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H.L. Berk and B.N. Breizman
Institute for Fusion Studies
The University of Texas at Austin
Austin, Texas 78712

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H.L. Berk and B.N. Breizman
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Abstract

The problem of a weak source of particles that forms a distribution function that is unstable to a discrete number of modes with the electrostatic bump-on-tail instability taken as a paradigm is considered. Over a wide range of parameters the system produces pulsations, where there are relatively brief bursts of waves separated by longer intervals of quiescent behavior. There are two types of pulsations; benign and explosive. In the benign phase, valid when particle motion is not stochastic, the distribution function is close to that predicted by classical transport theory, and the instability saturates when the wave trapping frequency equals the expected linear growth rate. If the field amplitude reaches the level where orbit stochasticity occurs, the particle diffusion leads to a further conversion of the distribution's free energy to wave energy. This leads to a rapid quasilinear relaxation (a phase space explosion) of the distribution function. Hence the overall response of the system is characterized by a relatively long time interval where the source needs to build up the distribution to its unstable shape as well as provide a sufficient amount of free energy for the instability to grow to the stochastic threshold of particle motion. The particle distribution is then flattened by the quasilinear diffusion in a relatively short time interval to regenerate the cycle.

1 Introduction

When energetic particles are present in a plasma their distribution is either readily predictable from classical collisional slowing down and scattering processes or, if instability is sufficiently virulent, the distribution is drastically different from the one that is predicted from classical theory. To assess when anomalous transport arises when there is instability, one has to study a self-consistent nonlinear problem for the evolution of unstable modes simultaneously with the evolution of the fast particle distribution. This generic problem has been a subject of extensive theoretical studies and one of the motivations for developing quasilinear theory[1,2] and weak turbulence theory.[3–5]

This problem has applications to a topic of current interest, the confinement of alpha particles under ignition conditions when Alfvén instabilities are present.[6–12] A steady-state alpha particle distribution forms as a result of the balance between the weak source of high energy particles (produced by the fusion reaction) and the slowing down of these particles by drag with the background plasma. This steady-state distribution has a destabilizing shape that can cause the excitation of Alfvén waves, via the free energy drive of the universal instability. The Alfvén waves that are excited are discrete eigenmodes.

With the motivation of the alpha particle problem, we discuss in this paper the following aspects of the wave-particle interaction: the discreteness of the mode spectrum, damping of the waves from dissipation of the background plasma, the presence of a particle source, and the effects from collisions which bring particles to or take them from the resonant region in a phase space where the particles interact with the waves.

A dominant nonlinear effect in our consideration will be the flattening of the particle distribution function near the resonances. This means that we will neglect the wave-wave nonlinear interactions between the modes and will take into account only the mode interaction with the resonant energetic particles. Another interesting part of the problem is whether one should expect the saturation of the instability at a stationary level or quasiperiodic bursts of waves.

To attempt to understand this problem clearly we will consider a simpler physical problem as a paradigm. We study the bump-on-tail instability, where we include the physical features mentioned above. Namely, the energetic particle distribution is fed continuously, and the modes have discrete phase velocities approximately given by $v_{ph} = \omega_p/k_m$ where ω_p is the plasma frequency and $k_m = 2\pi m/L$ with m an integer and L the plasma length.

2 Saturation of Isolated Modes

The first question we address is whether to expect a steady or bursting response when only a single mode is unstable. This problem has been studied in Refs. 10 and 13. In this section, we reproduce the essence of the arguments presented there.

The bump-on-tail instability requires the slope of the energetic particle distribution, F, be positive in the vicinity of the Cherenkov resonance between the particle and the excited wave, i.e.

$$\frac{\partial F(v)}{\partial v} > 0 \tag{1}$$

at $kv = \omega$, with k the wave number, ω the wave frequency, and v the energetic particle velocity.

In Ref. 10 a steady-state nonlinear wave was predicted when classical relaxation of energetic particles is accounted for. The solution allows for a balance between the nonlinear particle instability drive and plasma dissipation. As it was shown in Ref. 13, such a solution requires the background damping to be sufficiently weak whereas for stronger background damping rates, the steady-state nonlinear solution is unstable. In this case a new nonlinear scenario emerges. The system no longer maintains a steady-state solution. Instead the response is that of pulsations.

In what follows we will concentrate on the limiting case where the instability manifests itself most clearly. Namely, we assume $\gamma_L \gg (\gamma_d, \nu_{\text{eff}})$, where γ_L is the linear growth rate associated with the distribution function formed from classical relaxation processes in the absence of excitations, γ_d the dissipation rate of the excited wave caused by the background plasma, and ν_{eff} is the rate of reconstruction of the unperturbed distribution function after it has been flattened in phase space by a nonlinear wave. Several mechanisms determine ν_{eff}. Frequently, pitch angle diffusion is the dominant process, and in this case $\nu_{\text{eff}} \approx \nu\omega^2/\omega_b^2$, where ν is the 90° velocity pitch angle scattering rate, and ω_b the bounce frequency of resonant particles trapped in the wave. If drag determines ν_{eff}, then $\nu_{\text{eff}} \approx \nu_d\,\omega/\omega_b$ with ν_d the drag rate; while if particle annihilation determines ν_{eff} , then $\nu_{\text{eff}} \approx \nu_a$, with ν_a the particle annihilation rate.

Let us first suppose that $\nu_{\text{eff}} \gg \gamma_d$. In this case the steady-state solution is appropriate. In steady state a wave is found where the power, P, which is transferred from the fast particles to the wave, is given by

$$P \approx \gamma_L \left(\frac{\nu_{\text{eff}}}{\omega_b}\right) WE \ , \tag{2}$$

where WE is the energy of the wave. (For electrostatic plasma waves, $WE = \overline{|\delta\mathbf{E}|^2}/4\pi$, where the bar refers to time average, $\delta\mathbf{E}$ is the perturbed electric field, and equal energy contributions are taken into account for perturbed electric field energy and perturbed kinetic energy).

Generically, ω_b is proportional to the square root of the wave amplitude and specifically for plasma waves $\omega_b^2 = (e/m)k|\delta\mathbf{E}|$. This power is absorbed by background dissipation, $P_d = 2\gamma_d WE$. Hence, with $P - P_d = 0$, the saturated wave amplitude satisfies

$$\omega_b \approx \frac{\gamma_L \nu_{\text{eff}}}{\gamma_d} \ . \tag{3}$$

As we assumed $\gamma_d < \nu_{\rm eff}$, we see that the relaxation process pumps the wave to an amplitude that gives a bounce frequency higher than the linear growth rate.

If $\nu_{\rm eff} \ll \gamma_d$, the predicted bounce frequency in Eq. (3) is lower than γ_L. In this case the nonlinear steady-state distribution function found in Ref. 10 is unstable, basically to the same linear instability that exists in the unperturbed state. This observation readily follows from closely examining the response of linear theory. The linear growth rate, γ_L, is given by the following expression:

$$\gamma_L = \frac{-2\omega\pi\, e^2}{|k|m}\, \mathrm{Im} \int dv\, \frac{1}{\omega - kv} \frac{\partial F}{\partial v}\ . \tag{4}$$

For a smooth distribution function formed in the absence of nonlinear waves, γ_L reduces to

$$\gamma_L = \frac{2\omega\pi^2\, e^2}{|k|m} \int dv\, \frac{\partial F}{\partial v} \delta(\omega - kv)\ . \tag{5}$$

In the case $\nu_{\rm eff} \ll \gamma_d$, the nonlinear distribution function found in Ref. 10 only differs from the unperturbed one in a small resonance region where particles are trapped in the wave. There the distribution is flattened over an area

$$\delta v \approx \omega_b/k \equiv v_b\ . \tag{6}$$

Outside this region virtually the same F is obtained as in the unperturbed case. Hence, if one attempts to evaluate $\gamma_L(\omega)$ in Eq. (4), with this locally flattened distribution function, one finds that though $\gamma_L(\omega_0) \to 0$ with ω_0 the real frequency of the background oscillation, the value for γ is hardly changed from the value γ_L found in the smooth case (the difference is $\mathcal{O}(\omega_b/\gamma_L)$). Hence the steady-state solution is unstable for sufficiently large γ_d, viz., $\gamma_d \gg \nu_{\rm eff}$.

This result indicates that the nonlinear response in the $\gamma_d \gg \nu_{\rm eff}$ limit cannot be a steady state. Instead the following pulsation scenario is envisaged. Suppose the linear bump-on-tail instability with the smooth F distribution develops at the rate γ_L. The distribution function would initially look like the thick solid line in Fig. 1, just when instability begins.

Then, as basic and straightforward arguments indicate, the wave amplitude will grow until the bounce frequency of the trapped particles reaches the linear growth rate γ_L. The wave flattens the distribution function in the resonant region, which destroys the resonant particle drive in the manner described by O'Neil[14] and Mazitov,[15] and it is depicted by the thin solid curve in Fig. 1. However, with background dissipation present, this wave will now damp according to the equation $dWE/dt = -2\gamma_d WE$. Simultaneously, the classical transport mechanism attempts to reconstitute the unstable distribution function in the flattened region, $\delta v/v \approx \omega_b/\omega \approx \gamma_L/\omega$, at a rate ν_{eff}. Thus the time for the wave energy to disappear is $1/\gamma_d$, while the time for reconstitution is $1/\nu_{\text{eff}}$. After a time $1/\nu_{\text{eff}}$ the distribution is again ready to excite waves to an amplitude where $\omega_b \sim \gamma_L$. During intermediate times $1/\gamma_d < t < 1/\nu_{\text{eff}}$, precursor instability may arise, for example when the distribution is shaped like the dashed curve in Fig. 1. Low amplitude saturation will then occur due to particle trapping with a trapping frequency $\omega_{b1} \approx \gamma_L \nu_{\text{eff}} t < \gamma_L$. However these precursor waves do not destroy the free energy of the distribution in the velocity range

$$\frac{\omega_{b1}}{k} < \left| v - \frac{\omega}{k} \right| < \frac{\gamma_L}{k} \, .$$

Thus, low level precursor waves are expected prior to the largest "crash." After the largest crash, when $\omega_b \approx \gamma_L$, the distribution is again flattened over the interval $\delta v \approx |\delta v_b|$, with $\delta v_b \sim \gamma_L/k$, and then the process described repeats itself with an overall period ν_{eff}^{-1}.

The need for a pulsation scenario can also be explained in terms of energy balance, which shows that it is energetically impossible to sustain a steady excitation level if $\nu_{\text{eff}} < \gamma_d$. Over a long time scale, the average background dissipation can be estimated as $\gamma_d \overline{WE}$, with $\overline{WE}$ the time-averaged wave energy. This dissipation must be balanced by the free energy that is brought to the resonant region by collisions. In a time $1/\nu_{\text{eff}}$ the free energy of the particles is built up and then converted to the maximum wave energy WE_{max} determined from the condition $\omega_b \approx \gamma_L$. This free energy comes from the particle distribution and is equal to the difference in kinetic energy in the distributions (a) and (c) in Fig. 1. Hence the estimate for the feed power into the wave is $\nu_{\text{eff}} WE_{\text{max}}$. Equating the feed power to the average dissipative power gives

$$\overline{WE} = \left(\frac{\nu_{\text{eff}}}{\gamma_d}\right) WE_{\text{max}} \,. \tag{7}$$

Since ν_{eff} is assumed to be much less than γ_d, the average wave energy is much less than the maximum. Such a condition can only be achieved with relaxation oscillations, as depicted in the solid curves in Fig. 2. However, as previously discussed,[10] for $\nu_{\text{eff}}/\gamma_d > 1$, the wave energy saturates at a stationary level $WE^* = (\nu_{\text{eff}}/\gamma_d)WE_{\text{max}}$, as depicted by the dashed line of Fig. 2.

3 Multiple Modes and Phase Space Explosion

It follows from the previous section that a single unstable mode can only modify the particle distribution function locally. A different picture may arise when there are many unstable modes in the system with the fluctuation level exceeding the stochasticity threshold. This is because particles now really diffuse in phase space and there are no longer barriers to maintain an overall "inverted population" in the vicinity of the resonance region. This regime of the bump-on-tail instability is illustrated in Fig. 3. Below the critical amplitudes for mode overlapping, the situation is depicted in Fig. 3a, where the distribution flattens locally in the shaded regions, with an energy release proportional to $N \lesssim \frac{\omega}{\gamma_L}$, the number of modes. The picture changes drastically, as shown in Fig. 3b, when the resonances overlap. Then all the free energy of the inverted gradient is available to pump the waves to yet higher levels, and to cause strong particle diffusion.

When the amplitudes of excited modes exceed the threshold of resonance overlap, the effect of the waves on the particles is described as quasilinear diffusion. The corresponding diffusion equation for the particle distribution function then has the form:

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial v} D \frac{\partial F}{\partial v} - \nu_a F + Q(v) \; . \tag{8}$$

Here, the diffusion coefficient, $D(v)$, is related to the spectral density of the wave energy, $W(k)$, by

$$D(v) = \frac{4\pi^2 e^2}{m^2 v} W(\omega_p/v) \; . \tag{9}$$

The function $W(k)$ is normalized by

$$\int W(k)dk = U \tag{10}$$

where U is the wave energy per unit volume. The second and third terms on the right-hand side of Eq. (8) describe the source and the annihilation of the fast particles. We choose the source, $Q(v)$, and the annihilation rate, ν_a, to meet the requirement that the "classical" stationary solution of Eq. (8)

$$F = \frac{Q(v)}{\nu_a} \tag{11}$$

has a sufficiently large positive derivative $\partial F/\partial v$ to drive the bump-on-tail instability in the presence of a background damping. In order to simulate the feature that all potentially unstable modes are in a certain interval of phase velocities ranging from $v_{\min}$ to $v_{\max}$, we set $D = 0$ outside this interval.

After discussing Eq. (8) we will also consider a modified version of this equation in which the annihilation term is replaced by the collisional slowing down term and the source is localized outside the area of quasilinear diffusion (at a velocity v_0 larger than $v_{\max}$)

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial v} D \frac{\partial F}{\partial v} + \frac{\partial}{\partial v}(\nu v F) + Q(v) \; . \tag{12}$$

In this equation, the effective collision frequency ν will be taken to decrease with increasing v faster than $1/v$, to provide an instability of the "classical" stationary solution corresponding to a constant particle flux in the velocity space q, so that $Q(v) = q\delta(v - v_0)$:

$$F = \frac{q}{\nu v} \; . \tag{13}$$

To Eq. (8), we add the equation for evolution of wave energy,

$$\frac{\partial W(k)}{\partial t} = 2\gamma\, W(k) - 2\gamma_d\, W(k) \tag{14}$$

where the first term with

$$\gamma = \frac{2\pi^2\, e^2\, \omega_p}{k^2 m} \frac{\partial F}{\partial v} (v = \omega_p/k)$$

describes the wave excitation by energetic particles, while the second term takes into account background damping. The damping rate, γ_d, is assumed to be much less than the typical linear growth rate produced by the unperturbed stationary distribution (11) at the interval $(v_{\rm min},\, v_{\rm max})$. Thus the distribution (11) is strongly unstable. The stable stationary solution of Eqs. (8) and (14) differs from (11) due to quasilinear diffusion and is determined from the following equations:

$$\frac{2\pi^2\, e^2\, v^2}{m\omega_p} \frac{\partial F}{\partial v} - \gamma_d = 0 \tag{15}$$

$$\frac{\partial}{\partial v} D \frac{\partial F}{\partial v} - \nu_a F + Q(v) = 0\ . \tag{16}$$

By integrating Eq. (15), $F(v)$ is obtained to within a constant. This constant is found by integrating Eq. (16) from $v_{\rm min}-\varepsilon$ to $v_{\rm max}+\varepsilon$ with the boundary conditions $D(v)\, \frac{\partial F}{\partial v}|_{v=v_{\rm max}+\varepsilon} = D(v)\, \frac{\partial F}{\partial v}|_{v=v_{\rm min}-\varepsilon} = 0$, and we obtain

$$\int_{v_{\rm min}}^{v_{\rm max}} \nu_a F\, dv = \int_{v_{\rm min}}^{v_{\rm max}} Q\, dv\ . \tag{17}$$

With this condition taken into account, the solution for $F(v)$ is

$$F = F_1 + F_2 \tag{18}$$

where

$$F_1 = \frac{\displaystyle\int_{v_{\rm min}}^{v_{\rm max}} Q\, dv}{\displaystyle\int_{v_{\rm min}}^{v_{\rm max}} \nu_a\, dv} \tag{19}$$

$$F_2 = \frac{m\omega_p}{2\pi^2 e^2} \frac{\displaystyle\int_{v_{\min}}^{v} \gamma_d \frac{dv}{v^2} \int_{v_{\min}}^{v_{\max}} \nu_a \, dv - \int_{v_{\min}}^{v_{\max}} \nu_a \, dv \int_{v_{\min}}^{v} \gamma_d \frac{dv_1}{v_1^2}}{\displaystyle\int_{v_{\min}}^{v_{\max}} \nu_a \, dv} . \qquad (20)$$

The ratio of F_1 to F_2 is roughly of the order of γ_L/γ_d where γ_L is the linear growth rate for the unstable "classical" distribution function (11). When γ_L is assumed to be much larger than γ_d, one can neglect the velocity dependent contribution to F and then the distribution is nearly constant,

$$F = \frac{\displaystyle\int_{v_{\min}}^{v_{\max}} Q \, dv}{\displaystyle\int_{v_{\min}}^{v_{\max}} \nu_a \, dv} . \qquad (21)$$

We now combine Eqs. (15), (16), and (9) to find

$$W(\omega_p/v) = \frac{mv^3}{2\omega_p \gamma_d} \int_{v_{\min}}^{v} (\nu_a F - Q) dv . \qquad (22)$$

For the simplified distribution (21) we obtain

$$W(\omega_p/v) = \frac{mv^3}{2\omega_p \gamma_d} \int_{v_{\max}}^{v} \left(\nu_a \frac{\displaystyle\int_{v_{\min}}^{v_{\max}} Q \, dv_1}{\displaystyle\int_{v_{\min}}^{v_{\max}} \nu_a \, dv_1} - Q \right) dv . \qquad (23)$$

This equation shows that in quasilinear theory the wave energy density W scales linearly with Q. However, when the source is very weak, the wave energy is insufficient to provide mode overlapping and Eq. (23) is not applicable. In this case, each unstable mode forms a separate island in the phase space and quasilinear diffusion really does not arise, since island-to-island transitions are strongly suppressed.

Let us then study more carefully the cases when most of the time there is no mode overlap. Let E_i be the electric field amplitude of the i-th discrete mode. Then the energy density of this single mode, $E_i^2/8\pi$, can be estimated as

$$\frac{E_i^2}{8\pi} \sim \frac{\omega_p}{vN} W \qquad (24)$$

where N is the total number of modes. To overlap the neighboring resonances one needs

$$\frac{\Delta v}{v} > \frac{1}{N} \qquad (25)$$

where Δv is the velocity perturbation of the particle that resonates with the i-th mode. For Δv we have

$$\Delta v = \sqrt{\frac{evE_i}{m\omega_p}} \,. \tag{26}$$

By combining Eqs. (24)–(26) we find the following criterion of resonance overlapping:

$$W > \omega_p \frac{m^2\, v^2}{8\pi\, e^2\, N^3} \,. \tag{27}$$

Taking into account Eq. (23) we rewrite Eq. (27) as a restriction on the particle source

$$Q > \frac{\gamma_d\, m\omega_p^2}{4\pi\, e^2\, N^3 v} \,. \tag{28}$$

We then conclude that quasilinear stationary solution (21), (22) breaks when the intensity of the source is below the critical value given by Eq. (28). The "classical" stationary solution (11) is also inappropriate since we have chosen it to be strongly unstable. This indicates again that the system does not reach a stationary state but rather creates bursts which explosively release the free energy built up by the particle source.

In order to estimate the energy of a burst, we first neglect the particle source and the wave damping. As long as the excited discrete modes do not overlap (Fig. 3a) each of them saturates when the bounce frequency of a resonant particle trapped by the mode reaches the linear growth rate γ. In this regime, one has

$$\frac{\Delta v}{v} = \frac{\gamma}{\omega_p} \,. \tag{29}$$

As time progresses the source causes the slope of the distribution to build up so that γ increases and $\Delta v/v$ eventually reaches the value $1/N$. At this critical value of γ, the total free energy of the unstable distribution becomes available for the burst (Fig. 3b). This energy can be estimated as the energy that is released through global flattening of the distribution with $\gamma = \gamma_{\text{crit}} \equiv \dfrac{\omega_p}{N}$:

$$U_{\text{burst}} \sim \frac{m^2\omega_{pe}^2}{24\pi^2\, e^2\, N} \left(v_{\max} - v_{\min}\right)^3 \frac{1}{v_{\max} + v_{\min}} \,. \tag{30}$$

This consideration shows that a weak source is unable to build up a particle distribution with a free energy exceeding the value given by Eq. (30).

By comparing Eqs. (27) and (30) we may note that the wave energy required for mode overlapping is much less than U_{burst}:

$$U_{\text{overlap}} \sim \frac{U_{\text{burst}}}{N^2} \; . \tag{31}$$

Therefore, the burst is well described by quasilinear theory. This theory predicts complete flattening of the particle distribution within the time of the inverse critical growth rate ω_p/N. During a relatively short time the wave energy builds up to the level given by Eq. (30). Then the waves damp at a rate γ_d, with the distribution function remaining flat since the source is too weak to change the particle distribution within the damping timescale. The third, longer phase, is building up the free energy required for the next burst. The time interval, τ_{rst}, of the restoration, is determined by the energy balance. Hence, τ_{rst} is inversely proportional to the intensity of the source:

$$\tau_{\text{rst}} \sim \frac{U_{\text{burst}}}{mQv_{\text{max}}^3} \; . \tag{32}$$

It is interesting to note that when $\gamma_{\text{crit}} \ll \gamma_L$ the average power transfer from the particles to the waves is rather insensitive to whether the system reaches quasilinear stationary state or creates bursts. This result is straightforward to observe when one writes the average power transfer to the waves, $\overline{P}_w$, as a difference between the power supplied from the source and the dissipation from annihilation:

$$\overline{P}_w = \frac{1}{T}\int_0^T dt \int_{v_{\text{min}}}^{v_{\text{max}}} \frac{mv^2}{2}(Q - \nu_a F)dv \; , \tag{33}$$

where the averaging period, T, is over many burst periods. This expression only depends on the particle distribution function which, when $\gamma_{\text{crit}} \ll \gamma_L$, is close to plateau (19) in both cases. The bursts of the wave energy are obviously easier to observe than the corresponding small deviations of the particle distribution function from the plateau. It should also be noted that, most of the time between bursts, the distribution function is metastable. Therefore, if an appropriate triggering mechanism is available, the system bursts before the accumulated free energy reaches the critical value given by Eq. (30). Such a mechanism occures naturally in the "slowing down" model described by Eq. (12), as we will now show.

Proceeding to the analysis of the "slowing down" model, we note that, as the "classical" solution (13) of Eq. (12) is assumed to be strongly unstable, the constant flux solution of this equation must incorporate the quasilinear flux:

$$D\,\frac{\partial F}{\partial v} + \nu v\, F = q\ . \tag{34}$$

This equation must be solved together with Eq. (15), and the solution must satisfy a boundary condition that $F(v)$ is a continuous function at $v = v_{\rm min}$ since otherwise the flux would be singular at at $v = v_{\rm min}$. Then we obtain

$$F = \frac{q}{\nu(v_{\rm min})v_{\rm min}} + \frac{m\omega_p}{2\pi^2\, e^2}\int_{v_{\rm min}}^{v} \gamma_d\,\frac{dv}{v^2} \tag{35}$$

$$D = \frac{2\pi^2\, e^2\, v^2\, q}{m\gamma_d\,\omega_p}\left(1 - \frac{\nu v}{\nu(v_{\rm min})v_{\rm min}}\right) - \frac{\nu v^3}{\gamma_d}\int_{v_{\rm min}}^{v} \gamma_d\,\frac{dv}{v^2}\ . \tag{36}$$

Using the same arguments as for the annihilation model, we can neglect the last terms in F and D when the factor γ_d/γ_L is small. Thus, the solutions (35) and (36) simplify to

$$F = \frac{q}{\nu(v_{\rm min})v_{\rm min}} \tag{37}$$

$$D = \frac{2\pi^2\, e^2\, v^2\, q}{m\gamma_d\,\omega_p}\left(1 - \frac{\nu v}{\nu(v_{\rm min})v_{\rm min}}\right) \tag{38}$$

and we note again that the distribution function is almost flat in the range $v_{\rm min} < v < v_{\rm max}$ due to quasilinear diffusion.

Using arguments similar to those used to obtain Eq. (28), we find that the quasilinear solution breaks down when

$$q < q_{\rm crit} \equiv \gamma_d\,\frac{m\omega_p}{4\pi e^2\, N^3}\ . \tag{39}$$

In this regime, the wave energy comes out in bursts. However, the typical energy of a single burst differs from that given by Eq. (30). There is a clear trend that the bursts are initiated near $v = v_{\max}$ where the time averaged distribution function shown in Fig. 4 has a discontinuity. Though the distribution function shown in Fig. 4 always has positive slopes outside the region $v_{\min} < v < v_{\max}$, it does not excite an instability during a time interval between the bursts since our model limits the phase velocities of all unstable modes to the region $v_{\min} < v < v_{\max}$. However, the instability eventually starts as the discontinuity in the distribution shifts to lower velocities due to collisions. The number of particles n^* involved in a single burst can be estimated from the condition of the nonlinear saturation of the mode at the upper edge of the spectrum which interacts with this discontinuity and periodically flattens the distribution function near $v = v_{\max}$. This estimate gives

$$n^*/n \approx \left(\frac{\gamma_L}{\omega_p}\right)^{3/2} \tag{40}$$

where n is the plasma density. The burst spreads the particles down to $v = v_{\min}$ releasing the energy of the order of

$$\frac{n^* m v_{\max}^2}{2} \, . \tag{41}$$

The excited waves then damp at a rate γ_d, and the instability "waits" until collisions bring a new portion of particles close enough to $v = v_{\max}$to create a new burst. As before, when $\gamma_{\text{crit}} \ll \gamma_L$, the average power transfer to the waves, $\overline{P}_w$, is insensitive to whether one has bursts or a truly stationary quasilinear regime. For $\overline{P}_w$ we have

$$\overline{P}_w = q \, \frac{m v_{\max}^2 - m v_{\min}^2}{2} - q \int_{v_{\min}}^{v_{\max}} m v^2 \, \frac{\nu}{\nu(v_{\min}) v_{\min}} \, dv \, . \tag{42}$$

It should be noted that the discontinuity which triggers the bursts only exists because the particle source is located outside the region where waves resonate with particles. A source located inside this region should not cause direct triggering. In this case a larger free energy, up to the estimate of the annihilation model, can be accumulated between bursts.

4 Conclusions

We have considered a system where the distribution of energetic particles is formed from the balance of a weak source and a weak relaxation mechanism. The resulting steady-state distribution is assumed to have a shape which tries to destabilize a discrete spectrum of waves. In the absence of energetic particles these waves are supported by the background plasma and are weakly damped. This is a generic problem for many physical cases, and in this paper we discussed in detail the relatively simple case where the waves are electrostatic plasma waves and the beam source forms a bump-on-tail instability.

The critical question in this class of problems is whether the stored energy of the beam is close to the stored energy predicted by the transport properties in the absence of waves. Another interesting question is whether one sees a steady noise level or a pulsating response. Several scenarios have been described where different noise patterns and stored energy are obtained.

In one case we obtain a "benign" scenario where there can be bursting, but the stored energetic particle energy is nearly the same as in the case without excited fields. Then the discrete modes do not cause stochasticity and the distribution builds up to essentially the level predicted by instability free transport theory. If the source strength is large enough stationary steady waves are established. The energy that is being fed to the background plasma through wave dissipation is coming from the particle source. The main alteration of the energetic particle distribution function is in the resonance region. Though the distribution is flattened there, a flux of energetic particles flow through this region because of classical transport processes.

However, if the source strength is too weak, the source cannot maintain a steady-state wave, because at perturbed field levels required for saturation too much energy would be drained by dissipation to the background plasma. In this case "benign" bursts arise that only flatten the distribution function locally but the waves cannot tap the overall free energy source of the energetic particles. The waves only grow up to a natural saturation level where the bounce frequency of particles in the wave equals the linear growth rate. At this stage, the local free energy drive is saturated, and no further energy can be extracted from energetic particles by the wave. Subsequently, the wave damps due to background dissipation, and a time interval determined by classical transport processes needs to elapse before waves re-excite. As the resonant particles cannot move beyond the island boundary of the wave, the overall global distribution function is still close to the one predicted from the simplest transport theory.

If at the estimated level of wave saturation, where the particle bounce frequency equals the growth rate, the resonances of neighboring modes overlap, an entirely different scenario is established. Then particles really diffuse in phase space as described by quasilinear theory. When instability occurs, the particle distribution rapidly flattens to a new constant value over the region of phase space that resonates with the allowable discrete modes. For a sufficiently strong source the noise level can be steady as predicted by quasilinear theory. However, for a weaker source the system is quiescent most of the time. The particle distribution builds up from its flattened state until a certain point is reached where the saturated modes are about to overlap. During this period there can be precursors, but they just lead to the benign saturation previously described, with the overall increase in the distribution function continuing as if there were no oscillations. However, near the point of criticality often determined by mode overlap, the distribution will "explode," and again relax to the flattened quasilinear state, where the cycle repeats. Sometimes the critical point can be determined by other trigger mechanisms.

The characteristic growth and damping rates are: γ_d the damping rate of the background plasma, γ_L the growth rate of the instability as predicted from classical transport, and γ_c the growth rate when overlap occurs. For simplicity we have assumed $\gamma_d \ll \gamma_L$. If $\gamma_c \ll \gamma_L$

we showed that the average properties of the particle distribution is the same for the case when steady-state quasilinear theory is applicable or when the bursting scenario applies. However, if γ_c/γ_L is a finite number (less than unity) there can be considerable difference in the average properties of the bursting scenario compared to the prediction of quasilinear theory. We define the overall free energy, WF, as the difference between the energy stored in a steady-state distribution and the stored energy of the flattened distribution (the latter energy is essentially the energy predicted to be stored in the quasilinear calculations). If mode overlap is the trigger mechanism, then the average free energy that can be stored in the bursting scenario is roughly $WF\gamma_c/2\gamma_L$. If $\gamma_c/\gamma_L \ll 1$, very little free energy can be stored. However if $\gamma_c \approx \gamma_L$ the system can store an energy comparable to the classical prediction, just before the explosion flattens the distribution function. Between explosions the system builds up to the critical state.

The application of this picture to more complicated problems, such as Alfvén instabilities, is clear. An analysis for the parameters of this problem has been discussed elsewhere.[16] In the Alfvén problem, the classical transport mechanisms involve drag and pitch angle scattering in velocity space, whereas the quasilinear relaxation primarily involves spatial diffusion. The phase space explosions then imply rapid radial diffusion, which can lead to direct and rapid energetic particle loss to the plasma edge. Such an interpretation is quite compatible with experimental observations.[17,18] Specific predictions as to how alpha particles evolve for a given case will require determining the detailed instability growth rates for the mode spectrum as well as analyzing the mechanisms for particle resonance. Work is in progress to develop quantitative models for the alpha particle confinement when Alfvén instabilities exist.

Acknowledgments

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Figure Captions

1. Time behavior of the bump-on-tail distribution function near the resonant mode phase velocity. The thick solid curve (a) indicates distribution just before its relaxation; the thin solid curve (c) is just after the relaxation; and the dashed curve (b) is at an intermediate time during which the distribution is being reconstituted.

2. Relaxation oscillations. If $\nu_{\text{eff}} < \gamma_d$, relaxation oscillations arise as shown by solid curves. If $\nu_{\text{eff}} > \gamma_d$, the wave energy saturates in steady-state at a level $WE^* = (\nu_{\text{eff}}/\gamma_d)WE_{\text{max}}$.

3. Effect of resonance overlapping. In (a) modes do not overlap, and the relaxed distribution just has local flattening, with the general shape of the inverted equilibrium distribution preserved. When there is mode overlapping as in (b), the distribution flattens completely over the entire spectrum, with a much larger conversion of free energy to wave energy.

4. Stationary distribution function formed by quasilinear diffusion in the "slowing down" model (solid line). "Classical" stationary distribution is shown by the dashed line. When the particle flux is very small, the discontinuity in the distribution function at $v = v_{\text{max}}$ triggers bursts of the wave energy. The shaded area corresponds to the particles which initialize the bursts. The density of these particles is given by Eq. (40).

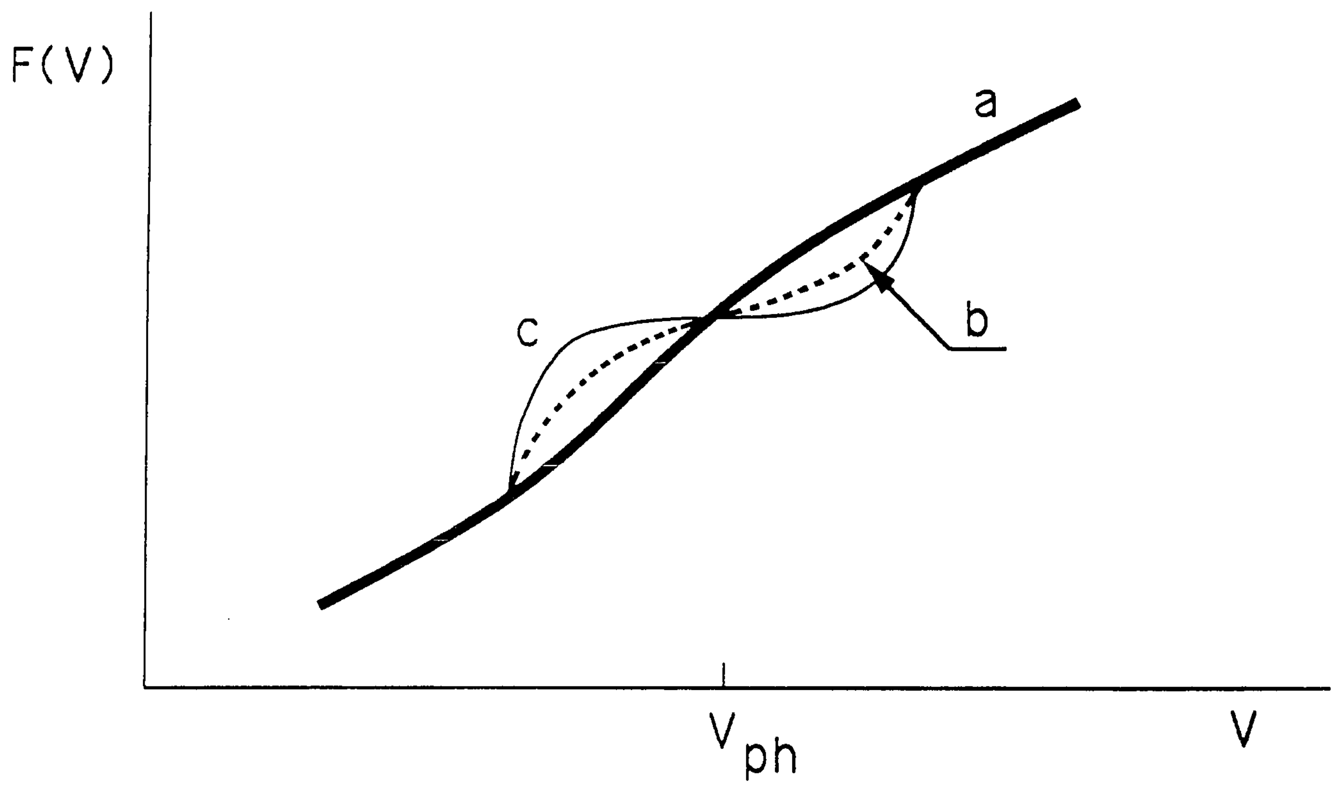

Figure 1

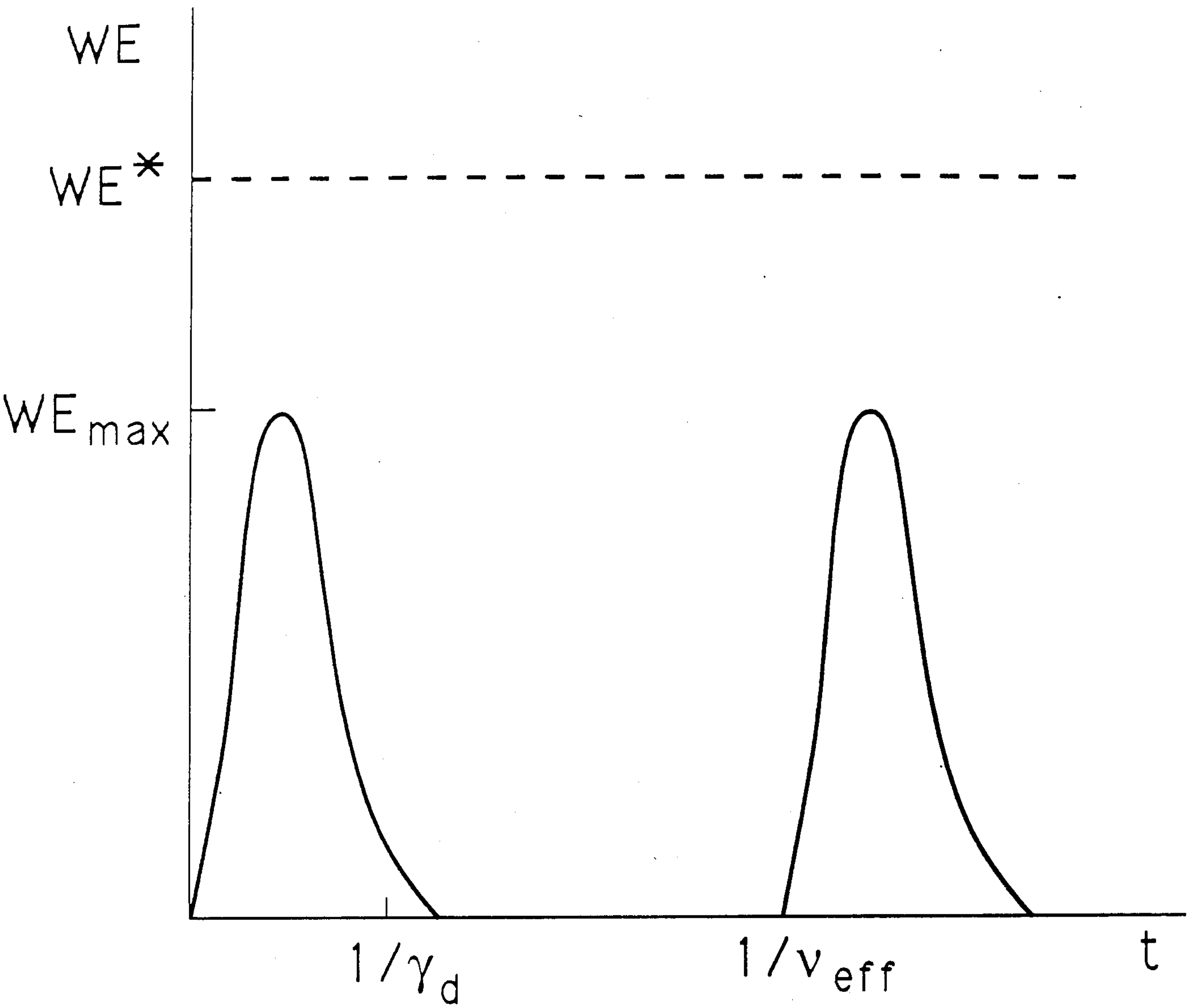

Figure 2

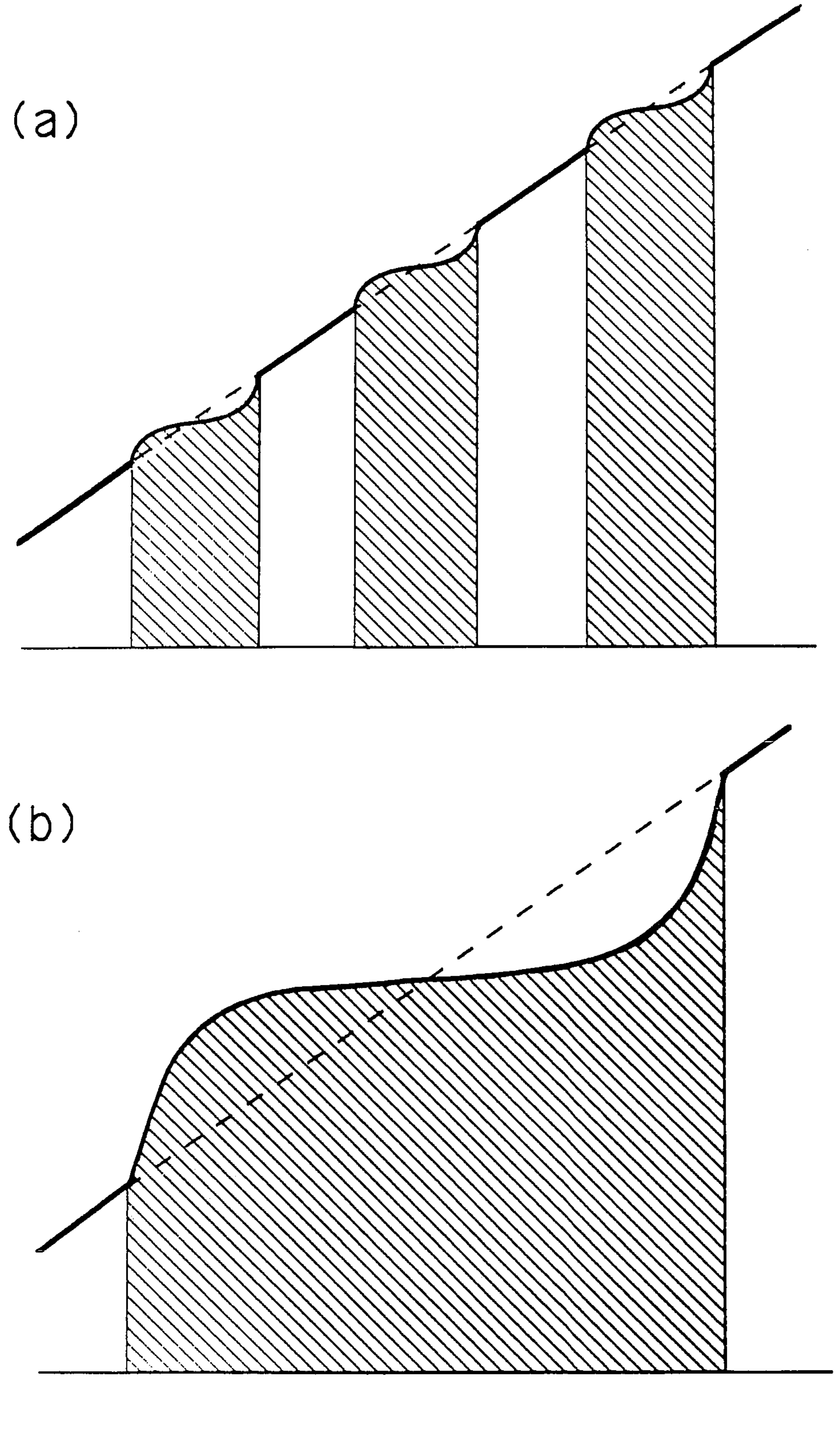

Figure 3

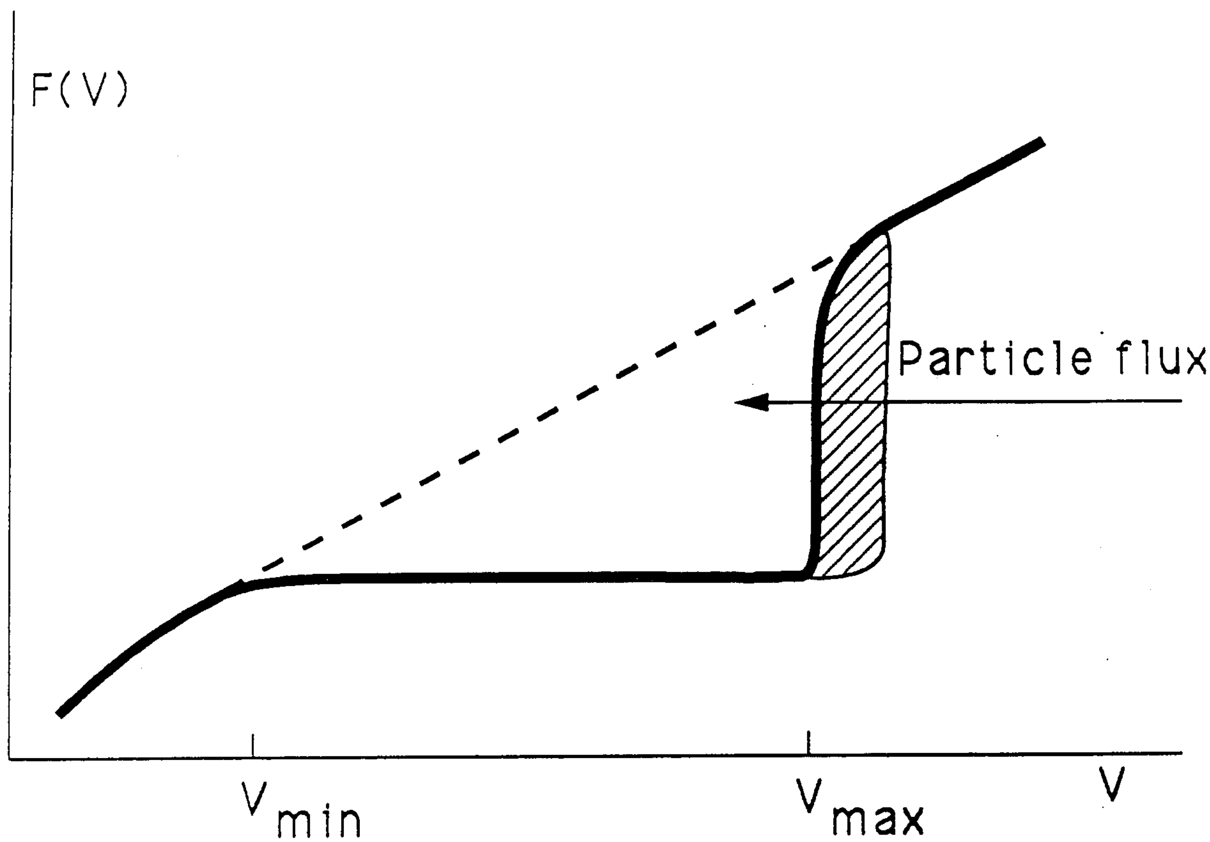

Figure 4