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Generalized relaxation theory and vortices in plasmas

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We present a generalization of the relaxation theory based in the canonical momentum of each species fluid in a multicomponent plasma. The generalized helicity, as a topological quantity, has a lifetime larger than the lifetime of the energy. The proposed variational principle suggests vortices structures. We study localized solutions, assuming the existence of a separatrix. Two-dimensional and three-dimensional solutions are studied for an electron-positron-proton plasma. Ideal magnetohydrodynamic (MHD) three-dimensional localized vortices are studied as well. Possible cosmological implications are discussed.

I. INTRODUCTION

A. Motivation

The observations of Galactic Magnetic Fields (GMF) of order 10^{-6} G in contrast from the intergalactic magnetic field $< 2 \times 10^{-8}$ G for $N_e < 4 \times 10^{-5}$ electrons/cm^3 brought to the astrophysicist an outstanding problem which concerns the origin and nature of these fields. For a review see Asseo and Sol report[1] and some of 918 references therein, and one by Kronberg.[2] Essentially there are two schools of thought, one believing in a primordial cosmic magnetic field from the Big Bang and another one in a generating magnetic field from a certain dynamical mechanism such as dynamos. Over the long period of time (about 10^{13} s) between the symmetry breaking of GUTs and before the recombination epoch in which electromagnetic plasma existed in the expanding and cooling universe, it is of considerable appeal to study if plasma phenomena played a role in the formation of structures that led to seeds of what we see today, such as galaxies and clusters of galaxies. Anderson and Kulsrud[3] discussed that it is difficult to make presently observed galactic magnetic fields through the dynamo mechanism after the recombination epoch. Ratra[4] discussed magnetic field generation during the inflation epoch. Tajima *et al.*[5,6] and Coles[7] discussed about magnetic fields during the plasma epoch. In particular, Tajima *et al.*[5] found that a large amount of magnetic fields with a minuscule spatial size exist even in a primordial plasma (such as $t \sim 10^{-2}$ sec) in a thermodynamical equilibrium, i.e. magnetic fields form tiny "bubbles." Furthermore, the recent work by Lai and Tajima[6] indicates that in the electron (positron) neutrino (antineutrino) plasma in an even earlier epoch (such as $t \sim 10^{-4} - 10^{-2}$ sec) electrons and neutrinos tend to separate their phases to form "bubbles." It is thus of interest to consider what are natural forms of magnetic and plasma topologies for such a plasma in a relaxed state.[8,9] That is, what is the likely direction of evolution of tiny bubbles of plasma magnetic fields created in such a plasma over some relaxation time (somewhat shorter, sometimes substantially shorter, than the collisional time). Therefore we consider the formation of structures such as vortices and/or solitary waves of magnetic fields.

During the radiation era (after the leptonic era) ionized matter consisted essentially of protons, electrons and positrons. Electromagnetic forces are dominant over gravitational force during this period. We assume slight asymmetry between electrons and positrons density, balanced by a background fluid of protons. In this work we also assume that the fluid velocities of electrons and positrons are equal due to strong coupling of photons to these two species of particles. Because of the observed isotropy and uniformity of the microwave background radiation emitted at the recombination epoch, it is reasonable for us to assume that the plasma is in isothermic equilibrium. As one of the main features of the cosmological plasma we assume no external field boundary conditions (no field at infinity).

B. Formulation of the problem

We seek localized solutions of electromagnetic fields in spatial magnetic field that do not cause charge separation. The localizability is required in virtue of the no external field boundary condition. Mathematically, at spatial infinity the fields must vanish. We may require the existence of a separatrix beyond which the fields decrease fast enough to have the total field energy finite. These solutions must be stable. The relaxation theory is appropriate to investigate this problem because of its self-organization feature which is in the spirit of self-generated and/or self-maintained configurations in a cosmological plasma, as we expect no 'external' energy source or sink.

Section II presents a generalized theory of relaxation for a multicomponent plasma and conclude that the formation of vortices is a possible equilibrium configuration.

In Sec. III we study static and stationary vortices in an electron-positron-proton (ee^+P or eeP for short) plasma and present possible solutions. That is, we assume the existence of a separatrix beyond which the magnetic field vanishes (in some cases asymptotically vanishes). A similar technique[11] has been proposed to find vortices of electron magnetohydrodynamic (EMHD) fluid in a uniform background proton plasma. In Sec. IV we present possible three-dimensional solitary vortices in ideal magnetohydrodynamics (MHD). These solutions have a preferred direction of interaction so that in an ensemble they have the tendency to form filamentary structures. In this way the theory provides a possible framework that there is a hierarchy of formation of large-scale structures in a plasma, beginning with spatial scale of thermal fluctuations in an eeP plasma up to large scales in an MHD plasma. Discussion of applications to cosmology is presented in the final section.

II. GENERALIZED RELAXATION THEORY FOR MULTICOMPONENT PLASMAS

A. Vorticity equation

The macroscopic equations of a plasma with N species are:

$$\nabla \cdot \mathbf{B} = 0 \tag{1}$$

$$\nabla \cdot \mathbf{E} = \frac{4\pi}{c} \sum_{a=1}^{N} q_a n_a \tag{2}$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \sum_{a=1}^{N} q_a n_a \mathbf{v_a} + \frac{\partial}{c\partial t}\mathbf{E} \tag{3}$$

$$\nabla \times \mathbf{E} = -\frac{\partial}{c\partial t}\mathbf{B} \tag{4}$$

$$\frac{\partial}{\partial t} n_a + \nabla \cdot (n_a \mathbf{v_a}) = 0 \ , \tag{5}$$

$$\nabla \cdot \mathbf{v_a} = 0 \ , \tag{6}$$

$$n_a m_a \left(\frac{d}{dt}\right)_a \mathbf{v_a} = q_a n_a \left(\mathbf{E} + \frac{\mathbf{v_a}}{c} \times \mathbf{B}\right) - n_a m_a \nabla \phi_G - \nabla P_a + \mathbf{R_a} \ , \tag{7}$$

where $(d/dt)_a \equiv \partial/\partial t + \mathbf{v_a} \cdot \nabla$ and

$$\mathbf{R_a} \equiv \mu_a \nabla^2 \mathbf{v_a} - m_a n_a \sum_b \nu_{ab}^c (\mathbf{v_a} - \mathbf{v_b}) \tag{8}$$

with the viscosity μ_a and the collision frequency for different fluids ν_{ab}^c. We assume an equation of state $P_a = P_a(n_a)$.

Let the electric and magnetic fields be given by their potentials $\mathbf{E} = -\boldsymbol{\nabla}\phi - \partial/c\partial t\,\mathbf{A}$ and $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. If we use the canonical momentum

$$\mathbf{p_a} \equiv m_a \mathbf{v_a} + \frac{q_a}{c}\mathbf{A} \tag{9}$$

of each of the species of the plasma and eliminate $\mathbf{A}$ in favor of $\mathbf{p_a}$ and $\mathbf{v_a}$, we get from the equation of motion (7)

$$\frac{\partial}{\partial t}\mathbf{p_a} = -\mathbf{v_a} \times \mathbf{\Omega_a} - \boldsymbol{\nabla}\epsilon_a + \mathbf{r_a} \; , \tag{10}$$

where the generalized vorticity is

$$\mathbf{\Omega_a} \equiv -\boldsymbol{\nabla} \times \mathbf{p_a} = -m_a \boldsymbol{\nabla} \times \mathbf{v_a} - \frac{q_a}{c}\mathbf{B} \; , \tag{11}$$

ϵ_a is the energy of the component a:

$$\epsilon_a \equiv \tfrac{1}{2} m_a \mathbf{v_a}^2 + q_a\phi + m_a\phi_G + \frac{P_a}{n_a} \; , \tag{12}$$

and

$$\mathbf{r_a} \equiv \frac{\mathbf{R_a}}{n_a} - \frac{P_a}{n_a}\boldsymbol{\nabla}(\log n_a) \; . \tag{13}$$

Applying the curl on (10) we get an equation for the vorticity:

$$\frac{\partial}{\partial t}\mathbf{\Omega_a} = \boldsymbol{\nabla} \times [\mathbf{v_a} \times \mathbf{\Omega_a}] - \boldsymbol{\nabla} \times \left(\frac{\mathbf{R_a}}{n_a}\right) \; . \tag{14}$$

In the limit of low-viscosity-high-density and low interfluid collision frequency

$$\frac{\mu_a}{n_a} \ll \min\left[Lm_a v_a; L^2 \frac{q_a}{c} B\right] \tag{15}$$

$$\nu_{ab}^c \ll \min\left[\frac{v_a}{L}, \frac{v_b}{L}; \frac{q_a B}{m_a c}\right] , \tag{16}$$

where L is a typical length in the problem, we can neglect $\mathbf{r_a}$ and $\mathbf{R_a}$. In this limit the equilibrium configuration will be the one in which ϵ_a is the level surface function for field lines of $\mathbf{v_a}$ and $\mathbf{\Omega_a}$:

$$\mathbf{v_a} \times \mathbf{\Omega_a} = \nabla \epsilon_a \tag{17}$$

$$\mathbf{v_a} \times \nabla \epsilon_a = 0 \tag{18}$$

$$\mathbf{\Omega_a} \cdot \nabla \epsilon_a = 0 . \tag{19}$$

The energy is constant along the streamlines of both the velocity and the vorticity fields.

Except possibly in subdomains where $\nabla \epsilon_a = 0$, the streamlines of $\mathbf{v_a}$ and $\mathbf{\Omega_a}$ lie on surfaces $\epsilon_a =$ const. The topology of these surfaces is determined by the topology of the sets of points at which $\nabla \epsilon_a = 0$: these points may be isolated, or they may fill three-dimensional subdomains.[11]

Let us consider a stationary solution propagating with some velocity $\mathbf{u}$, for which $\partial/\partial t = -\mathbf{u} \cdot \nabla$ and we get from (13)

$$[(\mathbf{v_a} - \mathbf{u}) \times \mathbf{\Omega_a}] = -\nabla(\epsilon_a - \mathbf{u} \cdot \mathbf{p_a}) \tag{20}$$

instead of (17). Now the divergenceless fields $\mathbf{\Omega_a}$ and $\mathbf{v_a} - \mathbf{u}$ lie on the level surfaces of $\epsilon_a - \mathbf{u}\cdot\mathbf{p_a}$. For localized solutions $\mathbf{v_a} \to 0$ and $\mathbf{B} \to 0$ as $\mathbf{r} \to \infty$. We conclude that outside a separatrix

$$\epsilon_a - \mathbf{u}\cdot\mathbf{p_a} = 0 \ , \tag{21}$$

and so

$$\mathbf{\Omega_a} = \alpha_a(\mathbf{r})(\mathbf{v_a} - \mathbf{u}) \ , \tag{22}$$

such that $\alpha_0(\mathbf{r})\mathbf{u}$ approaches zero at the spatial infinity. Note that $\alpha_a(\mathbf{r})$ is constant along $(\mathbf{v_a} - \mathbf{u})$ since the fields in (22) are divergenceless. So, a sufficient condition for existence of vortices is $\alpha = 0$ outside the separatrix.

B. Generalized relaxation theory

Now we relate the equations obtained above with a generalized version of the relaxation theory. The evolution of the fields, determined by (10) in the limit (15)–(16) and spatially constant density, preserves the generalized helicity:

$$I_a^h = \int \mathbf{p_a}\cdot\mathbf{\Omega_a} d^3x - \oint \mathbf{p_a}\cdot dl_1 \oint \mathbf{p_a}\cdot dl_2 \ , \tag{23}$$

where the integration is over the whole spatial volume and the line integrals appear for multiply-connected spaces. This definition is gauge independent.[9] Indeed the time derivative of (23) is

$$\frac{\partial}{\partial t} I_a^h = \oint \left[-\epsilon_a\mathbf{\Omega_a} + (\mathbf{p_a}\cdot\mathbf{\Omega_a})\mathbf{v}_a - (\mathbf{p_a}\cdot\mathbf{v_a})\mathbf{\Omega_a}\right]\cdot d\mathbf{S} \ . \tag{24}$$

So, for the boundary conditions $\mathbf{\Omega_a}\cdot\mathbf{n} = 0$ and $\mathbf{v_a}\cdot\mathbf{n} = 0$, we have $\partial/\partial t\, I_a^h = 0$.

If we include the dissipation terms we find, assuming the above boundary conditions,

$$\frac{\partial}{\partial t} I_a^h = 2\int \mathbf{r_a}\cdot\mathbf{\Omega_a} d^3x \, . \tag{25}$$

We promptly notice a special case for which $\mathbf{r_a}\cdot\mathbf{\Omega_a} = \lambda_a \mathbf{p_a}\cdot\mathbf{\Omega_a}$ so that the evolution of the helicity is $I_a^h(t) = I_a^h(0)\exp[2\int^t \lambda_a(t')dt']$. Therefore it can increase, decrease, or be constant, depending on the behavior of $\lambda_a(t)$ in time.

The time derivative of the total energy

$$E_{\text{total}} \equiv \int \tfrac{1}{2}\left[\sum_a n_a m_a v_a^2 + \frac{1}{4\pi}(E^2 + B^2)\right] d^3x \tag{26}$$

is

$$\frac{\partial}{\partial t} E_{\text{total}} = \int \sum_a \mathbf{R_a}\cdot\mathbf{v_a} d^3x \, . \tag{27}$$

The total energy decreases in time mainly because of the viscosity term in (8). In the ideal limit neglecting $\mathbf{R_a}$ the total energy is, of course, conserved. But in general both helicity and total energy can decay in time. From the expressions (25) and (27) we conclude that the rates of decay of helicity and total energy may be different. I_a^h are topological quantities and we have some reason[9] to believe that the helicity does not decay as fast as the total energy does.[10] Indeed, since changing in helicity involves changing in the topology of the lines, breaking and reconnecting them, it takes some time to happen while the dissipation of energy does not have such a constraint.

We estimate phenomenologically the lifetime for decreasing the total energy and the helicity as

$$\tau_{\text{energy}} = \min\left[1/\nu_a^c; n_a m_a L^2/\mu_a\right] \tag{28}$$

$$\tau_{\text{helicity}} = \max\left[1/\nu_a^c; n_a m_a L^2/\mu_a\right] . \tag{29}$$

The case $1/\nu_d^c \ll n_a m_a L^2/\mu_a$ is one in which the dissipation of the energy is through the interfluid collisions, usually at small scales compared to the other case $1/\nu_a^c \gg n_a m_a L^2/\mu_a$ in which the energy is dissipated through the viscosity of each fluid species.

In any case, given the motivations above, a variational principle is proposed as follows: minimize E_{total}, subject to the constraint that $\sum_a I_a^h =$ const. Let $\delta\phi, \delta\mathbf{A}, \delta\mathbf{p_a}$ be the general variations of the electrostatic potential, the vector potential and the canonical momentum, respectively. Then the 'stationarity' condition

$$\delta\left(E_{\text{total}} - \lambda \sum_a I_a^h\right) = 0 \tag{30}$$

leads to

$$\nabla \cdot \mathbf{E} = 0 , \tag{31}$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \sum_{a=1}^{N} q_a n_a \mathbf{v_a} , \tag{32}$$

$$\mathbf{\Omega_a} = -\frac{n_a}{2\lambda} \mathbf{v_a} . \tag{33}$$

The first two equations above are of no surprise. The last equation is a special case solution for Eq. (17). In this case the generalized vorticity field lines are frozen in the fluid.

To check the stability of these configurations, we make a second variation on (30), use Eqs. (31)–(33), and integrate by parts. We get

$$\delta^2 \left(E_{\text{total}} - \lambda \sum_a I_a^h \right) \Big|_{\text{extreme}} = \int d^3x \left[\sum_a \frac{n_a m_a}{2} (\delta v_a)^2 \right.$$

$$\left. + \frac{1}{4\pi} \left((\nabla \delta\phi)^2 + \sum_{ij} (\partial_j \delta A_1)^2 \right) - 2\lambda \sum_a \delta(\nabla \times \mathbf{p_a}) \cdot \delta \mathbf{p_a} \right] . \tag{34}$$

If the last term does not change sign, we can make the configuration stable by appropriate choice of λ. This term is called average perturbation spirality in connection to amplifications of vortex disturbances in planetary atmospheres.[13]

Combining (32), (33), and (10), we get

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \left[\left(2\lambda \sum_a \frac{q_a^2}{c} \right) \mathbf{B} + 2\lambda \sum_a (m_a q_a \nabla \times \mathbf{v_a}) \right] . \tag{35}$$

In the next section we study the more general equilibrium equation (17) with planar and axial symmetry for an electron-positron-proton plasma. We have been unsuccessful so far in finding an explicit solution for 3-d configurations. It may be that localized filamentary structures are preferred for this spatial scale.

III. ELECTRON POSITRON PROTON PLASMA

We assume that the electron and positron fluids have the same velocity field $\mathbf{v}$ due to the strong coupling with the isothermal photon pressure. Therefore the canonical momentum (8) and generalized vorticity (10) are:[1] $(\mathbf{v}_+ = \mathbf{v}_- = \mathbf{v})$

$$\mathbf{p}_\pm = m\mathbf{v} \pm \frac{e}{c}\mathbf{A} \tag{36}$$

$$\mathbf{\Omega}_\pm = -\nabla \times \mathbf{p}_\pm = -m\nabla \times \mathbf{v} \mp \frac{e}{c}\mathbf{B} \tag{37}$$

and similarly for the protons

$$\mathbf{p}_i = m_i\mathbf{v}_i + \frac{e}{c}\mathbf{A} \tag{38}$$

$$\mathbf{\Omega}_i = -\nabla \times \mathbf{p}_i = -m_i\nabla \times \mathbf{v}_i - \frac{e}{c}\mathbf{B}\ . \tag{39}$$

The displacement current is neglected: $\partial/c\partial t\,\mathbf{E} \ll \nabla \times \mathbf{B}$ and assume the quasineutrality condition: $n_i = n_- - n_+ \equiv \delta n^*$. We consider δn^* constant and the ions velocity much smaller than the electron-positron velocity: $\mathbf{v}_i \ll \mathbf{v}$. Therefore from Ampére's law (3):

$$\mathbf{v} = -\frac{c}{4\pi e\delta n^*}\nabla \times \mathbf{B}\ . \tag{40}$$

Configurations in which this relationship between v and B holds are called "magnetic vortices."[13] Let us use some appropriate units:

$$\text{spatial coordinates} = \frac{c}{\omega_p^*} \tag{41}$$

[1]for the electrons $(-)$, for the positrons $(+)$ and for the protons, or ions, (i).

$$\text{time} = \frac{mc}{eB_0} \tag{42}$$

$$\omega_p^* \equiv \sqrt{\frac{4\pi\delta n^* e^2}{m}} \tag{43}$$

$$\mathbf{H} \equiv \frac{\mathbf{B}}{B_0} \,. \tag{44}$$

Therefore the electron and positron vorticities (37) are:

$$\mathbf{\Omega}_\pm = \mp\mathbf{H} - \nabla^2\mathbf{H} \tag{45}$$

$$\frac{\partial}{\partial t}\mathbf{\Omega}_\pm = \nabla \times [\mathbf{\Omega}_\pm \times (\nabla \times \mathbf{H})] \,. \tag{46}$$

Adding and subtracting the previous equations we get:

$$\frac{\partial}{\partial t}\mathbf{H} - \nabla \times [\mathbf{H} \times (\nabla \times \mathbf{H})] \tag{47}$$

$$\frac{\partial}{\partial t}\nabla^2\mathbf{H} = \nabla \times [(\nabla^2\mathbf{H}) \times (\nabla \times \mathbf{H})] \,. \tag{48}$$

The first equation above tells us that $c\mathbf{E} = -\mathbf{v} \times \mathbf{B}$ and therefore the assumption $\partial/c\partial t\,\mathbf{E} \ll \nabla \times \mathbf{B}$ is validated. It also gives us the range of validity for the neutrality condition:

$$\frac{1}{4\pi e\delta n^*}\nabla \cdot \mathbf{E} = -\frac{B_0^2}{4\pi\delta n^* mc^2}(\mathbf{H} \cdot \nabla^2\mathbf{H} + (\nabla \times \mathbf{H})^2) \,. \tag{49}$$

Therefore the neutrality condition is satisfied either approximately for

$$\frac{B_0^2}{4\pi} \ll \delta n^* mc^2 \tag{50}$$

or exactly for

$$\mathbf{H} \cdot \nabla^2\mathbf{H} = -(\nabla \times \mathbf{H})^2 \tag{51}$$

Beltrami fields (the special case $\nabla \times \mathbf{H} = \alpha \mathbf{H}$), are fields in which the neutrality condition holds exactly.

2-d case

Now let us solve Eq. (47) for the case in which one spatial coordinate, say z, is ignorable. Physically it means that the typical length in the z-direction is much larger than the typical length in the x-y-plane. ($\partial A_x/\partial z \ll \partial A_x/\partial y$ and $\partial A_y/\partial z \ll \partial A_y/\partial x$). Therefore we can write the magnetic field as:

$$\mathbf{H} = [\nabla a(x,y,t) \times \hat{\mathbf{z}}] + h(x,y,t)\hat{\mathbf{z}} \ , \tag{52}$$

equations (47-48) for the magnetic field (52) become

$$\frac{\partial}{\partial t} h = 0 \ , \tag{53}$$

$$\frac{\partial}{\partial t} \nabla^2 h + (h, \nabla^2 h) = 0 \ , \tag{54}$$

$$\frac{\partial}{\partial t} a + (h, a) = 0 \ , \tag{55}$$

$$\frac{\partial}{\partial t} \nabla^2 a + (h, \nabla^2 a) = 0 \ , \tag{56}$$

where $(f,g) \equiv \hat{\mathbf{z}} \cdot [\nabla f \times \nabla g]$. From (53)–(54) we get

$$\nabla^2 h = P[h] \ .$$

For a vortex propagating in the x-y-plane with velocity u, the equations (55)–(56) give us:

$$\nabla^2 a = Q[h + \hat{\mathbf{z}} \cdot (\mathbf{u} \times \mathbf{r})] \tag{57}$$

$$a = R[h + \hat{\mathbf{z}} \cdot (\mathbf{u} \times \mathbf{r})] \ . \tag{58}$$

P, Q and R are arbitrary functions of the arguments in the brackets.

Let us solve a "linear" case $a = h + uy$, in which $\mathbf{u}$ is in the x-direction and Q is given by:

$$Q[a] = \begin{cases} -c^2(h+uy) & \text{for } r < r_0 \\ +d^2(h+uy) & \text{for } r > r_0 \ . \end{cases} \tag{59}$$

So we have to solve just

$$\boldsymbol{\nabla}^2 h = \begin{cases} -c^2(h+uy) & \text{for } r < r_0 \\ +d^2(h+uy) & \text{for } r > r_0 \ . \end{cases} \tag{60}$$

The general solution for h (continuous up to the first derivative) is given by:

- For $r < r_0$

$$h = u\left[\left(\frac{dr_0 K_2(dr_0)}{K_1(dr_0)} + B_1 d K_1'(dr_0)\right)\frac{J_1(cr)}{cJ_1'(cr_0)} - r\right]\sin\phi \tag{61}$$

- For $r > r_0$

$$h = u\left(\frac{r_0}{K_1(dr_0)} + B_1\right)K_1(dr)\sin\phi \tag{62}$$

where $J_1(cr_0) = 0$ and J_1, K_1 are Bessel functions.

Another possible choice of the arbitrary $Q[a]$ is:

$$Q[a] = \begin{cases} -c^2(h+uy) & \text{for } r < r_0 \\ 0 & \text{for } r > r_0 \end{cases} \tag{63}$$

gives us the solution:

$$h = \begin{cases} u\left(\dfrac{2}{cJ_1'(cr_0)}\, J_1(cr) - r\right)\sin\phi & \text{for } r < r_0 \\ -ur_0^2 \sin\phi / r & \text{for } r > r_0 \end{cases} \tag{64}$$

with $J_1(cr_0) = 0$.

Both solutions above are dipole-like solutions. The first solution has finite total energy while the second has a logarithmic divergence.

We emphasize that these solutions represent physically filamentary vortex structures. At a large enough scales this solutions are thin (r_0 very small) "strings" that may eventually close itself. A good ensemble of this filaments may form more complex structures in this large scale. As Petviashvili has shown,[13] MHD equations resemble a set of equations for filamentary vortices in unmagnetized plasmas.

3-d case

Now we present the basic steps toward a 3-d solution with azimuthal angle symmetry. Let us assume an axially symmetric 3-d configuration in which the magnetic field is given by

$$\mathbf{H} = \frac{1}{r}\left[\boldsymbol{\nabla}\psi(r,z,t) \times \widehat{\phi}\right] + \frac{1}{r}\, f(r,z,t)\widehat{\phi}\ . \tag{65}$$

Equations (47-48) become

$$\frac{d}{dt}(\mp\psi - \Delta^{*}\psi) = 0\ , \tag{66}$$

$$\frac{d}{dt}\left(\frac{\mp f - \Delta^* f}{r^2}\right) = \frac{1}{r}\left[\nabla\left(\frac{1}{r^2}\Delta^*\psi\right) \times \nabla(\mp\psi - \Delta^*\psi)\right]_\phi , \tag{67}$$

where

$$\Delta^* \equiv r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} , \tag{68}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{1}{r}[\nabla f \times \nabla]_\phi . \tag{69}$$

For stationary configuration, moving in the z-direction with speed u we have:

$$\frac{d}{dt} = \frac{1}{r}[\nabla \widetilde{f} \times \nabla]_\phi \tag{70}$$

$$\widetilde{f} = f + \frac{u}{2}r^2 \tag{71}$$

$$\Delta^*\psi \pm \psi = F_\pm[\widetilde{f}] \tag{72}$$

$$\Delta^* f \pm f + F'_\pm[\widetilde{f}]\Delta^*\psi = r^2 P_\pm[\widetilde{f}] . \tag{73}$$

$F_\pm[\widetilde{f}]$ and $P_\pm[\widetilde{f}]$ are arbitrary functions of their arguments.

If we add and subtract the equations above we get:

$$\Delta^*\psi = \frac{F_+[\widetilde{f}] + F_-[\widetilde{f}]}{2} \equiv G[\widetilde{f}] \tag{74}$$

$$\psi = \frac{F_+[\widetilde{f}] - F_-[\widetilde{f}]}{2} \equiv K[\widetilde{f}] \tag{75}$$

$$\Delta^* f + G[\widetilde{f}]G'[\widetilde{f}] = r^2\frac{P_+[\widetilde{f}] + P_-[\widetilde{f}]}{2} \equiv r^2 T[\widetilde{f}] \tag{76}$$

$$f + K'[\widetilde{f}]G[\widetilde{f}] = r^2\frac{P_+[\widetilde{f}] - P_-[\widetilde{f}]}{2} \equiv r^2 Q[\widetilde{f}] \tag{77}$$

which can be combined to form

$$\Delta^* f = r^2 \left(T[\tilde{f}] - \frac{G'[\tilde{f}]}{K'[\tilde{f}]} Q[\tilde{f}] \right) + f \frac{G'[\tilde{f}]}{K'[\tilde{f}]} \tag{78}$$

$$\Delta^* \psi = G[K^{-1}[\psi]] \equiv S[\psi] \tag{79}$$

where K^{-1} is the inverse function of K.

We were not successful in finding an explicit nontrivial solution for these equations. We can show that we need just to solve Eq. (79) for ψ so that $\tilde{f}$ becomes determined from ψ.

A. Relaxed state

The relaxed state configuration obeys variational principle given in Eqs. (32)–(33). We conclude that for the electron-positron-proton plasma case

$$\mathbf{H} = -\frac{\delta n^*}{4\lambda} \nabla \times \mathbf{H} \tag{80}$$

$$\nabla^2 \mathbf{H} = \frac{n_+ + n_-}{4\lambda} \nabla \times \mathbf{H} \, . \tag{81}$$

Compatibility of these equations gives $(4\lambda)^2 = \delta n^* (n_+ + n_-)$. Then we get a Helmholtz-like equation for $\mathbf{H}$:

$$\nabla^2 \mathbf{H} = -\frac{n_- + n_+}{n_- - n_+} \mathbf{H} \, . \tag{82}$$

The scales of these solutions are $\sqrt{\frac{n_- - n_+}{n_- + n_+}} \frac{c}{\omega^*} = c/\omega_p$ where ω_p is the plasma frequency for the density $n_- + n_+$, which means the size is $\sqrt{\frac{n_- + n_+}{n_- - n_+}}$ times the collisionless skin depth of ee^+ plasma.

It is well known that the general solution for the divergenceless fields satisfying (82) is

$$\mathbf{H} = \nabla \times (\widehat{\mathbf{m}}u) + \frac{1}{\alpha} \nabla \times (\nabla \times (\widehat{\mathbf{m}}u)) \tag{83}$$

where $\widehat{\mathbf{m}}$ is a unitary vector, and u satisfies the scalar Helmholtz equation:

$$\nabla^2 u = -\alpha^2 u$$

where $\alpha \equiv \sqrt{\frac{n_- + n_+}{n_- - n_+}}$. For the cosmological eeP, $\alpha \approx 10^4$. This estimate is based on the observed limits for the asymmetry of matter over anti-matter.

IV. IDEAL MHD

As we discussed before, the eeP structures can combine to form larger scale structures in MHD. Therefore it is appropriate to investigate localized solutions in MHD. The set of equations used are:

$$\mathbf{B} \cdot \nabla n = \mathbf{v} \cdot \nabla n = 0 \tag{84}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{85}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{86}$$

$$nm \left(\frac{d}{dt} \right) \mathbf{v} \approx + \frac{\mathbf{J}}{c} \times \mathbf{B} - \nabla P - mn \nabla \phi_G \tag{87}$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} \tag{88}$$

$$\frac{\partial}{\partial t} \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}) \ , \tag{89}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla}) \; . \tag{90}$$

The equation of motion (87) can be rewritten as

$$\frac{\partial}{\partial t}\mathbf{v} - [\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v})] + \frac{1}{4\pi mn}[\mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{B})] = -\boldsymbol{\nabla}\left(\frac{P}{mn} + \tfrac{1}{2}v^2 + \phi_G\right) \; . \tag{91}$$

We assume the density is constant. Then the time evolution equations have the following three conserved integral:

$$I_{\pm} = \int \left(\mathbf{v} \pm \frac{\mathbf{B}}{\sqrt{4\pi mn}}\right)^2 d^3\mathbf{r} \tag{92}$$

$$I_h = \int \mathbf{A} \cdot \mathbf{B} d^3\mathbf{r} \; . \tag{93}$$

Let us look for static solutions:

$$[\mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v})] - \frac{1}{4\pi mn}[\mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{B})] \tag{94}$$

$$= \boldsymbol{\nabla}\left(\frac{P}{mn} + \tfrac{1}{2}v^2 + \phi_G\right) \tag{95}$$

$$\boldsymbol{\nabla} \times [\mathbf{v} \times \mathbf{B}] = 0 \; . \tag{96}$$

There are three possible vortices, depending on how $\mathbf{v}$ and $\mathbf{B}$ are related. They are called parallel, magnetic and dynamic vortices. For parallel vortices

$$\mathbf{v} = \pm\frac{M}{\sqrt{4\pi mn}}\mathbf{B} \tag{97}$$

$$\frac{M^2 - 1}{4\pi mn}[\mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{B})] = \boldsymbol{\nabla}\left(\frac{P}{mn} + \tfrac{1}{2}v^2 + \phi_G\right) \; . \tag{98}$$

There is a degeneracy when the constant $M = 1$ and $P/mn + \frac{1}{2}v^2 + \phi_G =$ const which corresponds to Alfvén vortices.

Notice that

$$(M^2 - 1)\epsilon \equiv 4\pi \left(P + mn\,\tfrac{1}{2}\,v^2 + mn\phi_G \right) \tag{99}$$

is constant along the streamlines of the magnetic field. Let it be a 3-d axially symmetric field:

$$r\mathbf{B} = \widehat{\phi} \times \nabla\psi + \widehat{\phi} f[\psi] \tag{100}$$

and $\epsilon = \epsilon[\psi]$.

So we get the Grad-Shafranov equations:

$$\Delta^{*}\psi = -ff' - r^2\epsilon' \tag{101}$$

where prime means derivatives with respect to the argument and $\Delta^{*} \equiv r\,\partial/\partial r\, 1/r\, \partial/\partial r + \partial^2/\partial z^2$.

Let the separatrix be a sphere of radius a. Then we take $\epsilon[\psi]$ and $f[\psi]$ to be linear inside the sphere and zero outside of it. It turns out that ψ vanishes outside the separatrix. The inside equation and general solution[17] are as follows:

$$\Delta^{*}\psi = -k^2\psi + cr^2 \tag{102}$$

$$\psi = \frac{c}{k^2} r^2 + \sum_{n=2}^{\infty} A_n C_n^{-1/2} \left(\frac{z}{R} \right) \sqrt{R}\, j_{n-1/2}(kR)\ , \tag{103}$$

where $R \equiv \sqrt{(r^2+z^2)}$, $C_n^{-1/2}$ are Gegenbauer functions and $j_{n-1/2}$ are spherical Bessel functions. We impose continuity for ψ and its first derivative. This procedure leads us to:

$$A_n = -\delta_{n,2}\,\frac{c}{k^2}\,1/j(ka) \tag{104}$$

$$\tan\,ka = \frac{3ka}{3-(ka)^2}\ . \tag{105}$$

Therefore

$$\psi = \frac{c}{k^2}\left[1-\frac{j(kR)}{j(ka)}\right]^2 r^2 \tag{106}$$

$$j(\xi) = \frac{(\sin\,\xi - \xi\,\cos\,\xi)}{\xi^3}\ . \tag{107}$$

The first two roots of the transcendental are $ka = 5.76$ and $ka = 9.11$. This is an example of a localized solution in MHD. It is continuous up to second derivative of w which is zero outside the separatrix.

Other localized numerical solutions were found[13] for $f = \sqrt{2/(n+1)}\,\psi^{n+1/2}$ and $\epsilon' = -\psi$ for $n = 2, 3$. These solutions have a preferred direction of antiparamagnetic interaction along, say, the z-axis, and of antiparamagnetic in the plane normal to the axis. Therefore, this solitary vortices have a tendency to form linear polymer-like structure. In turn these "polymers" may form even larger structures and so on.[18]

V. DISCUSSION

We have found a series of localized relaxed solutions relevant for plasma structure formation. Localized solutions for eeP were found in the form of long strings (mathematically 2-d solutions). Filamentary eeP may form localized 3-d solution in MHD. The localized solutions in MHD may also form larger scale structures in a polymer-like shape. These solutions represent additional and perhaps more natural equilibrium structures than ones found in earlier work[19] in one-dimension in electron positron plasmas.

In the quasi-two-dimensional limit of three dimensions, i.e. with the structure being a long string but not strictly straight cylinder, such structures can meander in and weave through the plasma and occasionally crisscross each other. It is known[20,21] that the direction of such crisscross and thus the presence or lack of the strong magnetic field in the plane of contact and perpendicular to the reconnecting field lines are a crucial factor in determining the speed of possible reconnection of magnetic field lines. It is thus of much interest to pursue the study of the evolutionary outcome of such preferential reconnection in structure formation. Such interaction may be well described by the approach by Pumir and Siggia[14] in hydrodynamics and by Kinney *et al.*[16] in MHD. It is possible to speculate that a particular meandering and linking of such strings which originally did not carry an overall helicity can emerge to obtain a directed helicity as a result of reconnection.[15]

These structures may be of great importance to formation of isothermal perturbations during the radiation epoch of the universe. This scenario provides one possible way for formation of structures of later epochs that is consistent with the observed uniformity and isotropy of the Microwave Background Radiation.[5]

Moreover, it is often said that the effort of achieving a thermonuclear burning plasma is to copy astrophysical thermonuclear burning. (Conversely, the recent experimental progress[22] in tokamak fusion plasmas finds the presence of strong flows, a study of which may lead to more understandings of plasma vortices with flows as discussed in the present paper). The absence of external magnetic fields for these localized vortices suggests a possible path toward a fusion reactor without (so many) external coils. An attempt in that direction may be found in Ref. 23. More analysis is necessary, however, to check the feasibility of this option as a reactor.

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