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1/f Noise in Two-Dimensional Fluids

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Abstract

We derive an exact result on the velocity fluctuation power spectrum of an incompressible two-dimensional fluid. Employing the fluctuation-dissipation relationship and the enstrophy conservation, we obtain the frequency spectrum of a $1/f$ form.

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$1/f$ noise has been observed in a wide variety of physical systems including current fluctuations in metals and semiconductors,[1,2] changes in the volume of musical notes,[3] and the arrangements of bases in DNA strands,[4] to name a few. The $1/f$ spectrum is special in that it lacks any specific time scales of the system and it diverges in its infrared (low) and ultraviolet (high) energy integrals. Despite the numerous manifestations of this phenomenon, $1/f$ noise still defies general explanation. It is described successfully only in particular systems and, even then, only by phenomenological models. A review of some of the models of $1/f$ noise in metals and semiconductors is given in Ref. 2. Analytical approaches seem to be few and far between. Two examples of which we are aware are those of Klimontovich,[5] who derives $1/f$ noise in diffusive systems, after introducing a minimal physical volume over which physical quantities are averaged to obtain measurable quantities, and of Jensen,[6] who finds $1/f$ noise in diffusive volumes with white noise boundary conditions.

In this paper, we show that incompressible fluid films exhibit $1/f$ noise in their velocity power spectra. The result is purely analytical, making use of only the following assumptions: the linearized equations of motion of an incompressible fluid, the fluctuation-dissipation theorem, and certain unique but well established results of thermodynamics applied to two-dimensional fluid motion. First we outline our procedure for obtaining this result. Then we discuss possible applications of this result to systems which may be mathematically similar to the two-dimensional incompressible fluid.

We compute the frequency fluctuation spectra of the velocity in two-dimensional incompressible fluid motion. We follow typical fluctuation-dissipation theoretical procedures for dissipative, thermal equilibrium systems.[7,8] The starting point is the equation of motion of an incompressible, viscous fluid, driven by a stochastic thermal process:

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\boldsymbol{\nabla} p}{\rho} + \mu \nabla^2 \mathbf{v} + \mathbf{a}(\mathbf{x}, t) \ , \tag{1}$$

where μ is the fluid viscosity, ρ is assumed to be constant, and $\mathbf{a}(\mathbf{x}, t)$ is a random Gaussian field producing accelerations in the velocity and magnetic fields. The stochastic acceleration $\mathbf{a}(\mathbf{x}, t)$ represent a variety of external influences such as quantum mechanical fluctuations and thermal fluctuations from some heat bath in contact with the plasma and of internal influences arising from nonlinear coupling of the second term on the left-hand side of Eq. (1).

The spirit of the fluctuation-dissipation theorem is to look for the system's linear response in a thermal equilibrium, relating the linear response to system's fluctuations that include nonlinearities of the system.[9] The equation (1) is thus linearized to

$$\partial_t \mathbf{v} = -\frac{\boldsymbol{\nabla} p}{\rho} + \mu \nabla^2 \mathbf{v} + \mathbf{a}(\mathbf{x}, t) \ . \tag{2}$$

We now assume incompressibility. [This assumption is well fulfilled in low-frequency (infrared) regimes]. In this case the pressure gradient in the linear approximation serves only to cancel the compressive part of the stochastic accelerations $\mathbf{a}_{\mathrm{comp}}(\mathbf{x}, t)$. This can be seen by taking the divergence of both sides of the linearized momentum equation. Therefore, we can drop the pressure gradients and the compressive stochastic acceleration from Eq. (2) to get

$$\partial_t \mathbf{v}_{\mathrm{rot}} = \mu \nabla^2 \mathbf{v}_{\mathrm{rot}} + \mathbf{a}_{\mathrm{rot}}(\mathbf{x}, t) \ , \tag{3}$$

where the subscript "rot" indicates the rotational (*i.e.* divergence-free) part. From hereon, it is understood that we are dealing with an incompressible fluid, so the subscript "rot" will be dropped.

Fourier transformation of Eq. (3) leads to

$$(-i\omega + \mu k^2)\mathbf{v}(\mathbf{k}, \omega) = \mathbf{a}(\mathbf{k}, \omega) \ .$$

We multiply both sides of this equation by their respective complex conjugates and ensemble average the result. The relation obtained is

$$(\omega^2 + \mu^2 k^4) \left\langle v^2 \right\rangle_{\mathbf{k},\omega} = \left\langle a^2 \right\rangle_{\mathbf{k},\omega} \ . \tag{4}$$

(Here, v^2 can refer to the total velocity magnitude or the magnitude of one component of the velocity; the same can be said for a^2. The spectra to be calculated will differ only by a factor of two, as long as the two quantities are defined consistently with each other. We take v^2 to represent the magnitude of one component, for example $v^2 = v_x^2$.)

We assume that the random accelerations have no correlation in time. They, therefore, exhibit a white noise frequency spectrum independent of ω:

$$(\omega^2 + \mu^2 k^4) \left\langle v^2 \right\rangle_{\mathbf{k},\omega} = \left\langle a^2 \right\rangle_{\mathbf{k},\omega} = \left\langle a^2 \right\rangle_{\mathbf{k}} . \tag{5}$$

We wish to find $\langle v^2 \rangle_{\mathbf{k},\omega}$. To accomplish this, we first find $\langle a^2 \rangle_{\mathbf{k}}$.

$\langle a^2 \rangle_{\mathbf{k}}$ can be related to $\langle v^2 \rangle_{\mathbf{k}} = 1/(2\pi) \int d\omega \, \langle v^2 \rangle_{\mathbf{k},\omega}$. Specifically, from Eq. (5) and the definition of $\langle v^2 \rangle_{\mathbf{k}}$, we have

$$\left\langle a^2 \right\rangle_{\mathbf{k}} \times \int \frac{d\omega}{2\pi} \frac{1}{\omega^2 + \mu^2 k^4} = \frac{\langle a^2 \rangle_{\mathbf{k}}}{2\mu k^2} = \left\langle v^2 \right\rangle_{\mathbf{k}} .$$

The quantity $\langle v^2 \rangle_{\mathbf{k}}$ need not be calculated: it is given, very generally, by 'thermodynamics' of discrete or continuous fields. For example, $\langle v^2 \rangle_{\mathbf{k}}$ will often follow the equipartition law so that $\langle v^2 \rangle_{\mathbf{k}} = T/\rho$, where T is the temperature of the fluid. (If v^2 is referred to the total velocity magnitude, then $\langle v^2 \rangle_{\mathbf{k}}$ would equal $2T/\rho$, hence the aforementioned difference of a factor of two.) $\langle a^2 \rangle_{\mathbf{k}}$ is then determined. Substituting it back into Eq. (5), we find

$$\left\langle v^2 \right\rangle_{\mathbf{k},\omega} = \frac{2\mu k^2}{\omega^2 + \mu^2 k^4} \left\langle v^2 \right\rangle_{\mathbf{k}} . \tag{6}$$

We are now in a position to calculate the frequency spectrum $\langle v^2 \rangle_{\omega} = 2/(2\pi)^n \int d^n k \, \langle v^2 \rangle_{\mathbf{k},\omega}$, n being the dimensionality of the fluid. (The leading factor of 2 is included to compensate for negative frequencies, which are experimentally indistinguishable from positive frequencies and will contribute equally to any measured value.) If equipartition is assumed, $\langle v^2 \rangle_{\omega}$ behaves qualitatively as follows: We specify two different scale lengths L and l. L represents the spatial extent of the fluid and l represents the scale at which the fluid theory breaks down.

For instance, l could be a multiple of the average inter-molecular spacing in the fluid. It can be shown that if $\omega \ll (2\pi)^2\mu/L^2$, the spectrum is nearly constant. Also, if $\omega \gg (2\pi)^2\mu/l^2$, the spectrum falls off as ω^{-2}. In the range $(2\pi)^2\mu/l^2 \gg \omega \gg (2\pi)^2\mu/l^2$, the result depends on the dimensionality of the fluid. Specifically, if $n = 1$, $\langle v^2 \rangle_\omega$ falls off as $\omega^{-1/2}$; if $n = 2$, $\langle v^2 \rangle_\omega$ is nearly constant; if $n = 3$, $\langle v^2 \rangle_\omega$ increases as $\omega^{1/2}$. Conventional equipartition does *not* produce a $1/f$ noise spectrum.

However, thermodynamics does not always require that $\langle v^2 \rangle_{\mathbf{k}}$ equal T/ρ. The equipartition law rests on the assumption that energy, momentum, and directly related quantities are the only conserved quantities in the system to be studied. A two-dimensional fluid, however, has an additional conserved quantity. This quantity, denöted "enstrophy," is given by

$$\Omega = \int d^2x (\nabla \times v)^2 .$$

When thermodynamics is treated so as to account for this additional conserved quantity, the resulting wavevector spectrum is

$$\left\langle v^2 \right\rangle_{\mathbf{k}} = \frac{1}{(\beta + 2\alpha k^2)\rho} , \tag{7}$$

where $\beta = 1/T$ and α is the inverse of a generalized temperature corresponding to the enstrophy.[10,11] This hydrodynamic mode approach is appropriate for fluids in the low frequency limit.[9]

Since equipartition is so familiar and very much expected in this type of study, perhaps we should justify our departure from it before presenting our frequency spectrum results. The equipartition distribution is derived from the partition function

$$Z = \sum_r e^{-\beta \epsilon_r} ,$$

where r is a label indicating the possible states of the system under study, and ϵ_r is the energy of the state r. The partition function is commonly derived by considering the interaction of a system under experimental study with a larger heat bath. The energy of the total system (*i.e.* studied system + heat bath) is taken to be constant. This returns the above partition function.

In the case of a two-dimensional fluid, however, the equations of motion not only allow, but actually *demand* that a second quantity, namely enstrophy, be conserved. It can be shown that enstrophy conservation changes the partition function to

$$Z = \sum_{\mathbf{k}} e^{-(\beta\epsilon_{\mathbf{k}} + \alpha k^2 \epsilon_{\mathbf{k}})} ,$$

where $\mathbf{k}$ is the wavenumber of a Fourier mode of fluid motion. A derivation of the energy distribution, which parallels the standard derivation of the equipartition law, will show that the new energy distribution is given by Eq. (7).

If we wished, we could average over an ensemble of systems (or over various domains) with differing values of enstrophy and regain equipartition. This is, however, unnecessary. More to the point, an average over enstrophy results in a loss of useful information, since a single two-dimensional fluid conserves its own particular value of enstrophy.

The frequency spectrum can now be found by substituting Eq. (7) into Eq. (6) and integrating as

$$\left\langle v^2 \right\rangle_\omega = \frac{2}{\rho} \int \frac{d^2k}{(2\pi)^2} \frac{2\mu k^2}{\omega^2 + \mu^2 k^4} \frac{1}{\beta + 2\alpha k^2} . \tag{8}$$

The integral can be evaluated exactly by means of partial fractions. The end result is

$$\left\langle v^2 \right\rangle_\omega = \frac{1}{4\alpha\rho} \frac{\omega - \frac{\mu\beta}{\pi\alpha} \ln\left(\frac{2\alpha\omega}{\mu\beta}\right)}{\omega^2 + \frac{\beta^2\mu^2}{4\alpha^2}} . \tag{9}$$

Note that, in the regime $\omega \gg \mu\beta/\alpha$, ω is dominant over $(\mu\beta/\pi\alpha)\ln(2\alpha\omega/\mu\beta)$, where we find

$$\left\langle v^2 \right\rangle_\omega = \frac{1}{4\rho\alpha\omega} . \tag{10}$$

Also, if the viscosity μ is set to zero, we are led to Eq. (10) exactly for all ω. We have, therefore, analytically uncovered $1/f$ noise behavior in a two-dimensional, incompressible fluid. The smaller the viscosity of the fluid is, the more dominant the $1/f$ noise becomes.

The limit of Eq. (10) can also be obtained by taking the limit $\mu \to 0$ before performing the integral in Eq. (8). When the integral is written in terms of $\nu = \omega/k^2$, the result is

$$\left\langle v^2 \right\rangle_\omega = \frac{4}{\rho} \int_0^\infty \frac{d\nu}{4\pi} \frac{\mu}{\nu^2 + \mu^2} \frac{1}{\beta\nu + 2\alpha\omega} .$$

In the limit $\mu \to 0$, $(\beta\alpha + 2\alpha\omega)^{-1}$ can be approximated by $1/2\alpha\omega$, since its contribution to the integral will be cut off at larger ν. The integral can then be rewritten to give

$$\left\langle v^2 \right\rangle_\omega = \frac{1}{2\alpha\rho\omega} \int_0^\infty \frac{d\nu}{\pi} \frac{\mu}{\nu^2 + \mu^2} .$$

This gives our previous result of $\langle v^2 \rangle_\omega = 1/4\alpha\rho\omega$.

These considerations actually open our result to a possible criticism: In the limit $\mu \to 0$, the high k contributions (corresponding to the small ν contributions in the above integral) become increasingly important in determining $\langle v^2 \rangle_\omega$. But the hydrodynamic model we have adopted throughout our calculations is inapplicable beyond some maximum k value $k_{\max}$ being the inverse of length at which the particular nature of the fluid becomes apparent.[8] We must, instead, integrate in k from $k = 0$ to $k = k_{\max}$. The result, to first order in $1/k_{\max}^2$ ($k_{\max}^2 \gg \beta/\alpha, \omega/\mu$) is

$$\left\langle v^2 \right\rangle_\omega = \frac{1}{4\rho\alpha} \frac{\omega\left(1 - \frac{\beta}{\pi\alpha k_{\max}^2}\right) - \frac{\mu\beta}{\pi\alpha} \ln\left(\frac{2\alpha\omega}{\mu\beta}\right) - \frac{2\omega^2}{\pi\mu k_{\max}^2}}{\omega^2 + \frac{\mu^2\beta^2}{4\alpha^2}} .$$

A $1/f$ spectrum will be found, therefore, in the range

$$\mu k_{\max}^2 \gg \omega \gg \mu\beta/\alpha . \tag{11}$$

Therefore, for $1/f$ noise to exist, $k_{\max}^2$ must be much larger than β/α.

$1/f$ noise is well documented in thin metal films.[1,2] If the conduction electrons in the film can be modeled, on the right spatial and temporal scales, in a manner similar to the incompressible fluid, the present work may find applicability here. It is of interest in this regard that the expression for the $1/f$ noise given here is inversely proportional to the fluid density ρ, while the $1/f$ noise in metals is often inversely proportional to the number of charge carriers in the sample. Hall effect $1/f$ noise is also measured in metal films[12] and may have common ground with the present work. Experiments have shown that high T_c superconductors exhibit $1/f$ voltage noise.[13] If the two-dimensional vortex model of superconductivity in such materials proves correct, our work may have relevance here as well. In addition, Fukuda's tokamak experiments[14] appear to show magnetic field fluctuation spectra with behavior close to $1/f$, where we note that the main tokamak magnetic fluctuations are two-dimensional in nature.[10] With regard to plasma motion, the work of Taylor[15] has interesting similarities with the results presented here. He uses the fluctuation dissipation theorem to derive the electric field power spectrum of a magnetized plasma:[16]

$$\frac{\langle E^2\rangle_{\mathbf{k},\omega}}{8\pi} = \frac{T}{2\pi}\frac{k_D^2}{k^2+k_D^2}\frac{Dk^2}{\omega^2+(Dk^2)^2},$$

where k_D is the Debye wavenumber and D is a diffusion coefficient. This expression integrates over d^2k to give $1/f$ behavior. In fact, it is mathematically identical to the integrand in Eq. (8). The physical difference is that, in Taylor's work, the small wavelength cutoff comes from Debye screening, while in our work it is a consequence of enstrophy conservation. In addition, if the fluid (or plasma) in three dimensions for whatever reasons contains additional invariant(s), the appropriate statistics of thermodynamics of the fluid will deviate from those yielding the equipartition and the white ω-spectrum. Topological invariants in high Reynolds (or other order parameters) fluids are such examples of additional invariants.[11,17]

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