

where now $\psi = r \Sigma u$, and

$$\mathcal{S} = -r \left\{ \left[r^2 (\Omega - \nu) \frac{\Sigma}{c_s^2} h \right]' + mr \frac{\zeta \Sigma}{c_s^2} h \right\}. \qquad (47)$$

This is the generalization of the inelastic equation (44), and, on writing h and h' in terms of ψ and its derivative, corresponds to equation (26) of the shallow water example. In analogy with the inelastic equation, we can divide (46) by $\Omega - \nu$, treating the singular term by its principle value, and add a delta function. Eventually we solve a Fredholm problem for ψ.

As in the shallow water problem, equation (46) contains two types of modes that are easily distinguished in the inelastic limit. In the disk problem, one is tempted to call the vortical modes either Rossby waves or r modes, in analogy with the nomenclature of theory of geophysical fluid dynamics or stellar pulsation. These form a continuum delimited by the range of rotation speed, but, as suggested by Schutz and Verdaguer,[22] there may also be some discrete modes as a result of topographical influences (gradients in surface density),

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which break the Laplacian structure of the left-hand side of (30). Moreover, when the potential vorticity reverses sign, we violate the generalization of Rayleigh's criterion[23] and unstable/decaying mode pairs may appear.

VIII.   THE USES OF SINGULAR EIGENFUNCTIONS

In previous sections we have constructed eigenfunctions of the singular continuum for a variety of idealized problems. These eigenfunctions are characterized by irregular shapes; principal-part singularities and delta-functions. As a result it is not possible to add a single continuum mode with any finite amplitude onto the basic equilibrium state without immediately leaving the linear regime and introducing singular behaviour at the critical layer. An integral superposition of singular modes, however, with a distribution of amplitudes, $A(x_*)$ say, such as

$$S(x) = \mathcal{P} \int_{\mathcal{D}} \frac{A(x_*)\mathcal{M}_x\phi(x;x_*)}{x - x_*} \, dx_* + \left[1 - \mathcal{P} \int_{\mathcal{D}} \frac{\mathcal{M}_{x'}\phi(x';x)}{x' - x} \, dx' \right] A(x), \qquad (48)$$

need not be so pathological[3,4] (principal-value integrals are well-behaved functions). We have introduced a second dependence on x_* into the arguments of ϕ in equation (48) to explicitly reveal its implicit dependence through the Fredholm problem (6), and set $\Lambda = 1$ to make the form of the superposition more transparent.

Integral superpositions like (48) can be used to represent an initial condition, which enables us to consider the initial-value problem. In fact, methods borrowed from singular integral theory[10] allow us to invert the integral relation (48), and to write amplitude distribution, $A(x)$, in terms of the initial condition, $S(x)$. This procedure is typically complicated by the presence of discrete modes, but often we can prove that the combination of continuum and discrete modes can represent the initial disturbance.[5] This establishes that the combination of the discrete and continuous eigenfunctions form a complete set of basis functions.

Once we have a superposition like (48) to represent an initial condition, we can determine the evolution for all subsequent time and show the equivalence with the solution of the problem using Laplace transforms.[4] This amounts to reinstating the temporal dependence, $\exp(-ikct)$ or $\exp(-im\nu t)$ within the integral superposition (48). Integrals of various physical quantities over the domain (such as the total vorticity across the channel) then contain factors of the form, $\exp ikU(x)t$ within their integrands. By the Riemann-Lebesgue lemma, these integrals must vanish as $t \to \infty$ (unless there is some additional irregularity), revealing the usual phase mixing property of an ideal system. In many situations, the integrands can be analyzed further to estimate the asymptotic temporal dependence. If this is exponential, we observe the fluid analogue of Landau damping, but in general, that phenomenon is overshadowed by algebraic decay.

Another application of a complete set of singular eigenfunctions is in perturbation theory. Superpositions like (48) can be posed as approximate solutions about which we can open asymptotic expansions. We can then attempt weakly nonlinear theory and investigate the ideal limit of some dissipative systems. These amount to avenues we intend to explore in the future.

A final issue that we have not mentioned until now is Hamiltonian structure. The ideal fluid or plasma equations can be recast as Hamiltonian field theories. Typically, these theories are not canonical in the sense that they do not have a standard Poisson bracket.[24] However, by defining a transformation to Hamiltonian coordinates based on the amplitudes of the singular eigenfunctions, the Poisson bracket can be transformed into a canonical form. Moreover, in these linear "normal" coordinates, the Hamiltonian itself is diagonal and of action-angle form.[25,26,5] This indicates that the singular eigenfunctions are in some sense the intrinsic degrees of freedom of the linear fluid or plasma system, like the normal coordinates that describe the modes of vibration in the classical triatomic molecule. However, the degrees of freedom are not discrete in our case; we have a continuum analogue in an infinite-dimensional

Hamiltonian system.

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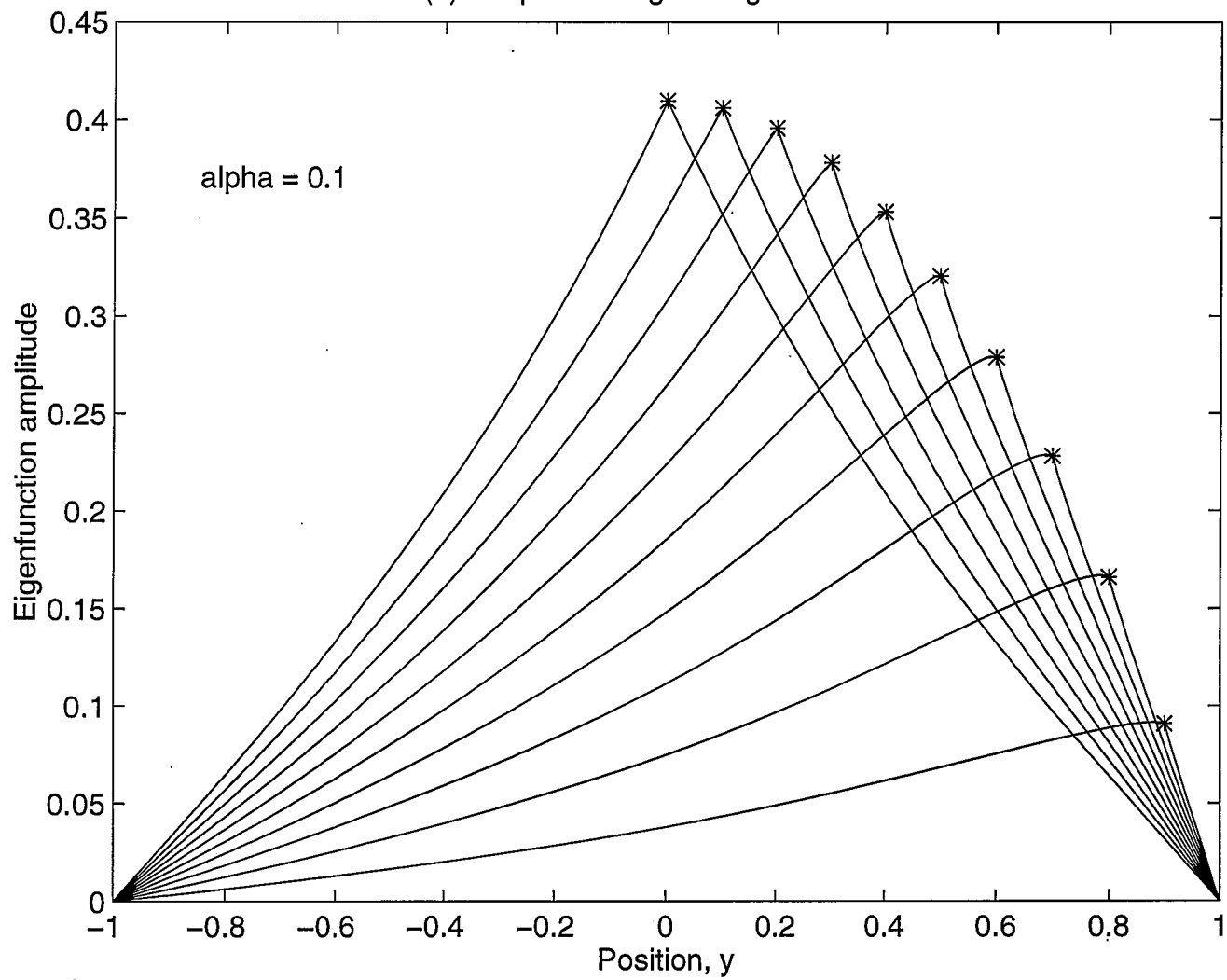11. Tricomi, F.G. *Integral Equations*, (Dover, New York, 1985).

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FIGURE CAPTIONS

Fig. 1. A selection of singular eigenfunctions, $\psi(y)$, for $U(y) = y + \alpha y^3$, with (a) $\alpha = 0.1$ and (b) $\alpha = 1$. Also, $k = 1$. Streamfunctions of various modes with different critical layers are displayed. The critical-layer amplitudes are indicated by stars.

(a) Shapes of singular eigenfunctions

(b) Shapes of singular eigenfunctions

alpha = 1.