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Analytical Studies of the Effects of Charge-Exchange on a Magnetized Plasma (THESIS)

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Analytical Studies of the Effects of Charge-Exchange on a Magnetized Plasma

by

Mark Darren Calvin, B.S.

DISSERTATION

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

Doctor of Philosophy

THE UNIVERSITY OF TEXAS AT AUSTIN

May, 1994

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Approved by

Dissertation Committee:

To my wife.

In general, I would like to acknowledge the Institute for Fusion Studies at The University of Texas at Austin for employment and resources during my graduate studies, and the Department of Energy for its support of the IFS. I am grateful to Herb Berk for being first to offer me support and guidance at the IFS. I would also like to thank the members of my doctoral committee for their assistance in accomplishing my academic goals. Specifically, I would like to acknowledge my supervising professor, Richard Hazeltine, as an inspiring teacher, a brilliant and prolific physicist, and a dynamic and conscientious administrator whom I can only dream of emulating.

I am also indebted to members of the Fusion Research Center at The University of Texas at Austin for their invaluable aid and instruction. I thank Prashant Valanju for his infinite patience, his willingness to consider any problem, and his practice of joyfully applying his vast yet understated experience to guide the student. Emilia Solano has shown me the importance of continually keeping sight of empirical reality in the critical development of physical theory. I also thank Bill Rowan and Andy Meigs for taking my work seriously enough to experimentally evaluate it.

Lastly, my wife has supported me in this effort in every conceivable way. I can only thank her with my support and devotion. She is my life.

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The University of Texas at Austin, 1994

Supervisor: Claude W. Horton

We analytically calculate the neutral particle distribution and its effects on ion heat and momentum transport in three dimensional magnetized plasmas with arbitrary temperature and density profiles. A general variational principle taking advantage of the simplicity of the charge-exchange (CX) operator is derived to solve self-consistently the neutral-plasma interaction problem. To facilitate an extremal solution, we use the short CX mean-free-path (λ_x) ordering. Further, a non-variational, analytical solution providing a full set of transport coefficients is derived by making the realistic assumption that the product of the CX cross-section with relative velocity is constant. The effects of neutrals on plasma energy loss and rotation appear in simple, sensible forms. The presence of ionized impurities in the plasma are also considered, and the effects of CX drag and ion-impurity collisions on plasma flows in the short λ_x

regime are presented. Finally, the long λ_x regime is analyzed in slab geometry by finding an appropriate Green's function for the neutral kinetic equation and by solving recursively for the neutral distribution function. Our results are found to agree favorably with previous work.

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Chapter 1.

Introduction

Conventional transport theory of high-temperature plasmas treats only completely ionized plasmas with no significant numbers of neutrals. However, the bulk transport is significantly affected by the edge, where the plasma is not completely ionized. Edge neutrals, whose motion is not affected by the magnetic field, interact with ions through charge-exchange (CX) and impact ionization. During a CX collision a neutral and ion exchange identities. It thus appears as if the neutral has scattered, suddenly changing its speed and direction. A neutral may continue this CX "scattering" many times before it is finally ionized due to impact with an electron. As Sacharov [1] first pointed out, it is in this way that low energy neutrals near the wall can gain energy through frequent CX scatterings and penetrate into the plasma interior, where they may affect plasma transport and rotation. Furthermore, CX may be an important mechanism for cooling the divertor plasmas of future fusion reactors.

Although it would be desirable to solve the full plasma-neutral interaction problem analytically, the near equality of the plasma and hot neutral density scale lengths makes it very hard to do so. Most analytic solutions available [1-7] work only in a one-dimensional slab geometry. They also assume prespecified, fixed, uniform plasma density and temperature profiles, as well as a uniform external neutral source. Neither the effects of the neutrals on the plasma nor the changes in the external neutral source due to changes in plasma flux are calculated. These restrictions severely limit their use in experiments.

We will show that the scope of the analytical solutions can be broadened considerably. We can solve the neutral-plasma interaction problem in a self-consistent manner without assuming any spatial symmetry, and without neglecting the effects of neutrals on the plasma. This is achieved as follows:

After reviewing neutral kinetic theory, we will show that CX obeys an H-theorem. Then the simplicity of the CX collision operator is used to set up a general variational principle for the solution of the neutral kinetic equation. Although general, this method is somewhat cumbersome since the forms involved are spatially nonlocal and not self-adjoint.

In the short CX mean-free-path (λ_x) regime, the process of finding an extremal solution is then simplified by using the smallness of the ratio of the ionization to the charge exchange mean-free-path. To each order in this small parameter, variationally accurate transport coefficients for neutral particle, energy and momentum fluxes can be read off from the extremized entropy production rate.

Furthermore, upon making the realistic assumption that the product of CX cross-section with relative velocity $\sigma_x|\mathbf{v} - \mathbf{v}'|$ is constant, we calculate the neutral entropy production and present the full set of neutral transport coefficients. A similar calculation is given by Vekshtein and Ryutov [8].

We then discuss the effects of CX on ion fluid behavior and transport. Momentum and energy moments of the ion and neutral kinetic equations are analyzed in the short λ_x regime. The effects of neutrals on plasma energy loss and rotation appear in particularly simple, intuitively sensible forms. After examining the contribution of neutrals to ion viscosity, we find that neutral viscosity dominates ion viscosity everywhere, and in the edge region by a large factor.

Since the rotation measured in tokamaks is that of ionized impurities and not that of hydrogen ions, we include impurities in the neutral-plasma interaction problem and present a simple drift-kinetic derivation of expressions for the poloidal ion and impurity flows in the presence of CX drag and ion-impurity collisions in the short λ_x regime.

Finally, we examine the plasma-neutral problem in the case where the ratio of the ionization to the CX mean-free-path is large. After finding a Green's function for the exact neutral kinetic equation in slab geometry, we introduce suitable expansions and calculate recursively the neutral distribution function in this long CX mfp limit. We find that our results compare favorably with previous work.

Chapter 2.

Neutral Kinetic Theory

To begin we briefly review neutral kinetic theory [9]. Neutral particles are subject to three inelastic processes:

1. Charge-exchange (CX) collisions locally conserve both ions and neutrals; in effect, only energy and momentum are exchanged. We write the CX operator as

$$X(f,g) \equiv \int d^3v'\, \sigma_x |\mathbf{v}' - \mathbf{v}| \{f(\mathbf{v})g(\mathbf{v}') - f(\mathbf{v}')g(\mathbf{v})\} \ , \tag{2.1}$$

where f and g are the ion and neutral particle distribution functions respectively. Charge-exchange occurs with frequency ν_x. σ_x is of course the CX cross-section. Notice that

$$X(g,f) = -X(f,g) \ . \tag{2.2}$$

2. Impact ionization, a neutral sink, is proportional to g and to n_e, the electron density (impact ionization due to ions is negligible). We suppress dependence on the electron distribution and write the impact ionization rate as $\nu_z g$.

3. Recombination, a neutral source, is proportional to the product of f and n_e; we denote it by $\nu_r f$. Other neutral sources, such as from gas puffing, are distinctly local.

Neutrals are not subject to mean forces, and their elastic collision rate is negligible. Hence the kinetic equation for g is

$$\frac{\partial g}{\partial t} + \mathbf{v} \cdot \nabla g = -X(g, f) - \nu_z g + \nu_r f \ . \tag{2.3}$$

This is instructively compared to the corresponding ion kinetic equation,

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} - C(f) = -X(f, g) + \nu_z g - \nu_r f \ . \tag{2.4}$$

Here $\mathbf{a}$ is the acceleration due to electric and magnetic fields and C the Coulomb collision operator. The point is that ion population changes due to recombination or impact ionization precisely balance neutral changes.

In discussing charge-exchange, it is advantageous to track the charge flow, rather than individual particles. Thus a CX event is viewed as an exchange of momentum and energy between a neutral and an ion, each particle maintaining its species identity. Because the initial momenta of the colliders are not correlated, CX yields large-angle scattering events. From this point of view a neutral will survive any number of CX "scatterings"; it disappears only upon impact ionization (or wall interaction), so that the neutral lifetime is $1/\nu_z$.

Chapter 4 of the present work provides a variational principle for the general solution of (2.3). However, most of our results, including the explicit transport formulae, pertain only in the special case

$$\nu_z \ll \nu_x \ , \tag{2.5}$$

in which each neutral suffers many charge-exchanges in its lifetime. In typical tokamak experiments ionization is indeed slower than CX although not always by a large factor. For example, in TEXT [10] ν_x exceeds ν_z by factors of 3 to 5. (The recombination rate ν_r is typically somewhat smaller still.)

The rates ν_x and ν_z are conveniently measured in terms of the corresponding mean-free-paths, $\lambda = v_n/\nu$, where v_n is the thermal speed of the neutral population. We shall refer to $\lambda_x = v_n/\nu_x$ as the CX mean-free-path; it is measured in centimeters in most tokamak discharges. The impact ionization length λ_z is a measure of the total path travelled by a neutral; notice that in our case this path is far from linear since (2.5) describes a neutral that changes direction several times before ionization.

The third length of interest is the scale length L for neutral density variation. A random walk argument [9] using (2.5) shows that

$$L \sim (\lambda_x \lambda_z)^{\frac{1}{2}} \tag{2.6}$$

and therefore that

$$\lambda_x \ll L \ll \lambda_z \, . \tag{2.7}$$

In other words, the ordering (2.5) is consistent with the short CX mean-free-path regime. In Chapter 5 we focus attention on the short mean-free-path limit. Our argument will show that these are self-consistent orderings.

Chapter 3.

H-theorem

Here we adopt classical arguments to establish some important properties of the CX operator (2.1).

Consider the bilinear form

$$\Theta[G_1, G_2] \equiv \int d^3v \, \frac{G_1}{n_n \widehat{f}} X(G_2, f) \ , \tag{3.1}$$

where G_1 and G_2 are any two neutral distributions, n_n is the neutral density and $\widehat{f}$ is an ion distribution normalized to have unit particle density:

$$\widehat{f} = \frac{f}{n_i} \ .$$

It will appear presently that if g is the solution to (2.3), then $\Theta[g, g]$ is the rate of neutral entropy production. Our notation is to use g for the solution to (2.3); upper-case G will refer to an arbitrary neutral distribution function or trial function. We refer to $\Theta[G_1, G_2]$ as the "CX bilinear form."

First we show that

$$\Theta[G_1, G_2] = \Theta[G_2, G_1] \ . \tag{3.2}$$

This symmetry forces the neutral transport coefficients to have Onsager symmetry.

The demonstration follows a conventional pattern. We use "Boltzmann notation," $f' \equiv f(v')$, so that (2.1) becomes

$$X(f, g) \equiv \int d^3v \, \sigma_x |\mathbf{v}' - \mathbf{v}| (f g' - f' g) \ .$$

From (2.1) and (2.2) we have

$$\Theta[G_1, G_2] = -\int d^3v\, d^3v' \frac{\sigma_x}{n_n} |\mathbf{v}' - \mathbf{v}| \frac{G_1}{\widehat{f}} (f\, G_2' - f' G_2) \ ,$$

or, after relabeling integration variables,

$$\Theta[G_1, G_2] = -\int d^3v\, d^3v' \frac{\sigma_x}{n_n} |\mathbf{v}' - \mathbf{v}| \frac{G_1'}{\widehat{f}'} (f'\, G_2 - f G_2') \ ,$$

and symmetrizing,

$$\Theta[G_1, G_2] = \frac{1}{2} n_i \int d^3v\, d^3v' \frac{\sigma_x}{n_n} |\mathbf{v}' - \mathbf{v}| \widehat{f}\widehat{f}' \left(\frac{G_1}{\widehat{f}} - \frac{G_1'}{\widehat{f}'} \right) \left(\frac{G_2}{\widehat{f}} - \frac{G_2'}{\widehat{f}'} \right) \tag{3.3}$$

from which (3.2) follows.

We also use (3.3) to express the quadratic form $\Theta[G, G]$ as

$$\Theta[G, G] = \frac{1}{2} n_i \int d^3v\, d^3v' \frac{\sigma_x}{n_n} |\mathbf{v}' - \mathbf{v}| \widehat{f}\widehat{f}' \left(\frac{G}{\widehat{f}} - \frac{G'}{\widehat{f}'} \right)^2 \tag{3.4}$$

showing that it can vanish only if $g/\widehat{f}$ is independent of velocity. Thus we have

$$\Theta[g, g] = 0 \Longrightarrow G(\mathbf{x}, \mathbf{v}) = N(\mathbf{x}) \widehat{f}(\mathbf{x}, \mathbf{v}) \tag{3.5}$$

for an arbitrary spatial function N.

It is obvious from (2.1) that the CX operator vanishes when $g = N\widehat{f}$; the interesting feature of (3.5) is that $g = N\widehat{f}$ is the only solution to $X(g, f) = 0$.

Finally (3.4) yields the inequality

$$\Theta[G, G] \geq 0 \qquad \text{for any } G\ . \tag{3.6}$$

This with (3.5) completes the entropy theorem ("H-theorem") for CX. We see that G relaxes, through charge-exchange, to have the velocity dependence of the (not necessarily Maxwellian) ion distribution. Θ is evidently the entropy production rate.

Chapter 4.

General Variational Principle

With some simplification (for example, the assumption of constant CX cross-section), the uniform temperature, one-dimensional version of (2.3) can be solved analytically for any mean-free-path ordering by singular eigenfunction techniques [6]. It is likely that a generalization of these methods could treat three-dimensional cases [11]. However, singular eigenfunctions are not easily adapted to allow for temperature variation.

We present here a complementary approach to neutral particle physics, based on a variational principle. The variational approach allows for arbitrary temperature variation — an important improvement in realism since temperature variation can be steep near the tokamak edge. It also allows for arbitrary energy dependence of the CX cross-section as well as three-dimensional geometry. More importantly it provides relatively simple asymptotic formulas for various quantities of interest in limiting parameter regimes.

On the other hand, it should be emphasized that the variational principle has its own complications, especially at long mean-free-path, where global trial functions are called for.

We derive the general variational principle in this section. Its specialization to the short-λ_x case — a much simpler and more obviously practical formalism — is considered in Chapter 5.

We begin by introducing the scalar product,

$$\{G_1, G_2\} \equiv \int d^3x\, d^3v \frac{G_1 G_2}{n_n \widehat{f}} = \{G_2, G_1\} \ , \tag{4.1}$$

where $\widehat{f}$ is the normalized ion distribution as in Chapter 2. We also introduce the spatially integrated CX bilinear form

$$\int d^3x \ \Theta[G_1, G_2] \equiv \overline{\Theta} = \{G_1, X(G_2, f)\} \ .$$

Now consider the steady-state version of (2.3):

$$\mathbf{v} \cdot \nabla g + X(g, f) + \nu_z g = \nu_r f \ . \tag{4.2}$$

It is combined with its adjoint

$$- f\mathbf{v} \cdot \nabla \left(\frac{g^\dagger}{f}\right) + X(g^\dagger, f) + \nu_z g^\dagger = \nu_r f \ , \tag{4.3}$$

in a conventional way

$$\{g^\dagger, \mathbf{v} \cdot \nabla g\} - \left\{g, f\mathbf{v} \cdot \nabla \left(\frac{g^\dagger}{f}\right)\right\} + 2\overline{\Theta}[g^\dagger, g] + 2\nu_z\{g^\dagger, g\} = \nu_r\{f, g + g^\dagger\} \ ,$$

to construct the form

$$S[G^\dagger, G] = V[G^\dagger, G] + \overline{\Theta}[G^\dagger, G] + \nu_z\{G^\dagger, G\} - \nu_r\{f, G + G^\dagger\} \ ,$$

where

$$V[G^\dagger, G] \equiv \left\{\frac{1}{2}G^\dagger, \mathbf{v} \cdot \nabla G\right\} - \left\{\frac{1}{2}G, f\mathbf{v} \cdot \nabla \left(\frac{G^\dagger}{f}\right)\right\} \ .$$

Here G and $G^\dagger$ are to be viewed as trial functions for the solutions to (4.2) and (4.3) respectively. After evaluation at the exact solutions g and $g^\dagger$, we obtain

$$S[g, g^\dagger] \equiv S^* = V[g^\dagger, g] + \overline{\Theta}[g^\dagger, g] + \nu_z\{g^\dagger, g\} = -\frac{1}{2}\nu_r\{f, g + g^\dagger\} \ .$$

This relation is essentially an entropy production law; the terms involving $\overline{\Theta}$, ν_z and ν_r measure entropy production by CX, ionization and recombination respectively. The first term, V, describes entropy flow.

After recalling (4.1) we find that

$$\frac{\delta V}{\delta G^\dagger} = \frac{1}{n_n \hat{f}} \mathbf{v} \cdot \nabla G \ ,$$

and similarly

$$\frac{\delta V}{\delta G} = \frac{1}{n_n \hat{f}} f \mathbf{v} \cdot \nabla \left(\frac{G^\dagger}{f} \right) \ .$$

Moreover, because of (3.2),

$$\frac{\delta \overline{\Theta}}{\delta G^\dagger} = \frac{1}{n_n \hat{f}} X(G, f) \ , \frac{\delta \overline{\Theta}}{\delta G} = \frac{1}{n_n \hat{f}} X(G^\dagger, f) \ ,$$

whence

$$\frac{\delta S}{\delta G^\dagger} = \frac{1}{n_n \hat{f}} [\mathbf{v} \cdot \nabla G + X(G, f) + \nu_z G - \nu_r f] \ ,$$

$$\frac{\delta S}{\delta G} = \frac{1}{n_n \hat{f}} \left[-f v \cdot \nabla \left(\frac{G^\dagger}{f} \right) + X(G^\dagger, f) + \nu_z G^\dagger - \nu_r f \right] \ .$$

Thus the variational principle

$$\frac{\delta S}{\delta G^\dagger} = 0$$

reproduces (4.2), while the principle

$$\frac{\delta S}{\delta G} = 0$$

yields (4.3). We summarize these facts by writing

$$\delta S = 0 \ . \tag{4.4}$$

A normalized version of (4.4) is easily constructed. One finds that

$$\delta \overline{H} = 0 \ , \tag{4.5}$$

where

$$\overline{H}(G, G^\dagger) \equiv \nu_r^2 \frac{\{f, G + G^\dagger\}^2}{V[G^\dagger, G] + \overline{\Theta}[G^\dagger, G] + \nu_z\{G^\dagger, G\}} .$$

This functional has the extremal value

$$\overline{H}^* = 4S^* .$$

Whichever form is used, the general variational theory has two disadvantages. First, both forms are spatially nonlocal so that trial functions must include both $\mathbf{x}$- and $\mathbf{v}$-dependence. Second, because the operator in (4.2) is not self-adjoint, trial functions for both g and $g^\dagger$ must be provided. Of course one prefers a variational principle involving only velocity integration, and using only symmetric (self-adjoint) bilinear forms.

Nonetheless, in any parameter regime that allows approximate analytic solution to (4.2) and (4.3), the variational principles (4.4) and (4.5) become directly useful. For then we can obtain higher-order information from quite simple integrals. For example, suppose that the CX rate is small: $\nu_x \ll v_n/L$. Then (4.2) and (4.3) are, asymptotically, first-order partial differential equations with well-understood Green's functions. Substitution of the solutions into H or S would provide an entropy balance law including CX effects through first order in ν_x.

Chapter 5.

Short Mean-Free-Path Theory

5.1. Ordered kinetic equations

We now turn attention to a case where the variational theory is both local and self-adjoint. Here we adopt the ordering (2.7). We define the small parameter

$$\Delta \equiv \frac{\lambda_x}{L} \approx \frac{L}{\lambda_z} ,$$

and expand the solution to (4.2) as

$$g = g_0 + g_1 \cdots + g_k = \mathcal{O}(\Delta^k) .$$

We also assume

$$\nu_r \sim \Delta^2 \nu_x \tag{5.1}$$

consistently with (2.5). Then, precisely as in Chapman-Enskog theory, we obtain a sequence of ordered equations for the g_k.

After writing (4.2) as

$$X(g, f) + \mathbf{v} \cdot \nabla g + \nu_z g = f \nu_r$$

we see that the first three orders are given by

$$\Delta^0\text{: } X(g_0, f) = 0 ; \tag{5.2}$$

$$\Delta^1\text{: } X(g_1, f) + \mathbf{v} \cdot \nabla g_0 = 0 ; \tag{5.3}$$

$$\Delta^2\text{: } X(g_2, f) + \mathbf{v} \cdot \nabla g_1 + \nu_z g_0 = f \nu_r . \tag{5.4}$$

The CX conservation law,

$$\int d^3v\, X(G,f) = 0\ , \tag{5.5}$$

for any G, provides a solubility condition in each order. Thus we must have, from (5.3),

$$\nabla \cdot \int d^3v\, \mathbf{v} g_0 = 0\ , \tag{5.6}$$

and from (5.4),

$$\nabla \cdot \int d^3v\, \mathbf{v} g_1 = \nu_r n_i - \nu_z n_0\ . \tag{5.7}$$

In view of (3.5), (5.2) has the unique solution

$$g_0 = n_0(\mathbf{x})\widehat{f}\ , \tag{5.8}$$

where n_0 is the lowest order neutral density.

5.2. Variational principle for short λ_x

To make further progress one must specify the ion distribution. We choose $\widehat{f}$ to correspond to a Maxwellian moving with velocity $\mathbf{V}_0$. Then

$$\widehat{f} = \left(\frac{m}{2\pi T_i}\right)^{\frac{3}{2}} \exp\left[-\frac{m(\mathbf{v}-\mathbf{V}_0)^2}{2T_i}\right] \tag{5.9}$$

with

$$\nabla \cdot (n_0 \mathbf{V}_0) = 0 \tag{5.10}$$

in order to satisfy (5.6). Thus $\widehat{f}$ depends upon position through $\mathbf{V}_0(\mathbf{x})$ and $T_i(\mathbf{x})$. While (5.9) is not exact, it closely approximates the observed ion distribution in most confinement experiments. The simple velocity shift, although strictly consistent with neoclassical theory only in the isothermal case, is not far from theoretical predictions and allows for the rapid rotation observed in the edge regions of some experiments. More elaborate non-Maxwellian corrections could be included in the present formalism with some loss of simplicity.

Allowing for a change of frames from the lab frame to one moving with neutral flow velocity $\mathbf{V} = \mathbf{V}_0 + \mathbf{V}_1$, we introduce the change of variables

$$\mathbf{w} = \mathbf{v} - \mathbf{V} ,$$

with the consequent transformation of the spatial derivative

$$\nabla|_v = \nabla|_w - (\nabla\mathbf{V}) \cdot \nabla_w .$$

Here $\nabla|_v$ is the spatial gradient evaluated at constant $\mathbf{v}$, while ∇_w is the gradient with respect to the velocity variable $\mathbf{w}$. Clearly, $\nabla_w = \nabla_v$. Note that the gradients in (5.2)–(5.4) should be interpreted as $\nabla|_v$.

Note also that we denote the neutral temperature by $T = T_0 + T_1$ where $T_0 = T_i$ due to (5.8).

Our first order equation has become

$$\mathbf{X}(g_1, f) = Q \equiv -\mathbf{v} \cdot \nabla g_0 , \tag{5.11}$$

where Q is found to be

$$Q = -n_0 \widehat{f}(\mathbf{w} - \mathbf{V}_0) \cdot \left[\nabla \ln p_n + \left(\frac{mw^2}{2T_0} - \frac{3}{2}\right) \nabla \ln T_0 + \frac{m}{T_0}(\nabla\mathbf{V}_0) \cdot \mathbf{w}\right] \tag{5.12}$$

and where we have used $p_n = n_n T_0$. We solve (5.11) via a straightforward specialization of the general variational method developed in Chapter 4. We recall the entropy production rate (3.1),

$$\Theta[G_1, G_2] = \int d^3v \left(\frac{G_1}{n_n \widehat{f}}\right) X(G_2, f)$$

and introduce the linear form

$$P[G] \equiv \int d^3v \, \frac{GQ}{n_n \hat{f}} \,, \tag{5.13}$$

to find that the functional

$$H[G,G] \equiv \frac{P^2[G]}{\Theta[G,G]}$$

is variational:

$$\delta H = 0 \,, \tag{5.14}$$

at $G = g_1$. Since (5.11) implies $\Theta[g_1, g_1] = P[g_1]$, the extremal value of H is

$$H[g_1, g_1] = P[g_1] = \Theta[g_1, g_1] \,, \tag{5.15}$$

the entropy production rate.

We now want to write H as a linear combination of thermodynamic forces and fluxes. First, we define the first order neutral particle, heat, and momentum fluxes by

$$\mathbf{\Gamma}_n = \int d^3v \, \mathbf{v} g_1 = n_1 \mathbf{V}_0 + n_0 \mathbf{V}_1 \tag{5.16}$$

$$\mathbf{q}_n = \int d^3v \, \frac{m}{2} w^2 \mathbf{w} g \tag{5.17}$$

$$\overleftrightarrow{\mathbf{p}}_n = \int d^3v \, m \mathbf{w}\mathbf{w} g \tag{5.18}$$

respectively. We define the neutral viscosity tensor as

$$\overleftrightarrow{\boldsymbol{\pi}}_n = \int d^3v \left(\mathbf{w}\mathbf{w} - \frac{1}{3} w^2 \mathbf{I} \right) g \,. \tag{5.19}$$

We will make use of the identity

$$\overleftrightarrow{\mathbf{p}}_1 : \nabla \mathbf{V}_0 = \frac{1}{2} \overleftrightarrow{\boldsymbol{\pi}}_1 : \overleftrightarrow{\mathbf{W}} + p_1 \nabla \cdot \mathbf{V}_0 \tag{5.20}$$

where

$$\overleftrightarrow{\mathbf{W}} \equiv (\nabla \mathbf{V}_0)^s - \frac{2}{3}(\nabla \cdot \mathbf{V}_0)\overleftrightarrow{\mathbf{I}}$$

is the rate of strain tensor, and where the superscript s denotes symmetrization; i.e.

$$(\nabla \mathbf{V})^s_{ij} \equiv \partial_i V_j + \partial_j V_i \ .$$

Using these definitions with (5.15), (5.13), and (5.12), we find

$$H[g_1, g_1] = -\frac{1}{T_0} n_0 T_0 \mathbf{V}_1 \cdot \left(\nabla \ln p_0 + \frac{m}{T_0}\mathbf{V}_0 \cdot \nabla \mathbf{V}_0\right) - \frac{1}{T_0}\mathbf{q}_n \cdot \nabla \ln T_0$$
$$-\frac{1}{T_0}\frac{1}{2}\overleftrightarrow{\pi}_1 : \overleftrightarrow{\mathbf{W}} - \frac{1}{T_0} n_0 T_1 \mathbf{V}_0 \cdot \nabla \left(\frac{3}{2}\ln T_0 - \ln n_0\right) . \qquad (5.21)$$

The second and third terms in (5.21) are conventional, and the first term lends itself easily to a force-flux interpretation. The unconventional fourth term is a bit tricky, however. Introducing the zeroth order entropy

$$s = \ln\left(\frac{T_0^{\frac{3}{2}}}{n_0}\right) ,$$

and using (5.10), we can write our generalized thermodynamic forces as

$$A_1 = \left(\nabla \ln p_0 + \frac{m}{T_0}\mathbf{V}_0 \cdot \nabla \mathbf{V}_0\right)$$

$$A_2 = \nabla \ln T_0$$

$$A_3 = \frac{1}{v_t}\overleftrightarrow{\mathbf{W}}$$

$$A_4 = \frac{1}{n_0 v_t}\left(\nabla \cdot n_0 s \mathbf{V}_0\right)\overleftrightarrow{\mathbf{I}} ,$$

and their conjugate generalized thermodynamic fluxes as

$$F_1 = n_0 \mathbf{V}_1$$

$$F_2 = \frac{1}{T_0}\mathbf{q}_n$$

$$F_3 = \frac{v_t}{2T_0}\overleftrightarrow{\boldsymbol{\pi}}_1$$

$$F_4 = n_0 v_t \frac{T_1}{T_0}\overleftrightarrow{\mathbf{I}}$$

in terms of which the neutral entropy production (5.21) becomes

$$H = \sum_{i=1}^{4} A_i F_i \tag{5.22}$$

where the product in (5.22) refers to complete tensor contraction.

In turn, near equilibrium the fluxes can be written in terms of the thermodynamic forces as

$$F_j = \sum_{j=1}^{4} L_{ij} A_j$$

where the transport coefficients L_{ij} are scalars. Substituting this into (5.22), we see that the entropy production can be written in the form

$$H = \sum_{i=1}^{4}\sum_{j=1}^{4} L_{ij} A_i A_j \ , \tag{5.23}$$

a quadratic form in the thermodynamic forces. The point is that once an expression of the form (5.23) is found for H, the transport coefficients can be read off by inspection. This formalism, like (5.21), follows the pattern of conventional transport theory.

Note that our unconventional fourth term in (5.21), call it H_4, may be written as

$$\Theta_4 = -\frac{T_1}{T_0}\nabla \cdot (n_0 s \mathbf{V}_0) \ . \tag{5.24}$$

Thus, we may interpret (5.24) in the following way: neutrals born at $T_n = T_0 + T_1$ move to produce a divergence of entropy density thereby contributing to the total entropy production.

In second order we need only the solubility condition, (5.7). Since the neutral particle flux is defined to be

$$\mathbf{\Gamma}_n = \int d^3v \, g_1 \mathbf{v} \ ,$$

in view of (5.10), (5.7) can be expressed as

$$\nabla \cdot \mathbf{\Gamma}_n = \nu_r n_i - \nu_z n_0 \ . \tag{5.25}$$

Since the ion parameters n_i and T_i are presumed given, (5.25) should be viewed as a constraint on the neutral density profile. We note that the time derivative term can be included straightforwardly in (5.25),

$$\frac{\partial n_n}{\partial t} + \nabla \cdot \mathbf{\Gamma}_n = \nu_r n_i - \nu_z n_0 \ ,$$

provided we use the self-consistent ordering $\partial/\partial t = \mathcal{O}(\Delta^2)$.

Our results (5.29) will show that

$$\mathbf{\Gamma}_n \approx \left(\frac{n_0}{\nu_x}\right) \frac{v_t^2}{L_n} \approx n_n \Delta \, v_n \ .$$

We substitute this estimate into (5.21) to obtain the ordering

$$\Delta \frac{v_t}{L} \approx \nu_x \Delta^2 \approx \nu_z \ ,$$

as anticipated in (5.1). Thus our orderings are internally consistent.

We conclude this section by showing that the extremum of H is in fact a maximum. Let $G = g + \delta g$ and define the quantities

$$\delta\Theta_1 \equiv \Theta(\delta g, g) + \Theta(g, \delta g) \ ,$$

$$\delta\Theta_2 \equiv \Theta(\delta g, \delta g) \ ,$$

$$\delta P \equiv P[\delta g] \ .$$

Then we have, without approximation,

$$\begin{aligned} \Theta[G,G] &= \Theta^* + \delta\Theta_1 + \delta\Theta_2 \ , \\ P[G] &= P^* + \delta P \ . \end{aligned} \tag{5.26}$$

Also, (5.15) implies $\Theta^* = P^*$ where the asterisks denote extremal values, and (5.14) implies

$$\delta\Theta_1 = 2\delta P \ .$$

We use these facts to evaluate $H[G,G]$, with the result

$$H = H^* \left\{ 1 - \frac{\delta\Theta_2}{\Theta^*} + \frac{1}{4}\left(\frac{\delta\Theta_1}{\Theta^*}\right)^2 \right\} \ , \tag{5.27}$$

correct through second order. As always, there is no first order term because of (5.15).

Now consider the quantity, $\Theta[xg+\delta g, xg+\delta g]$, where x is an arbitrary constant. For this choice (5.26) becomes

$$\Theta[xg+\delta g, xg+\delta g] = \Theta^* x^2 + \delta\Theta_1 x + \delta\Theta_2 \ .$$

Since (3.6) does not allow the value of Θ to cross the real axis, the quadratic equation

$$\Theta^* x^2 + \delta\Theta_1 x + \delta\Theta_2 = 0$$

cannot have two real roots. Therefore its discriminant cannot be positive:

$$4\Theta^*\delta\Theta_2 \geq (\delta\Theta_1)^2 \ ,$$

and (5.27) implies

$$H \leq H^* \ ;$$

the extremum is indeed a maximum.

5.3. Constant $\sigma_x|\mathbf{v}-\mathbf{v}'|$ assumption

In addition to the short λ_x approximation, the realistic assumption that $\sigma_x|\mathbf{v}-\mathbf{v}'|$ is nearly independent of velocity allows us to avoid the variational approach. Similar results are obtained by Vekshtein and Ryutov [8]. Our results differ in that we include flow in order to obtain an expression for the neutral momentum flux.

Assuming now that $\sigma_x|\mathbf{v}-\mathbf{v}'|$ is constant, we evaluate the CX operator (2.1) and find that in first order it reduces to

$$X(g_1, f) = \nu_x \int d^3v'(\widehat{f}' g_1 - \widehat{f} g_1') = \nu_x(g_1 - \widehat{f} n_1) \tag{5.28}$$

where we have taken

$$\sigma_x|\mathbf{v}-\mathbf{v}'| = \int d^3v\, \widehat{f} \sigma_x |\mathbf{v}-\mathbf{v}'| = \frac{\nu_x}{n_i} \; ,$$

and where n_1 is arbitrary. Combining (5.28) with (5.3), we find the first order correction to the neutral distribution function

$$g_1 = \left(\frac{n_1}{n_0} - \frac{1}{\nu_x}\mathbf{v}\cdot\nabla\right) g_0 \; , \tag{5.29}$$

where the lowest order neutral density n_0 is presumed known, and where n_1 is arbitrary. We will see that n_1 will not enter into the calculation of the neutral transport coefficients. However, n_1 would be needed if we wished to calculate the neutral particle flux. Recalling (5.25), where

$$\mathbf{\Gamma}_n = n_0 \mathbf{V}_1 + n_1 \mathbf{V}_0 \; ,$$

since we will calculate $\mathbf{V}_1$ presently, (5.25) could, in principle, be easily integrated for the only remaining unknown n_1.

At this point we could take velocity moments of (5.29) to calculate the neutral fluxes directly. Let us instead do something more instructive that gives the same results. We will calculate the neutral entropy production from it's definition. Since the entropy production will depend on the first order neutral temperature and flow velocity, let us calculate these quantities first in a novel way.

Denoting perturbations in the neutral density, temperature, and flow velocity via

$$n_n = n_0 + n_1 \ ,$$

$$T = T_0 + T_1 \ ,$$

$$\mathbf{V} = \mathbf{V}_0 + \mathbf{V}_1 \ ,$$

since (5.29) is the perturbation to a Maxwellian, we may write the neutral distribution function in the form of a general perturbed Maxwellian

$$g = g_0 + g_{M1}$$

where

$$g_{M1} = \left(n_1 \frac{d}{dn_0} \ln g_0 + T_1 \frac{d}{dT_0} \ln g_0 + \mathbf{V}_1 \cdot \frac{d}{d\mathbf{V}_0} \ln g_0 \right) g_0 \ . \tag{5.30}$$

Performing the derivatives in (5.30) and (5.29), and making the identification $g_{M1} = g_1$, we obtain the equation

$$\left[\frac{T_1}{T_0} \left(\frac{mw^2}{2T_0} - \frac{3}{2} \right) + \frac{m}{T_0} \mathbf{w} \cdot \mathbf{V}_1 \right] g_0 = -\frac{1}{\nu_x} (\mathbf{w} + \mathbf{V}_0) \cdot \Big[\nabla \ln n_0 + \left(\frac{mw^2}{2T_0} - \frac{3}{2} \right) \nabla \ln T_0 + \frac{m}{T_0} (\nabla \mathbf{V}_0) \cdot \mathbf{w} \Big] g_0 \ . \tag{5.31}$$

We now take three velocity moments of this equation. Integrating (5.31) over velocity space, we find

$$\nabla \cdot (n_0 \mathbf{V}_0) = 0$$

which is just (5.10). Next, multiplying (5.31) by $\mathbf{w}$ and integrating over velocity space, we find that the first order neutral flow velocity is

$$\mathbf{V}_1 = -\frac{T_0}{m\nu_x}\left(\nabla \ln p_0 + \frac{m}{T_0}\mathbf{V}_0 \cdot \nabla \mathbf{V}_0\right) . \tag{5.32}$$

Thirdly, multiplying (5.31) by w^2 and integrating over velocity space, we obtain for the first order neutral temperature

$$\begin{aligned} T_1 &= -\frac{T_0}{\nu_x}\left(\mathbf{V}_0 \cdot \nabla \ln T_0 - \frac{2}{3}\mathbf{V}_0 \cdot \nabla \ln n_0\right) \\ &= -\frac{2}{3}\frac{T_0}{n_0\nu_x}\nabla \cdot (n_0 s \mathbf{V}_0) . \end{aligned} \tag{5.33}$$

We will also need the zeroth order neutral heat flux. Expanding g_0 for small $\delta \equiv V_1/V_0$, we may write

$$g_0 = n_0\left(\frac{m}{2\pi T}\right)^{\frac{3}{2}}\left(1 - \frac{m}{T_0}\mathbf{w} \cdot \mathbf{V}_1\right)\exp\left(-\frac{mw^2}{2T}\right) . \tag{5.34}$$

So, the zeroth order neutral heat flux, a perturbed quantity, is

$$\mathbf{q}_0 \equiv \int d^3v\, \frac{m}{2}w^2\mathbf{w}g_0 = -\frac{5}{2}n_0T_0\mathbf{V}_1 . \tag{5.35}$$

We now proceed to calculate the neutral entropy production, defined as

$$\Theta_n \equiv -\int d^3v\, X(f,g)\ln g . \tag{5.36}$$

Substituting $g = g_0 + g_1$ into (5.36) using (5.28) and (5.29), we obtain

$$\begin{aligned} \Theta_n &= \frac{1}{T_0}\mathbf{V}_1 \cdot \mathbf{F} + \frac{1}{T_0}W \\ &+ \left(\nabla \ln p_0 - \frac{5}{2}\nabla \ln T_0\right) \cdot \int d^3v\left(\frac{n_1}{n_0}g_0 - g_1\right)\mathbf{v} \\ &+ \frac{1}{T_0}\nabla \ln T_0 \cdot \int d^3v\, \frac{m}{2}w^2(\mathbf{w} + \mathbf{V}_0)\left(\frac{n_1}{n_0}g_0 - g_1\right) \\ &+ \frac{1}{T_0}(\nabla \mathbf{V}_0) \colon \int d^3v\, m\mathbf{w}(\mathbf{w} + \mathbf{V}_0)\left(\frac{n_1}{n_0}g_0 - g_1\right) , \end{aligned} \tag{5.37}$$

where

$$\mathbf{F} \equiv \int d^3v\, m\mathbf{v} X(f,g) = \nabla P_0 = -mn_0\nu_x \mathbf{V}_1$$

is the momentum exchange, and where

$$W \equiv \int d^3v\, \frac{m}{2} w^2 X(f,g) = -\frac{3}{2} n_0 T_1$$

is the energy exchange between neutrals and ions. Note that the ion entropy production is

$$\Theta_i \equiv -\int d^3v\, X(g,f) \ln f = -\frac{1}{T_0}\mathbf{V}_1 \cdot \mathbf{F} - \frac{1}{T_0} W \,. \qquad (5.38)$$

Using the definitions (5.16)–(5.18), and using (5.36) and (5.38), we write the total entropy production $\Theta = \Theta_n + \Theta_i$ as

$$\begin{aligned}\Theta = &\left(\nabla \ln p_0 - \frac{5}{2}\nabla \ln T_0\right) \cdot (n_1\mathbf{V}_0 - \mathbf{\Gamma}_n) \\ &+ \frac{1}{T_0}\nabla \ln T_0 \cdot \left(\frac{n_1}{n_0}\mathbf{q}_0 - \mathbf{q}_1 + \frac{3}{2}\frac{n_1}{n_0} p_0 \mathbf{V}_0 - \frac{3}{2} p_1 \mathbf{V}_0\right) \\ &+ \frac{1}{T_0}(\nabla \mathbf{V}_0) : \left(\frac{n_1}{n_0}\overleftrightarrow{\mathbf{p}}_0 - \overleftrightarrow{\mathbf{p}}_1 + mn_1\mathbf{V}_0\mathbf{V}_0 - m\mathbf{\Gamma}_n\mathbf{V}_0\right) \,. \end{aligned} \qquad (5.39)$$

We want to write (5.39) in a form that exhibits Onsager symmetry. In our case, we will show that the transport matrix is diagonal. This is due to the simplicity of our CX operator. Using again the identity (5.20), along with (5.32), (5.33), and (5.35), we may write the total entropy production (5.39) as

$$\Theta = \frac{1}{T_0} mn_0\nu_x V_1^2 + \frac{1}{T_0}\frac{2}{5}\frac{m\nu_x}{n_0 T_0^2} q_n^2 + \frac{1}{T_0}\frac{1}{2}\frac{\nu_x}{n_0 T_0}\overleftrightarrow{\boldsymbol{\pi}} : \overleftrightarrow{\boldsymbol{\pi}} + \frac{1}{T_0}\frac{3}{2}\frac{n_0\nu_x}{T_0} T_1^2 \,, \qquad (5.40)$$

which shows explicitly that the transport matrix is diagonal.

We can also write (5.39) in force-flux form which just reproduces (5.21).

For completeness, the neutral heat flux is

$$\mathbf{q}_n \equiv \int d^3v\, \frac{m}{2} w^2 \mathbf{w} g = -\frac{5}{2}\frac{n_0 T_0^2}{m\nu_x} \nabla \ln T_0 \ , \tag{5.41}$$

and the neutral viscosity tensor is

$$\overleftrightarrow{\boldsymbol{\pi}}_n \equiv \int d^3v\, m \left(\mathbf{w}\mathbf{w} - \frac{w^2}{3} \overleftrightarrow{\mathbf{I}} \right) g = \overleftrightarrow{\boldsymbol{\pi}}_1 = -\frac{n_0 T_0}{\nu_x} \overleftrightarrow{\mathbf{W}} \ , \tag{5.42}$$

We can now present a full set of neutral transport coefficients

$$L_{lm} = \alpha_{lm} n_0 \nu_x \lambda_x^2 \tag{5.43}$$

where

$$\alpha_{11} = \tfrac{1}{2} \ , \alpha_{22} = \tfrac{5}{4} \ , \alpha_{33} = \tfrac{1}{2} \ , \alpha_{44} = \tfrac{2}{3}$$

and $\alpha_{lm} = 0$ for $l \neq m$.

The form of these neutral transport coefficients is consistent with our physical picture. In any realistic plasma there will always be a population of neutral particles that are not affected by the magnetic field. These neutrals may undergo many CX collisions before being ionized due to impact with an electron. Each CX collision results in a random change of neutral momentum. The result is that neutrals execute a random walk of step-size λ_x and frequency ν_x.

Chapter 6.

Charge-Exchange and Ion Transport

6.1. Moment equations

Our discussion of ion fluid behavior and transport is based on two moments of Eqs. (2.3) and (2.4): the momentum and energy moments. The momentum conservation law for ions has the form

$$\frac{\partial}{\partial t}(m_i n_i \mathbf{V}_i)+\nabla\cdot\overleftrightarrow{\mathbf{P}}_i-en_i(\mathbf{E}+c^{-1}\mathbf{V}_i\times\mathbf{B}) = \mathbf{F}_e-\mathbf{F}_x+\nu_z m_i n_n \mathbf{V}_n-\nu_r m_i n_i \mathbf{V}_i \ .$$

Here

$$n_i\mathbf{V}_i \equiv \int d^3v\,\mathbf{v}f$$

is the ion flow,

$$\overleftrightarrow{\mathbf{P}}_i \equiv \int d^3v\, m_i\mathbf{v}\mathbf{v}f$$

is the ion stress tensor,

$$\mathbf{F}_e \equiv \int d^3v\, m_i\mathbf{v}C$$

is the collisional friction force on ions due to Coulomb collisions with electrons, and

$$\mathbf{F}_x \equiv \int d^3v\, m_i\mathbf{v}\,X(f,g)$$

measures the effective friction due to charge exchange. It is convenient to define

$$\mathbf{F}_n \equiv -\mathbf{F}_x+\nu_z m_i n_n \mathbf{V}_n-\nu_r m_i n_i \mathbf{V}_i \ .$$

We refer to $\mathbf{F}_n$ as the "neutral friction." Then we have

$$\frac{\partial}{\partial t}(m_i n_i \mathbf{V}_i) + \nabla \cdot \overleftrightarrow{\mathbf{P}}_i - e n_i (\mathbf{E} + c^{-1}\mathbf{V}_i \times \mathbf{B}) = \mathbf{F}_e + \mathbf{F}_n \; . \tag{6.1}$$

The corresponding neutral force law,

$$\frac{\partial}{\partial t}(m n_n \mathbf{V}_n) + \nabla \cdot \overleftrightarrow{\mathbf{P}}_n = \mathbf{F}_n \; , \tag{6.2}$$

is obtained by changing subscripts, assuming $m_n = m_i \equiv m$ and noting that $e_n = 0$.

Next we consider pressure evolution. After multiplying (2.4) by $mv^2/2$ and integrating over velocity, we find that ion pressure,

$$p_i \equiv \int d^3v \, f \frac{m}{3}(\mathbf{v} - \mathbf{V}_i)^2 \; ,$$

evolves according to

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p_i + \frac{1}{2}m n_i V_i^2\right) + \nabla \cdot \mathbf{Q}_i = \mathbf{V}_i \cdot (\mathbf{F}_e + e n_i \mathbf{E}) + W_e + W_n \tag{6.3}$$

where

$$\mathbf{Q}_i \equiv \int d^3v \, f \mathbf{v} \, \frac{mv^2}{2}$$

is the ion energy flux,

$$W_e \equiv \int d^3v \, C \, \frac{mv^2}{2}$$

is the Coulomb energy exchange with electrons, and

$$\begin{aligned} W_n \equiv & \int d^3v \, X(f,g) \frac{mv^2}{2} + \mathbf{V}_i \cdot \mathbf{F}_n \\ & + \nu_z \left(\frac{3}{2}p_n + \frac{1}{2}m n_n V_n^2\right) - \nu_r \left(\frac{3}{2}p_i + \frac{1}{2}m n_i V_i^2\right) \; , \end{aligned} \tag{6.4}$$

is the energy gained by ions due to inelastic ion-neutral interaction. We denote the first term in (6.4) by

$$W_x \equiv \int d^3v\, X(f,g)\, \frac{mv^2}{2}\ ; \tag{6.5}$$

it usually dominates the sum. The neutral counterpart to (6.7) is

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p_n + \frac{1}{2}m_n n_n V_n^2\right) + \nabla\cdot\mathbf{Q}_n = -W_n\ . \tag{6.6}$$

Here $\mathbf{Q}_n$ is related to the neutral heat flux by

$$\mathbf{Q}_n = \mathbf{q}_n + \mathbf{V}_n\cdot\overleftrightarrow{\mathbf{p}}_n + \frac{1}{2}mn_nV_n^2\mathbf{V}_n + \frac{3}{2}p_n\mathbf{V}_n\ .$$

6.2. Perpendicular ion flow

Any change in ion momentum due to neutral interactions is balanced by a corresponding change in neutral momentum. How do the neutrals dispose of their changed momentum? In general, they might accelerate, or they might propagate the momentum change to the walls by viscous dissipation. In the short mean-free-path regime considered here, the net effect of momentum exchange is to allow the neutral pressure gradient to act on ions:

$$p_i \rightarrow p_i + p_n\ .$$

To see this conclusion explicitly, we add (6.1) and (6.2). Since $\mathbf{V}_i \cong \mathbf{V}_n$ while $n_n \ll n_i$, the general result is

$$\frac{\partial}{\partial t}(mn_i\mathbf{V}_i) + \nabla\cdot(\overleftrightarrow{\mathbf{P}}_i + \overleftrightarrow{\mathbf{P}}_n) - en_i(\mathbf{E} + c^{-1}\mathbf{V}_i\times\mathbf{B}) = \mathbf{F}_e\ . \tag{6.7}$$

A sharper result pertains in the short CX mean-free-path regime, where both stresses are approximately isotropic. Note isotropy of the neutral stress results from short CX-mean-free-path,

$$\overleftrightarrow{\mathbf{P}}_n = p_n \overleftrightarrow{\mathbf{I}} + \mathcal{O}\left(\frac{\lambda_x}{L}\right) ,$$

while that of the ion stress is an artifact of small gyroradius ρ:

$$\overleftrightarrow{\mathbf{P}}_i = p_i \overleftrightarrow{\mathbf{I}} + \mathcal{O}\left(\frac{\rho}{L}\right) .$$

By a conventional argument the acceleration and friction terms in (6.7) are also $\mathcal{O}(\rho/L)$. Hence, after solving (6.7) for $\mathbf{V}_i$ we have

$$\mathbf{V}_i = \mathbf{b}V_\parallel + \frac{c}{eBn_i}\mathbf{b} \times [en_i\mathbf{E} + \nabla(p_i + p_n)] \tag{6.8}$$

where $\mathbf{b} = \mathbf{B}/B$. Equation (6.8) shows that neutrals, although obviously unmagnetized, contribute like a magnetized species to the diamagnetic drift. Since the neutral and ion pressure gradients are opposed in much of the edge region, the observed effect of (6.8) is diminished ion diamagnetic rotation.

Since there is no corresponding effect on electron diamagnetism, the perpendicular plasma current is affected by neutrals in the obvious way:

$$\mathbf{J}_\perp = \frac{c}{eBn_i}\mathbf{b} \times \nabla(p_e + p_i + p_n) . \tag{6.9}$$

One implication of (6.9) is that experimental estimates of plasma beta must be performed with care whenever neutrals may be present.

A final conclusion from (6.8) is that neutrals are unlikely to affect ion particle transport. The radial particle flux in an axisymmetric system is proportional to the toroidal component of the friction force, F_T. Since

$$F_{nT} = -(\nabla p_n)_T = -\frac{1}{R}\frac{\partial p_n}{\partial \zeta} ,$$

where R is the major radius and ζ the toroidal angle, the axisymmetric effect of neutrals on Γ_i vanishes exactly. But even with asymmetry the effect appears small, because the flux-surface average will annihilate, or nearly annihilate $\mathbf{b} \times \nabla p_n$.

Observe next that, according to (6.7), not just the scalar pressure but the entire stress tensors of neutrals and ions act additively in ion dynamics. This circumstance is significant because measurements of ion viscosity in the tokamak are anomalously high. Thus the question arises as to whether CX can account for anomalous viscosity. From (5.42) and

$$\overleftrightarrow{\boldsymbol{\pi}}_n = -\eta_n \overleftrightarrow{\mathbf{W}}$$

the neutral viscosity is found to be

$$\eta_n = \left(\frac{1}{2}\right)\left(\frac{n_n T_i}{\nu_x}\right)$$

and the ion viscosity [12] is

$$\eta_i = \left(\frac{3}{10}\right)\left(\frac{n_i T_i}{\Omega_i^2 \tau_i}\right) .$$

Here Ω_i is the ion gyrofrequency and τ_i is the ion-ion collision time as defined by Braginskii [12]. Using typical profiles from TEXT, we find that the quantity

$$\frac{\eta_n}{\eta_i} \sim \left(\frac{n_n}{n_i}\right)\left(\frac{\Omega_i^2 \tau_i}{\nu_x}\right)$$

varies from 10^2 at the center to 10^6 at the wall 30 cm away. Since λ_x varies from 8 cm at the center to 13 cm at the wall, we are still within the window of validity for short λ_x theory. We therefore find that neutral viscosity dominates ion viscosity everywhere, and in the edge region by a large factor.

6.3. Ion energy transport

Similar physics applies to ion energy transport, except that here, in the absence of a conservation law analogous to (5.5), the effect is large. After CX delivers ion energy to the neutrals, it diffuses rapidly by neutral heat conduction, as described by (5.41). Since part of the neutral flux is proportional to the ion temperature gradient, the effect of ion-neutral energy exchange will appear as enhanced ion heat conduction together with convection.

To estimate the importance of this heat conduction process, we compare it to the neoclassical ion heat loss,

$$Q_{NC} \sim \nu_i \rho^2 \left(\frac{B}{B_p}\right)^2 \nabla p_i$$

where ν_i is the Coulomb collision frequency for ion-ion collisions, B_p is the poloidal magnetic field and a factor of $(r/R)^{1/2} \sim 1$ is suppressed. The corresponding measure of the new process is Q_n. Eq. (5.41) provides

$$\frac{Q_n}{Q_{NC}} \sim \frac{n_n}{n_i}\frac{\nu_x}{\nu_i}\left(\frac{\lambda_x}{\rho_p}\right)^2 , \tag{6.10}$$

where ρ_p is the poloidal gyroradius. This ratio exceeds unity in typical circumstances because its last factor is large.

The explicit calculation begins with the sum of (6.3) and (6.6). Since neutral pressure changes no faster than ion pressure,

$$\frac{\partial p_n}{\partial t} \sim \frac{n_n}{n_i}\frac{\partial p_i}{\partial t} \ll \frac{\partial p_i}{\partial t} ,$$

(at least after some relaxation time) and assuming that the speeds V_i and V_n are smaller than either thermal speed, the general result is

$$\frac{3}{2}\frac{\partial}{\partial t} p_i + \nabla \cdot (\mathbf{Q}_i + \mathbf{Q}_n) = \mathbf{V}_i \cdot (\mathbf{F}_e + e n_i \mathbf{E}) + W_e . \tag{6.11}$$

Here all the terms in (6.11) are conventional, pertinent to a neutral-free plasma, except Q_n. Thus, as anticipated, neutral energy transport simply adds, in the ion energy balance equation, to ion energy transport.

Chapter 7.

CX Damping of Rotation with Impurities

In this chapter we give a simple drift-kinetic derivation of the expressions for the poloidal ion and impurity flows in the presence of charge-exchange drag and ion-impurity collisions.

Recent observations [13] in DIII-D [14] have identified a sudden increase in plasma rotation as a signature of L to H transition. The discrepancies between experimental observations [15] and neoclassical predictions [16] of plasma flows are generally attributed to neutrals causing drag on ion rotation through charge-exchange (CX) collisions. Kim *et al.* [17] have derived neoclassical expressions for ion flows using the moment approach [18], but without the CX effects. We show that a simpler derivation (including the CX effects) is possible directly from the drift-kinetic equation without having to use the moment approach.

For simplicity, consider a plasma of ionized hydrogen (i) with a small concentration of neutrals (n), and one fully ionized impurity species (z). The results can be easily extended to multiple species. Also make the physically plausible assumptions that the ion density $n_i \gg n_z\ , n_n$, and that the CX mean-free-path λ_x and gyroradius $\rho \ll L$ the density scale length.

The steady-state force-balance equations for the three species are

$$\begin{aligned}
&\nabla \cdot \overleftrightarrow{\mathbf{P}}_i - en_i\left(\mathbf{E} + \tfrac{1}{c}\mathbf{V}_i \times \mathbf{B}\right) = \mathbf{F}_{iz} + \mathbf{F}_{in} \; , \\
&\nabla \cdot \overleftrightarrow{\mathbf{P}}_z - zen_z\left(\mathbf{E} + \tfrac{1}{c}\mathbf{V}_z \times \mathbf{B}\right) = -\mathbf{F}_{iz} \; , \\
&\nabla \cdot \overleftrightarrow{\mathbf{P}}_n = -\mathbf{F}_{in} \; .
\end{aligned} \tag{7.1}$$

where $\mathbf{F}$, $\overleftrightarrow{\mathbf{P}}$, and $\mathbf{V}$ are the friction force, pressure, and velocity. The neutral population is a mix of stationary, cold, primary neutrals generated from the wall, and secondary, hot neutrals created in charge-exchange with ions. These secondary neutrals have roughly the same temperature and fluid velocity as the ions. In this chapter, we restrict our attention to the hot neutrals. Hence our results are applicable only outside the very thin, cold-neutral-dominated, stationary boundary layer next to the wall. This layer is only a centimeter or so thick in most machines. The experimental observations in Ref. [15] were made more than 3 cm from the wall. In this hot-neutral-dominated region, we can approximate the pressure of each species by a scalar, indicated by p. Rearranging the force-balance equations we write the system to be solved as

$$\nabla(p_i + p_n) - en_i\left(\mathbf{E} + \frac{1}{c}\mathbf{V}_i \times \mathbf{B}\right) = \mathbf{F}_{iz} \; , \tag{7.2}$$

$$\nabla p_z - zen_z\left(\mathbf{E} + \frac{1}{c}\mathbf{V}_z \times \mathbf{B}\right) = -\mathbf{F}_{iz} \; . \tag{7.3}$$

For large ion-impurity collisionality, we approximate the ion-impurity friction force as

$$\mathbf{F}_{iz} = -\mathbf{F}_{zi} = m_z n_z \nu_z \Delta\mathbf{V} \; , \tag{7.4}$$

where $\Delta\mathbf{V} \equiv \mathbf{V}_i - \mathbf{V}_z$, and ν_z is the ion-impurity collision frequency. After appropriately manipulating (7.2)–(7.4), we obtain

$$(1 - q\mathrm{b}\times)\Delta\mathbf{V} = \mathbf{Q} \; , \tag{7.5}$$

where $\mathbf{b} \equiv \mathbf{B}/B$,

$$q = \frac{\Omega_z}{\nu_z}\left(1 + \frac{z n_z}{n_i}\right)^{-1} \approx \frac{\Omega_z}{\nu_z} ,$$

and

$$\mathbf{Q} \equiv \left(\frac{c}{eB}\right) q \left(\frac{1}{n_i}\nabla(p_i + p_n) - \frac{1}{z n_z}\nabla p_z\right) .$$

The solution to (7.5) can be written as

$$\Delta\mathbf{V} = (1 + q^2)^{-1}(1 + q\mathbf{b}\times)\mathbf{Q} .$$

We find that, assuming small parallel pressure gradients,

$$Q_\parallel = \left(\frac{c}{eB}\right) q \left[\frac{1}{n_i}\nabla_\parallel(p_i + p_n) - \frac{1}{z n_z}\nabla_\parallel p_z\right]$$

is very small for all q, and therefore that

$$V_{\parallel i} \approx V_{\parallel z} .$$

However, the perpendicular flows do not equilibrate. For the physical case of large q ($\Omega_z \gg \nu_z$), we have

$$\begin{aligned} \Delta\mathbf{V}_\perp &= \frac{1}{q}\mathbf{b} \times \mathbf{Q}_\perp + \mathcal{O}(q^2) \\ &\approx \left(\frac{c}{eB}\right) \mathbf{b} \times \left[\frac{1}{n_i}\nabla(p_i + p_n) - \frac{1}{z n_z}\nabla p_z\right] , \end{aligned}$$

whose poloidal component is given by

$$\Delta V_p \equiv (\Delta\mathbf{V}_\perp)_p = \frac{c}{eBn_i}\left(p_i' + p_n' - \frac{n_i}{z n_z}p_z'\right) , \tag{7.6}$$

where primes indicate radial gradients. This result corresponds simply to independent diamagnetic (and $\mathbf{E} \times \mathbf{B}$) motion of each species. The physical point is simple: large ν_z ($q \to 0$, $\mathbf{Q} \to 0$) wants the two species to move together, while large Ω_z ($q \to \infty$) wants the perpendicular velocities to be independent and diamagnetic.

Our goal is to calculate the net poloidal impurity flow

$$\mathbf{V}_{pz} = \mathbf{V}_{pi} - \Delta \mathbf{V}_p \ . \tag{7.7}$$

Since $\mathbf{F}_{iz}$ does not affect perpendicular ion motion, from (7.2) we have

$$\mathbf{V}_{\perp i} = \frac{c}{eBn_i}\mathbf{b} \times [\nabla(p_i + p_n) + en_i \nabla \Phi] \ . \tag{7.8}$$

Note that the neutral pressure gradient term contributes additively to the ion diamagnetic flow.

We also need $\mathbf{V}_{\parallel i}$ to compute $\mathbf{V}_{pi}$. Using the velocity variables ($v, \xi \equiv v_{\parallel}/v$), we write the linearized ion drift-kinetic equation as

$$\omega \frac{\partial h}{\partial \theta} + Mh - Ch - Xh = -\mathbf{V}_d \cdot \nabla f_M - Mf_d + Xf_d \ . \tag{7.9}$$

Here θ is the poloidal angle, $\mathbf{V}_d$ is the guiding-center drift,

$$f \equiv f_M + f_d + h$$

is the ion distribution function,

$$f_d \equiv \frac{2\xi v V_{\parallel i}}{v_{ti}^2} f_M$$

is the first order perturbation of a displaced Maxwellian,

$$\omega \equiv \frac{v\xi}{qR}$$

is the transit frequency,

$$Mf = -\frac{r}{2R}\frac{\omega}{\xi} \sin\theta (1 - \xi^2) \frac{\partial f}{\partial \xi}$$

is the mirror force,

$$Xf = -\nu_x \left(f - \frac{g}{n_n} \int d^3v \, f \right)$$

is the CX operator, ν_x is the ion CX collision frequency, and $C = C_{ii}$ is the ion-ion collision operator.

For plateau ions, we order our operators according to

$$\omega > C + X > M \ .$$

To lowest order, for small ν/ω, the solution to (7.9) is

$$h = (\nu^2 + \omega^2)^{-1}(\nu \sin\theta - \omega\cos\theta) Q_s f_M \ , \tag{7.10}$$

where $\nu \equiv \nu_c + \nu_x$, and

$$Q_s = \frac{v^2}{2\Omega R}(1 - \xi^2) \left[\frac{p_i'}{p_i} + \frac{e\Phi'}{T_i} + \left(\frac{v^2}{v_{ti}^2} - \frac{5}{2} \right) \frac{T_i'}{T_i} + \frac{2\Omega r}{qR} \frac{V_{\parallel i}}{v_{ti}^2} \right] \ .$$

Using (7.10), we return to the "exact" drift-kinetic equation (7.9), multiply by $m_i v \xi$, integrate over velocity and perform a flux-surface average (here equivalent to a θ-average). We obtain for the parallel ion flow

$$V_{\parallel i} = D \, U_{neo} \ , \tag{7.11}$$

with the CX damping coefficient, D, given by

$$D = \left[1 + \frac{2}{\sqrt{\pi}} \frac{\nu_x}{\omega} \left(\frac{R}{r} \right)^2 \right]^{-1} \ . \tag{7.12}$$

Here U_{neo} denotes the conventional [16] neoclassical parallel flow

$$U_{neo} = -\frac{T_i}{m_i \Omega_p} \left(\frac{p_i'}{p_i} + \frac{e\Phi'}{T_i} - k\frac{T_i'}{T_i} \right) \ , \tag{7.13}$$

$k = -1/2$ for plateau ions, and Ω_p is the ion poloidal gyrofrequency. From (7.8) and (7.11)–(7.13), we obtain the net poloidal ion flow

$$V_{pi} = \frac{c}{eBn_i}\left[p_n' + (1-D)(p_i' + en_i\Phi') + Dkn_iT_i'\right] , \tag{7.14}$$

and from (7.6), (7.7), and (7.14), the net poloidal impurity flow

$$V_{pz} = -\frac{c}{eB}\left[DT_i\frac{n_i'}{n_i} + (1-k)DT_i' + (D-1)e\Phi' - T_z\frac{n_z'}{zn_z} - \frac{1}{z}T_z'\right] .$$

We have therefore found that CX with neutrals causes the poloidal ion and impurity flows to depend on the radial electric field. Note that D approaches 1 in the limit of no charge-exchange. In this limit, our results agree with those of Kim, *et al.* [17].

Chapter 8.

Neutral-Plasma Interaction: Long CX mfp

Let $g(x, \mathbf{v})$ and $f(x, \mathbf{v})$ denote the neutral and ion distribution functions, respectively. In the half-space slab, $x \geq 0$, we consider a steady-state plasma in which ions and neutrals interact through charge-exchange (CX) and ionization. Assuming that the ionization frequency, ν_z, and the CX frequency, $\nu_x = n_i \langle \sigma_x v_r \rangle$, are constants in both x and $\mathbf{v}$, the neutral kinetic equation takes the simple form.

$$v_x \frac{\partial g(x, \mathbf{v})}{\partial x} + \nu_x [g(x, \mathbf{v}) - n(x) \widehat{f}(x, \mathbf{v})] + \nu_z g(x, \mathbf{v}) = 0 \ , \tag{8.1}$$

where

$$n(x) = \int d^3 v \, g(x, \mathbf{v})$$

is the neutral density, and

$$\widehat{f}(x, \mathbf{v}) = \frac{f(x, \mathbf{v})}{n_i}$$

is the normalized ion distribution.

We want a Green's function $G(x - x', v_x)$ which satisfies

$$- v_x \frac{\partial G(x - x', v_x)}{\partial x} + \nu G(x - x', v_x) = \delta(x - x') \ , \tag{8.2}$$

where $\nu = \nu_x + \nu_z$ and where $\delta(x - x')$ is the Dirac delta function. The general solution of (8.2) is

$$G(x - x', v_x) = \left[\frac{A(v_x) - \Theta(x - x')}{\nu_x} \right] \exp \left[\frac{(x - x')\nu}{v_x} \right] \ ,$$

where $\Theta(x-x')$ is the Heaviside step function and $A(v_x)$ is arbitrary.

Since we want $G(x-x', v_x)$ to remain finite over the range of its arguments, we choose $A(v_x) = \Theta(v_x)$, and our Green's function is

$$G(x-x', v_x) = \left[\frac{\Theta(v_x) - \Theta(x-x')}{\nu_x}\right] \exp\left[\frac{(x-x')\nu}{v_x}\right] . \tag{8.3}$$

Now, multiplying (8.1) by $G(x-x', v_x)$, (8.2) by $g(x, \mathbf{v})$, and integrating the difference over $0 \leq x \leq \infty$, we obtain for the neutral distribution function the integral equation

$$g(x,\mathbf{v}) = v_x G(-x, v_x) g(0, \mathbf{v}) + \nu_x \int_0^\infty dx'\, G(x'-x, v_x) n(x') \widehat{f}(x', \mathbf{v}) , \tag{8.4}$$

where

$$g(0, \mathbf{v}) = g_+(0, \mathbf{v})\Theta(v_x) + g_-(0, \mathbf{v})\Theta(-v_x)$$

is a boundary condition on the neutral distribution function. Note that g_+ is to be specified while g_- is to be calculated.

An instructive recursive procedure may be applied to (8.4). Using (8.2), we substitute into (8.4)

$$G(x'-x, v_x) = \frac{1}{\nu}\left[v_x \frac{\partial G(x'-x, v_x)}{\partial x'} + \delta(x'-x)\right] .$$

The result is the exact expression

$$\begin{aligned} g(x,\mathbf{v}) = {} & v_x G(-x, v_x) g(0, \mathbf{v}) + \frac{\nu_x}{\nu} n(x) \widehat{f}(x, \mathbf{v}) \\ & - \frac{\nu_x}{\nu} v_x G(-x, v_x) n(0) \widehat{f}(0, \mathbf{v}) \\ & - \frac{\nu_x}{\nu} \int_0^\infty dx'\, v_x G(x'-x, v_x) \frac{\partial}{\partial x'} [n(x') \widehat{f}(x', \mathbf{v})] . \end{aligned}$$

Repeating this substitution indefinitely, we obtain the series

$$g(x,\mathbf{v}) = v_x G(-x,v_x) g(0,\mathbf{v}) + \frac{\nu_x}{\nu}\sum_{k=0}^{\infty}\left(-\frac{v_x}{\nu}\right)^k \left\{\frac{\partial^k}{\partial x^k}[n(x)\widehat{f}(x,\mathbf{v})] \right.$$
$$\left. - v_x G(-x,v_x)\frac{\partial^k}{\partial x^k}[n(0)\widehat{f}(0,\mathbf{v})]\right\} \tag{8.5}$$

which converges if $|v_x| < \nu L_{nf}$, where L_{nf} is the scale length of $n(x)\widehat{f}(x,\mathbf{v})$. Expression (8.5) explicitly exhibits non-local transport since it involves all derivatives of the distribution function. Note that (8.5) must still be solved for $g(x,\mathbf{v})$ which is buried in $n(x)$.

For our present purpose, assume that we are far enough from the wall that the boundary terms in (8.5) have decayed away due to the form of $G(-x,v_x)$. Also assume that $\nu_z \ll \nu_x$. Keeping terms through $k=1$ we have

$$g(x,\mathbf{v}) = n(x)\widehat{f}(x,\mathbf{v}) - \frac{v_x}{\nu_x}\frac{\partial}{\partial x}n(x)\widehat{f}(x,\mathbf{v})\ .$$

This is precisely the 1-D specialization of the result (5.29) given by our previous short CX mfp theory.

Now, to make progress in solving (8.4), we choose

$$g_+(0,\mathbf{v}) = \frac{N_w}{\pi^{\frac{3}{2}}v_w^3}\exp\left(-\frac{v^2}{v_w^2}\right)$$

and

$$\widehat{f}(x,\mathbf{v}) = \frac{1}{\pi^{\frac{3}{2}}v_w^3}\left(1+\frac{2\mathbf{v}\cdot\mathbf{V}(x)}{v_t^2}\right)\exp\left(-\frac{v^2}{v_t^2}\right)\ ,$$

where $N_w \equiv n(0)$, $\mathbf{V}(x)$ is some ion flow velocity, and where the ion thermal speed $v_t \gg v_w$ the thermal speed of neutrals from the wall. If we then assume that the CX mean-free-path $\lambda_x \equiv v/\nu_x$ is much greater than the neutral density scale length L, then the first term in (8.4) dominates, and we may proceed recursively.

The zeroth recursive solution of (8.4) is

$$g^0(x,\mathbf{v}) = v_x G(-x, v_x) g(0,\mathbf{v}) \ ,$$

or explicitly,

$$g^0(x,\mathbf{v}) = \frac{N_w}{\pi^{\frac{3}{2}} v_w^3} \exp\left(-\frac{x\nu}{v_x} - \frac{v^2}{v_w^2}\right) \Theta(v_x) \ . \tag{8.6}$$

Integrating (8.6) over velocity space, we find for the zeroth neutral density

$$n^0(x) = \frac{N_w}{\sqrt{\pi}} \mathcal{F}_0\left(\frac{x\nu}{v_w}\right) \ , \tag{8.7}$$

where the functions

$$\mathcal{F}_m \equiv \int_0^\infty dt\, t^m \exp\left(-\frac{x}{t} - t^2\right)$$

are tabulated [19].

For the next recursion, we substitute (8.7) into (8.4) to get

$$g^1(x,\mathbf{v}) = g^0(x,\mathbf{v}) + \nu_x \int_0^\infty dx'\, G(x'-x, v_x) n^0(x') \widehat{f}(x,\mathbf{v}) \ .$$

Explicitly, the first correction to $g^0(x,\mathbf{v})$ is

$$g^1 - g^0 = \frac{N_w \nu_x}{\pi^2 v_t^3 v_x} \exp\left(-\frac{x\nu}{v_x} - \frac{v^2}{v_w^2}\right) \int_0^\infty dx' \left\{ \exp\left(\frac{x'\nu}{v_x}\right) \right.$$
$$\left. \cdot\ \mathcal{F}_0\left(\frac{x'\nu}{v_w}\right) \left(1 + \frac{2\mathbf{v}\cdot\mathbf{V}(x')}{v_t^2}\right) \left[\Theta(v_x) - \Theta(x'-x)\right] \right\} \ . \tag{8.8}$$

Using the relation

$$\mathcal{F}_n'(x) = -\mathcal{F}_{n-1}(x) \ ,$$

we can evaluate (8.8) by performing successive integrations by parts. This results in two expansions: one for large $|v_x|$ and one for small $|v_x|$. The convergence of these expansions provides a constraint on L_V, the scale length of $\mathbf{V}(x)$.

The expansion of $(g^1 - g^0)$ for large $|v_x|$ is

$$(g^1 - g^0)_> = \frac{N_w}{\pi^2 v_t^3}\left(\frac{\nu_x}{\nu}\right)\left(\frac{v_w}{v_x}\right)\exp\left(-\frac{v^2}{v_t^2}\right)\left\{\exp\left(-\frac{x\nu}{v_x}\right)\right.$$
$$\cdot \sum_{k=0}^{\infty}\mathcal{F}_{k+1}(0)\left(\frac{v_w}{v_x}\right)^k\left[1+\left(\frac{2}{v_t^2}\right)\mathbf{v}\cdot\sum_{l=0}^{k}\binom{k}{l}\left(\frac{v_x}{\nu}\right)^l\mathbf{V}^{(l)}(0)\right]$$
$$\left. - \sum_{k=0}^{\infty}\mathcal{F}_{k+1}\left(\frac{x\nu}{v_w}\right)\left(\frac{v_w}{v_x}\right)^k\left[1+\left(\frac{2}{v_t^2}\right)\mathbf{v}\cdot\sum_{l=0}^{k}\binom{k}{l}\left(\frac{v_x}{\nu}\right)^l\mathbf{V}^{(l)}(x)\right]\right\} \tag{8.9}$$

where $\binom{k}{l}$ are binomial coefficients, and where the superscript (l) refers to the l^{th} derivative with respect to x. The expansion (8.9) converges if $|v_x| > v_w$ and $L_V > v_w/\nu \equiv \lambda_w$.

The expansion of $(g^1 - g^0)$ for small $|v_x|$ is

$$(g_1 - g^0)_< = -\frac{N_w}{\pi^2 v_t^3}\left(\frac{\nu_x}{\nu}\right)\exp\left(-\frac{v^2}{v_t^2}\right)\left\{\exp\left(-\frac{x\nu}{v_x}\right)\sum_{k=0}^{\infty}\left(-\frac{v_x}{v_w}\right)^k\left[\mathcal{F}_{-k}(0)\right.\right.$$
$$\left. + \left(\frac{2}{v_t^2}\right)\mathbf{v}\cdot\sum_{l=0}^{k}\binom{k}{l}\left(-\frac{v_w}{\nu}\right)^l\mathcal{F}_{l-k}(0)\mathbf{V}^{(l)}(0)\right] - \sum_{k=0}^{\infty}\left(-\frac{v_x}{v_w}\right)^k\left[\mathcal{F}_{-k}\left(\frac{x\nu}{v_w}\right)\right.$$
$$\left.\left. + \left(\frac{2}{v_t^2}\right)\mathbf{v}\cdot\sum_{l=0}^{k}\binom{k}{l}\left(-\frac{v_w}{\nu}\right)^l\mathcal{F}_{l-k}\left(\frac{x\nu}{v_w}\right)\mathbf{V}^{(l)}(x)\right]\right\}. \tag{8.10}$$

The expansion (8.10) converges if $|v_x| < v_w$ and $|v_x| < \nu L_V$ for $x > 0$, but it has a logarithmic singularity at $x = 0$ for $k > 0$. Neutral-neutral collisions very near the wall would prevent such a singularity and would support the neutral

Maxwellian boundary distribution. However, their inclusion would make the problem nonlinear. So we will instead ignore the mild singularity near $x = 0$.

Keeping terms in (8.9) through order v_w/v_x and λ_w/L_V, we find

$$(g^1 - g^0)_> = \frac{N_w}{2\pi^2 v_t^3}\left(\frac{\nu_x}{\nu}\right)\left(\frac{v_w}{v_x}\right)\exp\left(-\frac{v^2}{v_t^2}\right)\left\{\Theta(v_x)\exp\left(-\frac{x\nu}{v_x}\right)\right.$$
$$\cdot\left[1 + \frac{2\mathbf{v}\cdot\mathbf{V}(0)}{v_t^2} + \frac{v_w\sqrt{\pi}\mathbf{v}\cdot\mathbf{V}'(0)}{\nu v_t^2}\right] - 2\mathcal{F}_1\left(\frac{x\nu}{v_w}\right)\left[1 + \frac{2\mathbf{v}\cdot\mathbf{V}(x)}{v_t^2}\right]$$
$$\left. - \left(\frac{4v_w}{\nu v_t^2}\right)\mathcal{F}_2\left(\frac{x\nu}{v_w}\right)\mathbf{v}\cdot\mathbf{V}'(x)\right\} \tag{8.11}$$

since $\mathcal{F}_1(0) = 1/2$ and $\mathcal{F}_2(0) = \sqrt{\pi}/4$.

Keeping terms in (8.10) of order unity, we have

$$(g^1 - g^0)_< = -\frac{N_w}{2\pi^2 v_t^3}\left(\frac{\nu_x}{\nu}\right)\exp\left(-\frac{v^2}{v_t^2}\right)\left\{\sqrt{\pi}\Theta(v_x)\exp\left(-\frac{x\nu}{v_x}\right)\left[1 + \frac{2\mathbf{v}\cdot\mathbf{V}(0)}{v_t^2}\right]\right.$$
$$\left. -2\mathcal{F}_0\left(\frac{x\nu}{v_w}\right)\left[1 + \frac{2\mathbf{v}\cdot\mathbf{V}(x)}{v_t^2}\right]\right\}, \tag{8.12}$$

since $\mathcal{F}_0(0) = \sqrt{\pi}/2$.

We have thus calculated a recursive approximation to the neutral distribution function whose contributions are (8.6), (8.11), and (8.12). In principle then, the neutral density, and the particle, heat, and momentum fluxes, can be estimated by performing appropriate velocity integrals of these contributions. Although these integrals are straightforward, they involve several special functions, and the appearance of the results is not particularly illuminating. However, we will present the first recursive neutral density here since it compares favorably with previous work.

Integrating (8.6), (8.11), and (8.12) over velocity space, using limits of integration appropriate to the parameter regimes, for $x > \lambda_w$, our estimate of the first recursive neutral density is

$$n^1(x) \approx \frac{N_w}{\sqrt{\pi}} \left\{ \mathcal{F}_0\left(\frac{x}{\lambda_w}\right) + \frac{1}{\sqrt{\pi}} \frac{\nu_x}{\nu} \frac{v_w}{v_t} \left[2\mathcal{F}_0\left(\frac{x}{\lambda_w}\right) \right.\right.$$
$$\left.\left. - \frac{\sqrt{\pi}}{2} E_2\left(\frac{x}{\lambda_w}\right) + \mathcal{F}_{-1}\left(\frac{x}{\lambda_t}\right) \right] \right\}$$

where $\lambda_t = v_t/\nu$ and where E_2 is the second-order exponential integral. Note that for $x \to \infty$ this looks like

$$n^1(x) \sim \left(\frac{x}{2\lambda_t}\right)^{-\frac{1}{3}} \exp\left[-3\left(\frac{x}{2\lambda_t}\right)^{\frac{2}{3}}\right] .$$

This expression agrees with Connor [6].

Chapter 9.

Summary

Taking advantage of the simplicity of the charge-exchange (CX) operator introduced in Chapter 2, we demonstrated in Chapter 3 that CX obeys an H-theorem. Then in Chapter 4 we developed a general variational principle for finding the neutral transport coefficients. This general variational method has the following advantages: it treats three dimensional plasmas with arbitrary temperature and density profiles; it includes the effects of neutrals on the plasma; it allows for arbitrary CX cross-section and mean-free-path; and it provides relatively simple asymptotic formulas for various quantities of interest in limiting parameter regimes. However, this general variational method also has the following disadvantages: it is spatially nonlocal so that the trial functions must include both **x**- and **v**-dependence; and it involves operators that are not self-adjoint, therefore requiring dual trial functions. Nonetheless, in any parameter regime that allows an approximate analytic solution, the general variational principle becomes directly useful in obtaining higher-order information about CX effects from quite simple integrals.

In Chapter 5 the special case of short CX mean-free-path (λ_x) was found to provide an ordering in which the variational theory is both local and self-adjoint. We found that this short λ_x specialization of the general variational method may be used to obtain expressions for the neutral entropy production from which variationally accurate neutral transport coefficients can be read off by inspection.

The realistic simplification that the product of CX cross-section with relative velocity ($\sigma_x|\mathbf{v} - \mathbf{v}'|$) is constant allowed us, in the case of short λ_x, to avoid the variational approach and to find the neutral distribution function directly. We then calculated the neutral entropy production rate and presented a full set of neutral transport coefficients. Their form confirms our physical picture of neutrals executing a random walk with step size λ_x.

In Chapter 6 our findings about neutral transport combined with analysis of the momentum and energy moments of the ion and neutral kinetic equations lead us to simple, sensible conclusions about the effects of neutrals on ion fluid behavior and transport.

We found that the neutral stress simply adds to the ion stress in the ion momentum balance equation. Since the ion and neutral pressure gradients oppose each other in much of the edge region, the observed effect on ion perpendicular flow is diminished ion diamagnetic rotation.

Furthermore, because of the effect of neutrals on the perpendicular plasma current (6.9), we concluded that experimental estimates of plasma beta must be performed with care whenever neutrals may be present.

We demonstrated CX causes neutral viscosity to contribute directly to ion viscosity. Neutral viscosity was therefore compared to classical perpendicular ion viscosity and was found to dominate everywhere, and in the edge region by a large factor. Thus CX appears to be related to measurements of anomalously high ion viscosity in the tokamak.

Similarly, we found that neutral energy flux simply adds to ion energy flux in the ion energy balance equation. Although ion particle transport was shown to be unaffected by CX, the effect on ion energy transport appears as enhanced ion heat conduction. The neutral heat flux was compared to the neoclassical ion heat flux (6.10) and was found to be larger in typical circumstances.

In Chapter 7 we gave a simple drift-kinetic derivation of the expressions for the poloidal ion and impurity flows in the presence of charge exchange drag and ion-impurity collisions. We found that CX damps ion and impurity rotation, and causes the poloidal ion and impurity flows to depend on the radial electric field.

Finally, in Chapter 8 we treated the neural-plasma interaction problem in the long CX mean-free-path regime in slab geometry. An appropriate Green's function was found for the neutral kinetic equation and a recursive procedure was used to find suitable expansions for the neutral distribution function. The results compared favorably with previous work.

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