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Low Beta Equilibrium and Stability for Anisotropic Pressure Closed Field Line Plasma Confinement Systems

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Abstract

A formalism has been developed to analyze the equilibrium and stability of low-beta anisotropic-pressure plasmas confined in closed field line magnetic systems. This formalism, based on the paraxial (long-thin) approximation, allows consideration of rather general magnetic systems with nonuniform axis curvature and longitudinal profiles of toroidal and multipole poloidal field. Strong pressure anisotropy, corresponding to enhanced plasma pressure in mirror cells of the system, may also be considered. Nonconventional features of anisotropic pressure equilibria have been revealed. Application of the above formalism to the recently proposed linked mirror neutron source (LMNS) confirms the basic principles of the LMNS concept, but calculations based on this formalism have appreciably corrected some LMNS parameters. The LMNS longitudinal pressure profile and magnetic field distribution are optimized.

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I. INTRODUCTION

Closed magnetic field line plasma confinement traps periodically attract attention as possible alternative approaches to fusion plasma confinement.[1–6] The main advantages of such usage would be steady state operation and the relative simplicity of corresponding magnetic systems in comparison with that of conventional stellarators. The main disadvantage would be the lack of a rotational transform which would otherwise create nested magnetic surfaces and preclude strong perturbation of the field lines by magnetic system imperfections. However, it seems possible that external field corrections may reduce, or even eliminate such perturbations. When residual imperfections are small, a self-consistent radial electric field appears to provide closed plasma particle drift surfaces and to preserve the plasma equilibrium.

Recently[7] a linked mirror system (LMS) with closed field lines has been proposed as a highly effective neutron source for testing materials in a fusion environment. This system has all the advantages inherent in previous mirror-based neutron source concepts,[8–10] and the system's toroidal linkage eliminates the longitudinal energy losses, thereby raising the electron temperature and enhancing the power efficiency of the source. The advantages of the LMS-based neutron source can be seen in two parameters. The first is the large ratio p_h/p_t ($p_h/p_t > 5$), where p_h is the pressure of the mirror confined hot ions and p_t is the pressure of the toroidally confined warm target plasma. The second parameter is a high ellipticity ($E > 10$) of the plasma cross-section in the toroidal linkage cells which appears as a result of high ellipticity of the mirror end fans.

Previous research[7] has revealed the main properties of plasma equilibrium and magnetohydrodynamic (MHD) stability in the LMS. Its conclusions were reached, however, using some simplifying assumptions. Thus, an advanced self-consistent analysis is needed to enhance the reliability of the LMS neutron source concept. The previous formalism[4] developed for isotropic low-beta paraxial (long-thin) plasma equilibrium and MHD-stability analysis of closed field line systems is not applicable to LMS due to the strong plasma pressure anisotropy. Here we develop a generalized formalism which allows analysis of the equilibrium and stability of the paraxial low-beta plasma with arbitrary pressure profile along field lines in a wide class of closed field line plasma confinement systems. We have obtained that the nonuniform longitudinal pressure profile results in some new qualitative features of plasma equilibrium and stability in the above systems. According to the previous study,[4] the isotropic low-beta plasma equilibrium and stability are completely determined by the magnetic system parameters. We show that existence of anisotropic pressure equilibria, and the position and the shape of the equilibrium plasma column, all essentially depend on the longitudinal pressure distribution; changing this pressure distribution can stabilize an unstable equilibrium without changes of the external magnetic system. We also reveal the very unusual feature that the stable and unstable equilibria are separated in parameter space by a region containing no anisotropic pressure equilibria. We apply our formalism to analyze the plasma equilibrium and stability in LMS and to optimize LMS parameters.

II. PLASMA EQUILIBRIUM

An anisotropic plasma equilibrium is determined by the following equations:

$$\nabla \cdot \mathbf{P} = \mathbf{j} \times \mathbf{B}, \tag{1}$$

$$\nabla \times \mathbf{B} = \mathbf{j}, \tag{2}$$

where $\mathbf{j}$ is the plasma current and $\mathbf{B}$ the magnetic field. The pressure tensor is of the form:

$$\mathbf{P} = p_\perp \mathbf{I} + (p_\parallel - p_\perp)\mathbf{bb}, \tag{3}$$

$\mathbf{b} = \mathbf{B}/B$ and $\mathbf{I}$ is the unit dynamic. The longitudinal component of Eq. (1) is automatically satisfied, if $p_\parallel$ and $p_\perp$ are calculated using a plasma particle distribution function which depends on the integrals of motion. From the transverse component of Eq. (1) it follows, using standard techniques,[11] that:

$$\mathbf{j}_\perp = \left[(p_\parallel - p_\perp)(\mathbf{b} \times \boldsymbol{\kappa}) + \mathbf{b} \times \nabla p_\perp\right]/B, \tag{4}$$

$$\boldsymbol{\kappa} = (\mathbf{b} \cdot \nabla) \cdot \mathbf{b} \equiv -\mathbf{b} \times (\nabla \times \mathbf{b}).$$

The longitudinal current $j_\parallel$ is found using $\boldsymbol{\nabla} \cdot \mathbf{j} = 0$ which leads to the relation

$$\mathbf{b} \cdot \nabla \left[\frac{1}{B^3} j_\parallel (B^2 + p_\perp - p_\parallel)\right] = -\frac{\mathbf{b} \times \boldsymbol{\kappa}}{B^2} \cdot \nabla (p_\parallel + p_\perp). \tag{5}$$

Equation (5) must satisfy the following solvability condition:

$$\oint \frac{ds}{B^2} (\mathbf{b} \times \boldsymbol{\kappa}) \cdot \nabla (p_\parallel + p_\perp) = 0, \tag{6}$$

where the integration is along a closed field line. Equation (6) is the most important condition of the equilibrium. Physically it corresponds to a closure of Pfirsch-Schlüter currents within the system considered.

Following the conventional procedure[4] used as the first step of plasma equilibrium and stability analysis we assume that β is asymptotically small and invoke a rather reasonable paraxial (long–thin) approximation. These assumptions[12,13] are justified for $\beta = 2p/B^2 < |\boldsymbol{\kappa}|^2 a^2 \ll 1$, where $a \approx p/|\nabla_\perp p|$ denotes the cross-field pressure gradient scale and $\boldsymbol{\kappa}$ is the field line curvature defined by Eq. (4). The assumptions allow us to neglect the magnetic field distortions caused by plasma currents. Therefore, we may substitute the vacuum (non-perturbed) magnetic field into Eq. (6), which appears to be the necessary and sufficient condition of plasma equilibrium under the above assumptions.

In the case of closed field lines it is possible to introduce Clebsh coordinates $\Psi, \Phi, \mathcal{X}$ to describe the vacuum magnetic field:

$$\mathbf{B} = \nabla \mathcal{X} = \nabla \Psi \times \nabla \Phi. \tag{7}$$

Below Ψ is an appropriately chosen magnetic flux, so that Φ is a corresponding poloidal angle. $\mathcal{X}$ plays a role of longitudinal coordinate. If the plasma pressure is isotropic ($p_\perp = p_\parallel$), we can always choose Ψ to provide $p = p(\Psi)$ due to relation $\mathbf{B} \cdot \nabla p = 0$. In the general case of an anisotropic plasma the effective pressure depends generally on all three coordinates: $P \equiv p_\perp + p_\parallel = P(\Psi, \Phi, \mathcal{X})$. The form of such a dependence is specified by the coordinate dependence of particle motion invariants. Nevertheless, we can again define Ψ to provide $P = 2p(\Psi)$ in the regions where the plasma pressure is isotropic. Such the definition of Ψ is possible, because the isotropic pressure component has to be constant along field lines. Under the above definition of Ψ the Φ-dependence of P has to be rather weak—it can only appear in small paraxial corrections to the main part of $P \approx P(\Psi, \mathcal{X})$. These corrections could be determined from the extremely complicated kinetic consideration accounting a self-consistent radial electric field which appears to unify the drift trajectories of different plasma particles[7] and also to reduce the Φ-dependence of P. Therefore, taking into account the above qualitative arguments we shall assume below that $\partial P/\partial \Phi = 0$. Of course, this assumption should not contradict the longitudinal pressure balance:

$$\frac{\partial}{\partial \mathcal{X}} \left(\frac{p_\parallel - p_\perp}{B^2} \right) + \frac{1}{B^2} \frac{\partial P}{\partial \mathcal{X}} - 0.$$

The solvability condition of this equation is the following:

$$\oint \frac{\partial P}{\partial \mathcal{X}} \frac{d\mathcal{X}}{B^2} = 0.$$

This condition can be satisfied by $P(\Psi, \mathcal{X})$ for a wide class of closed field line magnetic confinement systems. In particular, it can be identically satisfied for some symmetric configurations (e.g., linked mirror system considered below) in which P and B are even functions of $\mathcal{X}$ with respect to a certain plasma cross-section.

Under the above assumptions we can rewrite Eq. (6) in the following simple form:

$$\oint \frac{\partial P}{\partial \Psi} \frac{\partial B}{\partial \Phi} \frac{d\mathcal{X}}{B^3} = -\frac{1}{2} \oint \frac{\partial P}{\partial \Psi} \frac{\partial}{\partial \Phi} \left(\frac{d\mathcal{X}}{B^2} \right) = 0. \tag{8}$$

The Clebsh coordinates depend on parameters of the system considered. To determine these coordinates we suppose that the plasma column is located in a vicinity of a closed field line whose physical characteristics (magnetic field, curvature, etc.) are given. We denote this field line as a "magnetic axis." In the case of isotropic plasma pressure equilibrium it is reasonable to define the magnetic axis as the center of closed nested surfaces of constant $\oint dl/B$ so that it would appear as the center of the plasma column.[4] In the general case of anisotropic pressure, as will be shown below, the position of the plasma column axis depends on longitudinal pressure distribution, so the plasma column is not attached to the surfaces $\oint dl/B = \text{const}$. Thus we have a lot of freedom to concretize the magnetic axis. As a rule, a spatial symmetry inherent in the closed field line magnetic systems[1-6] allows us to choose the magnetic axis to be a rather simple untwisted planar curve.

Further, we introduce the local cartesian coordinates $\{s, x, y\}$ where s is the distance along the magnetic axis, x is the distance measured from the axis along its external normal, and y is chosen to complete the right triad.

Under the paraxial approximation it is sufficient to consider an expansion of $\mathcal{X}$ of up to third order in distances from the magnetic axis[4] the corresponding expression takes the form:

$$\mathcal{X} = \int_0^S \mathcal{B}(s)ds - \frac{\eta'\mathcal{B}}{4}(x^2 - y^2) - \frac{\mathcal{B}'}{4}(x^2 + y^2) - B_3(x^3 - 3xy^2) - \phi_k(x^3 + xy^2) + \ldots, \quad (9)$$

where $\mathcal{B}(s)$ is toroidal field at the axis, $\mathcal{B}\eta'(s)$ and $B_3(s)$ characterize quadrupole and hexapole poloidal field components correspondingly, prime denotes the derivative with respect to s, ϕ_k has the form:

$$\phi_k = \frac{5}{16}\mathcal{B}'k + \frac{1}{8}\mathcal{B}\,k' + \frac{1}{16}\mathcal{B}\,k\eta', \quad (10)$$

and $k(s)$ is the axis curvature. Using Eqs. (7) and (9) we calculate the magnetic field:

$$B_x = -\frac{x}{2}(\mathcal{B}' + \eta'\mathcal{B}) - x^2(3B_3 + 3\phi_k) + y^2(3B_3 - \phi_k) + \ldots,$$

$$B_y = -\frac{y}{2}\left(\mathcal{B}' - \eta'\mathcal{B}\right) + 2xy(3B_3 - \phi_k) + \ldots, \tag{11}$$

$$B_s = \frac{1}{1-kx}\left[\mathcal{B} - \frac{x^2}{4}\left(\mathcal{B}'' + (\eta'\mathcal{B})'\right) - \frac{y^2}{4}\left(\mathcal{B}'' - (\eta'\mathcal{B})'\right) + \ldots\right].$$

To obtain the appropriate expressions for Ψ and Φ we have to solve the field line equations:

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{ds}{B_s}. \tag{12}$$

Using renormalized coordinates:

$$X = x\mathcal{B}^{1/2}e^{\eta/2}; \qquad Y = y\mathcal{B}^{1/2}e^{-\eta/2}, \tag{13}$$

we can rewrite Eq. (12) as follows:

$$\frac{dX}{ds} = X^2\frac{e^{-\eta/2}}{\mathcal{B}^{1/2}}\left(-\frac{3B_3+3\phi_k}{\mathcal{B}} + k\frac{\mathcal{B}'}{\mathcal{B}} + k\eta'\right) + Y^2\frac{e^{3\eta/2}}{\mathcal{B}^{1/2}}\left(\frac{3B_3-\phi_k}{\mathcal{B}}\right) + \ldots, \tag{14}$$

$$\frac{dY}{ds} = XY\frac{e^{-\eta/2}}{\mathcal{B}^{1/2}}\left(2\frac{3B_3-\phi_k}{\mathcal{B}} + k\frac{\mathcal{B}'}{\mathcal{B}} - k\eta'\right) + \ldots .$$

The right-hand sides of Eq. (14) consist of second order terms only. Thus, the solution of Eq. (14), accurate up to the second order in distance from the magnetic axis, takes the form:

$$X = X_0 + X_0^2u(s) + Y_0^2v(s), \tag{15}$$

$$Y = Y_0 + X_0Y_0w(s),$$

where $X_0 = X|_{s=0}$; $Y_0 = Y|_{s=0}$, $s = 0$ is an arbitrary point at the axis, and

$$u = \int_0^S \left(k\eta' + k\frac{\mathcal{B}'}{\mathcal{B}} - 3\frac{B_3+\phi_k}{\mathcal{B}}\right)\frac{e^{-\eta/2}}{\sqrt{\mathcal{B}}}ds,$$

$$v = \int_0^S \frac{3B_3-\phi_k}{\mathcal{B}}\frac{e^{3\eta/2}}{\sqrt{\mathcal{B}}}ds, \tag{16}$$

$$w = \int_0^S \left(-k\eta' + k\frac{\mathcal{B}'}{\mathcal{B}} + 2\frac{3B_3-\phi_k}{\mathcal{B}}\right)\frac{e^{-\eta/2}}{\sqrt{\mathcal{B}}}ds.$$

The above expressions are used to determine the Clebsh variables Ψ and Φ. At the main order of the expansion, which corresponds to the paraxial approximation, a transversal cross-section of the surface $\Psi = \text{const}$ has to be a closed second order curve (ellipse). Therefore, we can write the expressions for Ψ and Φ in the following general form:

$$\Psi = (X_0 - \Delta)^2 + Y_0^2,$$

$$\Phi = \arctan\left(\frac{Y_0}{X_0 - \Delta}\right), \tag{17}$$

where Δ is a renormalized displacement of a plasma axis with respect to the magnetic axis, and Ψ is taken to be the magnetic flux. It is not difficult to verify that expressions (17) satisfy Eq. (7). Using Eqs. (7), (11), (13), (15), and (17) we obtain, by means of algebraic transformations, the following expression:

$$\begin{aligned}
\frac{dX}{B^2} = \frac{dX}{B^2(X)}\Bigg\{ & 1 - 2k\frac{e^{-\eta/2}}{B^{1/2}}\Delta + \Delta^2\left[-2k\frac{e^{-\eta/2}}{B^{1/2}}u + \left[k^2 - \frac{1}{4}\left(\frac{B'}{B}+\eta'\right)^2 + \frac{1}{2}\left(\frac{B'}{B}+\eta'\right)'\right]\frac{e^{-\eta}}{B}\right] \\
& + \sqrt{\Psi}\,\cos\Phi\left[-2k\frac{e^{-\eta/2}}{B^{1/2}} + \Delta\left[-4k\frac{e^{-\eta/2}}{B^{1/2}}u + 2\frac{e^{-\eta}}{B}\left(k^2 - \frac{1}{4}\left(\frac{B'}{B}+\eta'\right)^2 + \frac{1}{2}\left(\frac{B'}{B}+\eta'\right)'\right)\right]\right] \\
& + \Psi\left[-k\frac{e^{-\eta/2}}{B^{1/2}}(u+v) + \frac{e^{-\eta}}{2B}\left[k^2 - \frac{1}{4}\left(\frac{B'}{B}+\eta'\right)^2 + \frac{1}{2}\left(\frac{B'}{B}+\eta'\right)'\right]\right. \\
& \left. + \frac{e^{\eta}}{2B}\left[-\frac{1}{4}\left(\frac{B'}{B}-\eta'\right)^2 + \frac{1}{2}\left(\frac{B'}{B}-\eta'\right)'\right]\right] \\
& + \Psi\cos 2\Phi\left[-k\frac{e^{-\eta/2}}{B^{1/2}}(u-v) + \frac{e^{-\eta}}{2B}\left[k^2 - \frac{1}{4}\left(\frac{B'}{B}+\eta'\right)^2 + \frac{1}{2}\left(\frac{B'}{B}+\eta'\right)'\right]\right. \\
& \left. - \frac{e^{\eta}}{2B}\left[-\frac{1}{4}\left(\frac{B'}{B}-\eta'\right)^2 + \frac{1}{2}\left(\frac{B'}{B}-\eta'\right)'\right]\right]\Bigg\},
\end{aligned} \tag{18}$$

which has an appropriate form to be used in the equilibrium condition (8).

The high plasma pressure anisotropy in a steady state plasma confinement systems has to be sustained by some auxiliary heating (neutral beam, ion–cyclotron heating, etc). The heating conditions are very similar for each magnetic surface inside of the paraxial plasma column. Therefore, it is reasonable to expect that the pressure distribution along field lines must be approximately the same for the each magnetic surface. That is why we assume below that the pressure distribution can be written in the following form:

$$P(\Psi, \chi) = \mathcal{P}(\chi)\alpha(\Psi), \tag{19}$$

where $\alpha = 1$ at the magnetic axis and $\alpha = 0$ at the plasma edge. Substituting Eqs. (18) and (19) into Eq. (8) we suppose that s is independent of Ψ and Φ, because $\partial s/\partial\Phi$, $\partial s/\partial\Psi$ could only add terms of the order of $\Psi^{3/2}$ and higher to the equilibrium condition (8). Changing the integration along field lines by the integration along the axis we obtain the following equilibrium condition:

$$\begin{aligned}
&\sqrt{\Psi}\,\sin\Phi \oint \mathcal{P}(s)\left\{ -k\frac{e^{-\eta/2}}{\mathcal{B}^{3/2}} + \Delta\left[-2k\frac{e^{-\eta/2}}{\mathcal{B}^{3/2}}u + \frac{e^{-\eta}}{\mathcal{B}^2}\left(k^2 - \frac{1}{4}\left(\frac{\mathcal{B}'}{\mathcal{B}}+\eta'\right)^2 + \frac{1}{2}\left(\frac{\mathcal{B}'}{\mathcal{B}}+\eta'\right)'\right)\right]\right\}ds \\
&+ \Psi\sin 2\Phi \oint \mathcal{P}(s)\left\{ -2k\frac{e^{-\eta/2}}{\mathcal{B}^{3/2}}(u-v) + \frac{e^{-\eta}}{\mathcal{B}^2}\left[k^2 - \frac{1}{4}\left(\frac{\mathcal{B}'}{\mathcal{B}}+\eta'\right)^2\right.\right. \\
&+ \left.\frac{1}{2}\left(\frac{\mathcal{B}'}{\mathcal{B}}+\eta'\right)'\right] - \frac{e^{\eta}}{\mathcal{B}^2}\left[-\frac{1}{4}\left(\frac{\mathcal{B}'}{\mathcal{B}}-\eta'\right)^2 + \frac{1}{2}\left(\frac{\mathcal{B}'}{\mathcal{B}}-\eta'\right)'\right]\Bigg\}ds = 0.
\end{aligned} \tag{20}$$

Both the terms in Eq. (20) have to equal zero independently. The first and the second terms describe the closure of the dipole and the quadrupole Pfirsch-Schlüter currents correspondingly.

Introducing the useful notations

$$\begin{aligned}
J_1 &= -\oint ds\frac{\mathcal{P}(s)}{\mathcal{B}^2}\left\{e^{-\eta}\left[k^2 - \frac{1}{4}\left(\frac{\mathcal{B}'}{\mathcal{B}}+\tilde{\eta}'\right)^2 + \frac{1}{2}\left(\frac{\mathcal{B}'}{\mathcal{B}}+\tilde{\eta}'\right)'\right] - 2ke^{-\tilde{\eta}/2}\mathcal{B}^{1/2}\tilde{u}\right\}ds, \\
J_2 &= -\oint ds\frac{\mathcal{P}(s)}{\mathcal{B}^2}\left\{e^{-\eta}\left[-\frac{1}{4}\left(\frac{\mathcal{B}'}{\mathcal{B}}-\tilde{\eta}'\right)^2 + \frac{1}{2}\left(\frac{\mathcal{B}'}{\mathcal{B}}-\tilde{\eta}'\right)'\right] - 2ke^{-\tilde{\eta}/2}\mathcal{B}^{1/2}\tilde{v}\right\}ds,
\end{aligned} \tag{21}$$

where $\tilde{\eta} = \eta - \eta_0$, $\quad \eta_0 = \eta|_{s=0}$, and

$$\tilde{u} = \int_0^S \left[k\left(\tilde{\eta}' + \frac{\mathcal{B}'}{\mathcal{B}}\right) - 3\frac{B_3 + \phi_k}{\mathcal{B}} \right] \frac{e^{-\tilde{\eta}/2}}{\mathcal{B}^{1/2}} ds,$$

$$\tilde{v} = \int_0^S \left(\frac{3B_3 - \phi_k}{\mathcal{B}} \right) \frac{e^{3\tilde{\eta}/2}}{\mathcal{B}^{1/2}} ds, \tag{22}$$

we rewrite the equilibrium conditions in the following form:

$$\delta_0 = -\frac{1}{\mathcal{B}_0^{1/2} J_1} \oint \mathcal{P} \frac{k}{\mathcal{B}^{3/2}} e^{-\tilde{\eta}/2} ds, \tag{23}$$

$$E_0^2 \equiv e^{2\eta_0} = J_1 / J_2, \tag{24}$$

where $\delta_0 = \Delta e^{-\eta_0/2} / \mathcal{B}_0^{1/2}$ is the displacement of plasma axis from the magnetic axis, and E_0 is the ellipticity of the plasma cross-section at $s = 0$. When the pressure distribution $P(\Psi, \mathcal{X})$ cannot be presented in the form (19), the integrals J_1, J_2 and, correspondingly, δ_0, E_0 will slightly depend on Ψ.

Expressions (13), (15), (17), (23), and (24) completely determine the Clebsh coordinates Ψ and Φ in the main order of paraxial expansion, and as a result they completely determine the shape of equilibrium plasma column. In the general anisotropic pressure case the displacement δ_0 and the ellipticity E_0 are not inherent parameters of the magnetic system, so far as they also depend on longitudinal pressure profile. Nevertheless, an appropriate choice of the axis curvature in accordance with Eq. (23) allows us to design the magnetic system which provides $\delta_0 = 0$ for the basic regimes of the system operation with desirable longitudinal pressure profile. Integrals J_1, J_2 can change their signs due to variations of the pressure profile and magnetic system parameters. From Eq. (24) one can see that the plasma equilibrium exists if J_1 and J_2 have the same signs (both are positive, or both are negative). Either integral's changing sign implies a loss of equilibrium in terms of the paraxial approximation. Finally we should emphasize that the above procedure allows expansions in powers of distances from the axis to be expanded to arbitrary order. For example, if the system considered has a strong external hexapole magnetic field component, we should expect an appreciable triangularity in the plasma cross-section. In this case we have to consider a third order expansion in terms of these distances in Eq. (8). The corresponding parameter of triangularity appears to be calculated from the hexapole Pfirsch-Schlüter current closure condition (vanishing of the third order term in Eq. (8)).

III. Low-β MHD Stability

The MHD-stability problem for low-β plasmas confined in closed field line systems (CFLS) is reduced to the analysis of flute mode stability. Under paraxial approximation, as it follows from the energy principle, the flute-like mode stability criterion corresponds to positive definiteness of quadratic form:

$$W = -\frac{1}{2}\int \frac{(\boldsymbol{\xi}_\perp \nabla P)(\boldsymbol{\xi}_\perp \nabla B)}{B}\, dV > 0, \tag{25}$$

where $\boldsymbol{\xi}_\perp$ is cross-field flute plasma displacement. Using Clebsh coordinates and taking into account the equilibrium condition (8), we can rewrite Eq. (25) as follows:

$$W = \frac{1}{2}\int (\boldsymbol{\xi}_\perp \nabla\Psi)^2 d\Psi d\Phi \oint \frac{\partial P}{\partial \Psi}\frac{\partial}{\partial \Psi}\left(\frac{d\chi}{B^2}\right) > 0. \tag{26}$$

Here $(\boldsymbol{\xi}_\perp \nabla\Psi)^2$ is an arbitrary positively defined function of Ψ and Φ. Therefore, the stability criterion can be written in the following local form:

$$\oint \frac{\partial P}{\partial \Psi}\frac{\partial}{\partial \Psi}\left(\frac{d\chi}{B^2}\right) > 0, \tag{27}$$

which is similar to the equilibrium condition (8). Substituting Eqs. (18), (19) into Eq. (27) and taking into account notations (21) and (22), and equilibrium conditions (23) and (24), we obtain the following stability criterion:

$$G = J_1 e^{-\eta_0} + J_2 e^{\eta_0} > 0. \tag{28}$$

According to Eqs. (24) and (28), the MHD equilibrium of a low-β plasma column in CFLS is stable with respect to flute modes if the integrals J_1 and J_2 are both positive. If both are negative, the equilibrium exists but is unstable. It is obvious that any approaching the flute mode marginal stability condition (implying small values of either of integrals $J_{1,2}$) means breaking the paraxial equilibrium condition as well. A similar correlation between MHD equilibrium and stability conditions was noted earlier[14] for a tandem mirror geometry. This geometry can be considered as a specific limit of CFLS with $k(s) = 0$ in mirror cells and $\mathcal{P}(s) = 0$ in a toroidal linkage. In general, J_1 and J_2 do not change their signs simultaneously for the different pressure profiles and magnetic system parameters. Therefore, the stable and unstable regions in a parameter space of CFLS are separated by a zone where plasma equilibria are absent ($J_1 \cdot J_2 \leq 0$). Such a feature is not typical for equilibria in conventional plasma confinement systems. We shall discuss the aforementioned features of the equilibrium in detail, when we consider the linked mirror system in the next section.

IV. Equilibrium and Stability in a Toroidally Linked Mirror System

We now apply the results obtained in the previous sections to the analysis of low-β plasma equilibrium and stability in the Linked Mirror Neutron Source (LMNS).[7] The magnetic configuration to be discussed is shown in Fig. 1. It consists of four individual, or equivalently, two parallel pairs of minimum-B mirror cells (midplane cross-sections 2, 6, 8, 12) linked by two semitori with highly elliptical cross-sections (4,10). The LMNS plasma consists of two components. The first is a toroidally confined target plasma with isotropic pressure p_t constant along a field line. The second, localized near the mirror cell midplanes (2, 6, 8, 12), is a hot ion component which has an anisotropic pressure.

The LMNS magnetic system exhibits the reflection symmetry relative to the plane 1-7 and to the plane 4-10 (see Fig. 1). Therefore, it is sufficient to consider one quarter of the system and to perform all integrations between cross-sections 1 and 4. We also choose $s = 0$ at the mirror midplane (cross-section 2) and assume the following magnetic axis curvature: $k(s) = k_m$ in the mirror cell (1-3) and $k(s) = -k_t$ in the toroidal link cell (3-4). The toroidal and quadrupole magnetic field components in the mirror cell will be modeled by the reasonable analytical expressions[7]:

$$\mathcal{B}(s) = \mathcal{B}_0\left(1 + (R-1)\ \sin^2\pi\,\frac{s}{L}\right); \qquad \mathcal{B}\eta' \equiv q(s) = q_0\ \cos\pi\,\frac{s}{L}; \tag{29}$$

where R is mirror ratio, L is total length of mirror cell axis. The magnetic field in the toroidal cell has only toroidal component $\mathcal{B}(s) = \mathcal{B}_t = \mathcal{B}_0 R$, so $\eta'(s) = 0$ in this cell. The hexapole component is $B_3(s) = 0$ everywhere. Contrary to the previous LMNS study,[7] we do not assume the plasma cross-section is circular in the mirror cell midplane. The midplane ellipticity E_0 will be calculated self-consistently according to Eq. (24). Therefore, the toroidal cell ellipticity E_t depends on E_0 as follows: $E_t = E_0 E$, where relative ellipticity E is determined by the mirror cell magnetic field distribution (29):

$$E \equiv \exp\left\{\eta(L/2) - \eta_0\right\} = \exp\left\{\int_0^{L/2} \frac{q}{\mathcal{B}}\,ds\right\}. \tag{30}$$

To calculate the longitudinal pressure profile we shall assume the following hot ion distribution function:

$$f_h = f(|\mathrm{v}|)\left(1 - \frac{\cos^2\theta_0}{\cos^2\theta_h}\right), \qquad |\theta_0| \le \theta_h, \tag{31}$$

which is more smooth and realistic than that which was used in the previous study.[7] Here v and θ_0 are the velocity and the pitch angle of a plasma particle at the mirror midplane, respectively. Using (29) and considering $\sin^2\theta/\sin^2\theta_0 = R(s) \equiv \mathcal{B}(s)/\mathcal{B}_0$ (which describes conservation of particle magnetic moment) we come to the following hot ion pressure distribution:

$$\mathcal{P}_h(s) = \frac{p_{h0}}{R(s)}\left(\frac{R_h - R(s)}{R_h - 1}\right)^{3/2} \frac{6R_h - R(s)}{6R_h - 1}, \tag{32}$$

where $\mathcal{P}_h(s) = p_{h\|} + p_{h\perp}$, $R_h = R(l_h/2) = \sin^{-2}\theta_h$ and l_h is a hot ion component length. The total pressure distribution has the form $\mathcal{P} = \mathcal{P}_h(s) + 2p_t$. We shall also use an effective hot component length l_h^* defined as follows:

$$l_h^* = \frac{2}{p_{h0}} \int_0^{l_h/2} \frac{\mathcal{B}_0}{\mathcal{B}(s)} \mathcal{P}_h(s) ds, \tag{33}$$

to compare the results of this paper with the previous results[7] which were based on a step-like pressure distribution.

Even after the above simplifications and parameterization of the profiles the LMNS equilibrium and stability depend on six dimensionless parameters which determine the magnetic system and pressure distribution. They are: relative toroidal and mirror cell curvatures $K_t = k_t L$, $K_m = k_m L$; mirror ratio R; relative ellipticity E; target/hot plasma pressure ratio $\alpha = p_t/p_{h0}$; and relative hot component length $\lambda = l_h/L$. To analyze this problem, we should define some optimization criterions in order to reduce a number of free parameters.

We shall first assume that K_m is chosen from Eq. (23) to provide $\delta_0 = 0$ for any values of the other parameters. Thus, K_m becomes a dependent parameter. We then calculate dimensionless integrals $J_{1,2}$ (normalized by a factor $p_{h0}/\mathcal{B}_0^2 L$) and plot them as functions of $\alpha = p_t/p_{h0}$ in Fig. 2. The other parameters are fixed and chosen in accordance with the LMNS proposal[7] as follows: $K_t = 2$, $R = 2$, $E = 12$, $\lambda = 0.3$, $(\lambda^* \equiv l_h^*/L = 0.16)$. Fig. 2 shows that stable equilibria ($J_{1,2} > 0$) exist if $\alpha < 0.1$. This restriction does not allow $\alpha = 0.15$ which was presented in the initial proposal[7] as a desirable value. Moreover, in reality α has to be appreciably less than $\alpha = 0.1$, because the toroidal ellipticity E_t goes to infinity due to vanishing of J_2 when α approaches this critical value. Such behavior of E_t corresponds to the loss of equilibrium discussed in the previous section. According to Eq. (24), plasma equilibria ($J_2 < J_1 < 0$) exist when $\alpha > 0.15$, including the case of isotropic plasma pressure ($p_{h0} = 0$), however, these equilibria are unstable.

Singularity of E_t does not allow us to use $J_2 = 0$ as a marginal stability condition.

As plasma would not be allowed to touch the material walls of a real system, E_t must be restricted. Therefore, a reasonable procedure of LMNS magnetic system optimization is to maximize $\alpha = p_t/p_{h0}$ for a fixed (desirable) value of E_t. With E_t fixed, α has a maximum as a function of the parameters λ, R, E, and increases monotonically as toroidal curvature K_t decreases. Thus, we are able to optimize the parameters λ, R, E to provide the maximum of α. However, the low-β equilibrium and stability theory does not provide an algorithm for optimization of K_t. Therefore, we consider $K_t = 2$, as it has been chosen in the initial proposal.[7] We shall also assume a fixed value $\lambda = 0.3$, because α depends only slightly on λ in the vicinity of $\lambda = 0.3$, which is suitable for the high neutron flux production.[7]

A family of the curves $\alpha(E)$ for a set of different values E_t and $R = 2$ is shown in Fig. 3. The parameters α appears to be very sensitive to variations of E. According to the optimization procedure suggested above we choose $E = E_{\text{opt}}$ which corresponds to the maximum of $\alpha(E)$. Analogously we optimize the parameter R to maximize $\alpha(R)$. The optimized values of E and R as functions of E_t are presented in Fig. 4. Figure 5 presents the maximum values of α for a given E_t and corresponding "stability factor" S_f considered as a functions of E_t. The factor S_f is defined as the ratio of the local potential energy G, defined by Eq. (28), to a value $G = G_0$ calculated for the same magnetic system parameters with $p_t = 0$. The stability factor S_f characterizes a rigidity of the unstable mode anchoring by the hot ion component; the factor S_f seems important to ballooning mode stability.

We can see from Fig. 4 that contrary to the previous study's assumption the mirror cell midplane ellipticity $E_0 = E_t/E$ appreciably differs from unity: $E_0 \approx 1.7$. This result means that quadrupole Pfirsch-Schlüter current generated in the toroidal cell cannot be neglected. Fig. 4 and Fig. 5 also show that values of α and R appropriate to LMNS operation could be provided if $E_t \geq 15$. Finally we present in Table 1 the dimensionless MHD parameters of LMNS modified in accordance with the theory developed in this paper.

Table 1: Parameters of LMNS magnetic system. The first column presents the parameters of the initial proposal, the second column shows the optimized results of the present study.

Toroidal cell curvature $k_t L$	2	2
Effective hot component length l_h/L	0.15	0.16
Mirror ratio R	2	2
Mirror cell axis curvature $k_m L$	0.44	0.09
Toroidal cell ellipticity E_t	12	18
Mirror midplane ellipticity E_0	1	1.51
Targent/hot plasma pressure ratio p_t/p_{h0}	0.15	0.086
Stability factor	-	0.192

V. Conclusion

An appropriate formalism is developed to analyze the MHD equilibrium and stability of low-β anisotropic pressure plasmas contained in closed field line magnetic confinement systems. The formalism is shown to be completely self-consistent under paraxial approximation. It allows calculation of the principal parameters of the plasma column. In the calculations, all expansions in powers of distance from the magnetic axis can be extended up to arbitrary order. Regions of stable and unstable anisotropic pressure equilibria in parameter space are often separated by a regions in which no equilibria exist; this is noted as unusual.

Analysis of the linked mirror neutron source using this formalism not only confirms basic results of the preliminary study, but also highlights new features of plasma equilibrium which indicate necessary modification of LMNS parameters. We have demonstrated a way to optimize these parameters and thus improve the stability of plasma confinement in the LMNS. All of this allows us to consider this formalism as a fruitful and useful one.

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FIGURE CAPTIONS

FIG. 1. Structure of magnetic field lines in the toroidally linked mirror system. Cross-sections 2, 6, 8, 12 correspond to minimum-B mirror cell midplanes.

FIG. 2. Dependence of J_1 and J_2 on $\alpha = p_t/p_{h_0}$ for the LMNS magnetic system parameters.

FIG. 3. The family of the curves $\alpha(E)$ for $R = 2$ and $E_t = const.$ The curves 1, 2, 3, 4 correspond to the following values of E_t: 12, 15, 18, 21.

FIG. 4. The optimal values of E and R as the functions of E_t.

FIG. 5. The optimal value of α and "stability factor" S_f as the functions of E_t.

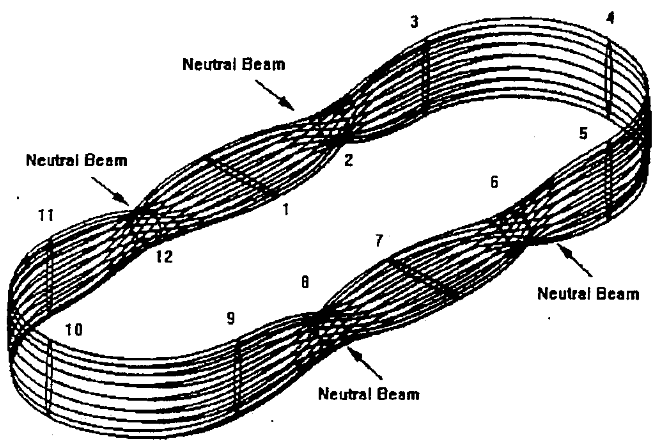
Neutral Beam
Neutral Beam
Neutral Beam
Neutral Beam
1
2
3
4
5
6
7
8
9
10
11
12

Fig. 1

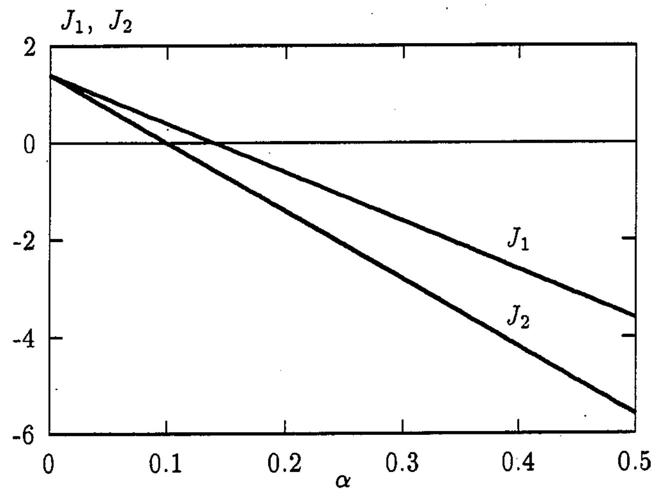

Fig. 2

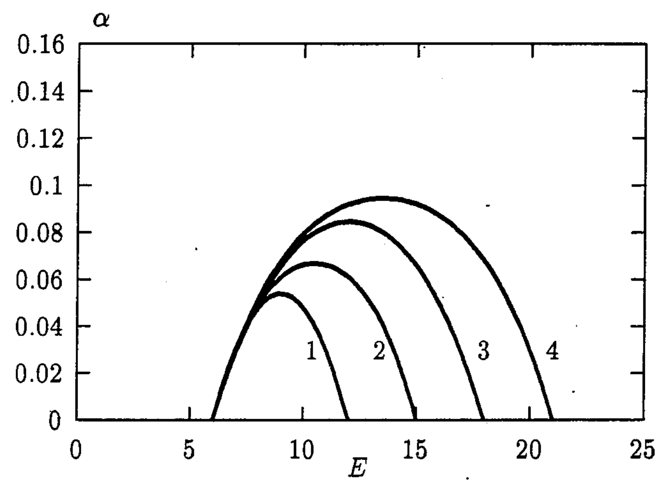

Fig. 3

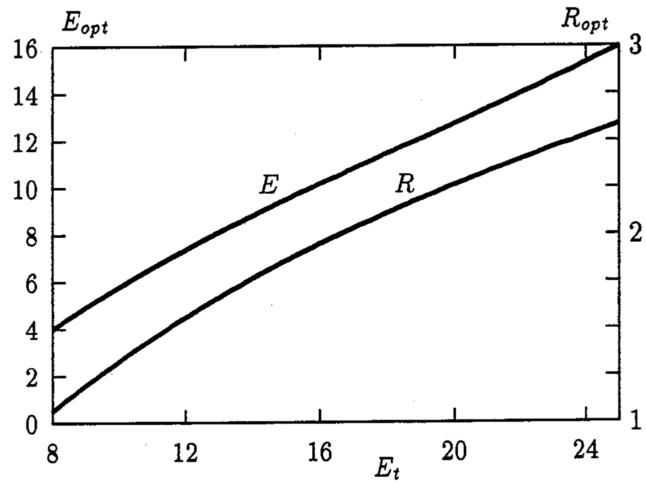

Fig. 4

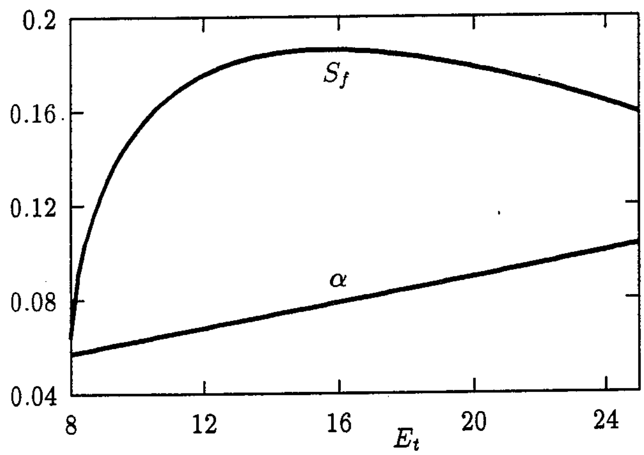

Fig. 5