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Intense Neutralized Ion Beam

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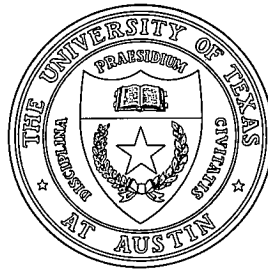
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Abstract

Experiments involving propagation of neutralized ion beams¹ across a magnetic field indicate a magnetic field penetration time determined by the Hall resistivity rather than the Spitzer or Pedersen resistivity. In magnetohydrodynamics the Hall current is negligible because electrons and ions drift together in response to an electric field perpendicular to the magnetic field. For a propagating neutralized ion beam, the ion orbits are completely different from the electron orbits and the Hall current must be considered. There would be no effect unless there is a component of magnetic field normal to the surface² which would usually be absent for a good conductor. It is necessary to consider electron inertia and the consequent penetration of the normal component to a depth c/ω_p . In addition it is essential to consider a component of magnetic field parallel to the velocity of the beam which may be initially absent, but is generated by the Hall effect. The penetration time is determined by whistler waves rather than diffusion.

Introduction

In magnetohydrodynamics Ohm's law is $\mathbf{E} + \frac{1}{c} \mathbf{V} \times \mathbf{B} = 0$ i.e. the fluid is considered to be a perfect conductor. To include finite resistivity Ohm's law is employed in the form $\mathbf{J} = \sigma (\mathbf{E} + \frac{1}{c} \mathbf{V} \times \mathbf{B})$. The Hall effect (or the antisymmetric part of the conductivity tensor) is omitted. This is appropriate because magnetohydrodynamics is based on an expansion in the mass to charge ratio m/e to the lowest order in which m and e do not appear explicitly. To this order the electron and ion $\mathbf{E} \times \mathbf{B}$ -drifts are the same and there is no Hall current. For a neutralized ion beam the ions have energies of hundreds of keV and the electrons tens to hundreds of eV. The electron and ion orbits are completely different and the ion orbits are not well described by the drift approximation. The Hall effect may not be neglected. Ion motion may often be neglected and this regime is called electron magnetohydrodynamics in the Soviet literature.³ In the present case the ion motion is assumed to be frozen. The problem of field penetration is treated with Maxwell's equations and Ohm's law in the form⁴

$$\frac{m}{ne^2} \frac{\partial \mathbf{j}}{\partial t} + \eta \mathbf{j} + \frac{\mathbf{j} \times \mathbf{B}}{nec} = \mathbf{E} + \frac{\mathbf{V} \times \mathbf{B}}{c} + \frac{\nabla P_e}{ne} \quad (1)$$

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$\eta = (m/ne^2\tau_e)$ is the resistivity, n = plasma density, τ_e = electron-ion collision time, \mathbf{B} = magnetic field, E = electric field, \mathbf{j} = current density, \mathbf{V} = fluid velocity, and P_e = electron pressure; this term will be neglected.

Combining Maxwell's equations $\nabla \times \mathbf{B} = (4\pi/c)\mathbf{j}$ (neglecting displacement current) and $\nabla \times \mathbf{E} = -\partial\mathbf{B}/c\partial t$ and Eq. (1) an equation is obtained for the magnetic field:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \frac{c^2\eta}{4\pi} \left[1 + \tau_e \frac{\partial}{\partial t} \right] \nabla^2 \mathbf{B} - \frac{c^2}{4\pi} \nabla \times \left[\frac{1}{nec} (\nabla \times \mathbf{B}) \times \mathbf{B} \right]. \quad (2)$$

In the following we shall consider solutions of Eq. (2) that correspond to neutralized ion beams. A slab model illustrates the fact that the classical diffusion rate obtains unless there is a normal component of the magnetic field. Then a cylindrical model of a beam propagating in a transverse magnetic field is treated, in which case there is a normal component with the result that the Hall effect is important in determining the penetration time of the magnetic field.

1. Slab Model

This model is illustrated in Fig. 1. All quantities depend only on x and t . $\mathbf{B} = \mathbf{e}_z B_z(x, t)$ and $\mathbf{V} = \mathbf{e}_y V_0$. Equation (2) for this case is

$$\frac{\partial B_z}{\partial t} = \frac{c^2\eta}{4\pi} \left[1 + \tau_e \frac{\partial}{\partial t} \right] \frac{\partial^2 B_z}{\partial x^2}. \quad (3)$$

For initial condition assume

$$\begin{aligned} B_z &= B_{ex} + b_z \\ B_{ex} &= B_0 \Theta(t) \\ b_z &= 0 \quad \text{when } t \leq 0. \end{aligned} \quad (3.1)$$

Rather than treat a beam with a front, we consider that the external magnetic field is turned on at $t = 0$. $\Theta(t)$ is, for example, a unit step function. The field, due to beam currents, is initially zero. It is convenient to introduce orthonormal functions that satisfy the boundary conditions $b_z(L/2, t) = b_z(-L/2, t) = 0$, i.e. $\Phi_n = \sqrt{2/L} \cos k_n x$, $k_n = (2n+1)\pi/L$, $n = 0, 1, 2, \dots$. Assume $b_z = \sum_n a_n(t) \Phi_n(x)$, substitute Eq. (3):

$$\begin{aligned} \frac{\partial a_n}{\partial t} [1 + Dk_n^2 \tau_e] + Dk_n^2 a_n &= -B_0 \delta(t) \langle \phi_n | 1 \rangle \\ b_z(x, t) &= -B_0 \sum_n \frac{\langle \phi_n | 1 \rangle}{1 + Dk_n^2 \tau_e} \exp \left\{ \frac{-Dk_n^2 t}{1 + Dk_n^2 \tau_e} \right\} \Phi_n(x) \quad \text{for } t > 0 \end{aligned} \quad (4)$$

$D = c^2\eta/4\pi$ and $\langle\phi_n|1\rangle = 2\sqrt{2/L}(-1)^n/k_n$. For $t = +0$ in Eq. (4), the series can be summed to give

$$B_z = B_0 + b_z = B_0 \left\{ \frac{[\cosh(x\omega_p/c)]}{[\cosh(L\omega_p/2c)]} \right\} \quad (5)$$

which is plotted in Fig. 1 along with the corresponding result when electrons inertia is neglected. This illustrates diamagnetism where the initial magnetic field penetrates to a depth of order $c/\omega_p = \sqrt{D\tau_e}$ which is about 2cm in a typical ion beam experiment. This can be a significant fraction of the beam radius.¹

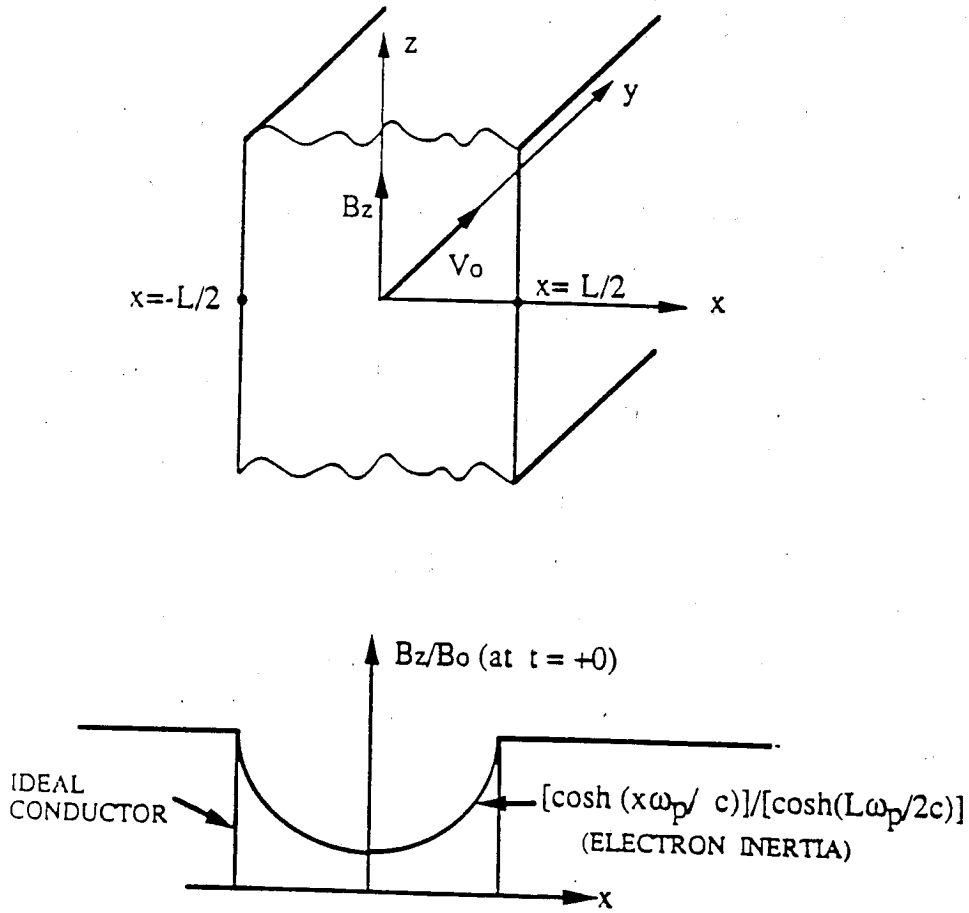


Fig. 1 Slab Model of a Neutralized Ion Beam

To consider the effect of a magnetic field normal to surface of the beam, assume

$$\mathbf{B} = \mathbf{e}_x B_\perp + \mathbf{e}_y B_y(x, t) + \mathbf{e}_z B_z(x, t)$$

B_\perp is assumed to be constant, and $\mathbf{V} = \mathbf{e}_y V_0$. Equation (2) reduces to two equations

$$\frac{\partial B_y}{\partial t} = \frac{c^2}{4\pi} \eta \left[1 + \tau_e \frac{\partial}{\partial t} \right] \frac{\partial^2 B_y}{\partial x^2} + \frac{c^2}{4\pi} \frac{B_\perp}{nec} \frac{\partial^2 B_z}{\partial x^2} \quad (6)$$

$$\frac{\partial B_z}{\partial t} = \frac{c^2}{4\pi} \eta \left[1 + \tau_e \frac{\partial}{\partial t} \right] \frac{\partial^2 B_z}{\partial x^2} - \frac{c^2}{4\pi} \frac{B_\perp}{nec} \frac{\partial^2 B_y}{\partial x^2} \quad (7)$$

The last term in Eqs. (6) and (7) are due to the Hall effect. Assume the same initial conditions as Eq. (3) — i.e. only B_z is switched on. Because of the Hall coupling term there will be a B_y component even if it is initially zero. The solutions are

$$B_y = B_{0z} \sum_n \frac{2}{k_n L} \frac{(-1)^n e^{-t/\tau_n}}{(1 + D k_n^2 \tau_e)} [\sin(\omega_n t - k_n x) + \sin(\omega_n t + k_n x)] \quad (8)$$

$$B_z = B_{0z} \left\{ 1 - \sum_n \frac{2}{k_n L} \frac{(-1)^n e^{-t/\tau_n}}{(1 + D k_n^2 \tau_e)} [\sin(\omega_n t - k_n x) + \cos(\omega_n t + k_n x)] \right\} \quad (9)$$

$\omega_n = D_H k^2 / (1 + D k_n^2 \tau_e)$ where $D_H = c^2 \eta_H / 4\pi = c B_\perp / 4\pi n e$; $\tau_n = (1 + D k_n^2 \tau_e) / D k_n^2$. For low n modes $n = 0, 1$, $\omega_n \cong (\Omega_\perp / \omega_p^2) (c k_n)^2$ where $\Omega_\perp = e B_\perp / mc$, and $\tau_n \cong \omega_p^2 \tau_e / (c k_n)^2$. Equations (8) and (9) describe whistler waves that traverse the beam and are slowly damped at the same rate as Eq. (4). If B_y, B_z are observed as a time average and $\omega_n \tau_n \gg 0$, after a few whistler periods $2\pi/\omega_n$, $B_y \cong 0$ and $B_z \cong B_{0z}$.

For a slab model the normal component $B_x = B_\perp$ must be constant and this does not correspond to the experiments with beams. We therefore consider a cylindrical model as illustrated in Fig. 2. The magnetic field $\mathbf{B} = B_0 \mathbf{e}_x$ is turned on at $t = 0$. For an ideal conductor there is no normal component of \mathbf{B} . If electron inertia is included as in Fig. 2 there will be a variable normal component to a depth of order c/ω_p at $t = +0$ and the normal component will penetrate along with other components. The differential equations corresponding to Eqs. (6) and (7) will be written in dimensionless variables by using the beam radius ' a ' for length units, the diffusion time $\tau_0 = 4\pi a^2 / \eta c^2$ for time units and B_0 for magnetic field units.

$$\frac{\partial A_z}{\partial \tau} = \left[1 + \left(\frac{c}{a\omega_p} \right)^2 \frac{\partial}{\partial \tau} \right] \nabla^2 A_z - \frac{\eta_H}{\eta} (\hat{\mathbf{B}} \cdot \nabla) \hat{B}_z - \rho \sin \theta \delta(\tau) \quad (10)$$

$$\frac{\partial \hat{B}_z}{\partial \tau} = \left[1 + \left(\frac{c}{a\omega_p} \right)^2 \frac{\partial}{\partial \tau} \right] \nabla^2 \hat{B}_z + \frac{\eta_H}{\eta} (\hat{\mathbf{B}} \cdot \nabla) \nabla^2 A_z \quad (11)$$

$\hat{\mathbf{B}} \cdot \nabla = \hat{B}_\rho \frac{\partial}{\partial \rho} + \frac{\hat{B}_\theta}{\rho} \frac{\partial}{\partial \theta}$ and $\nabla^2 = \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \theta^2}$, $\eta_H = \frac{c B_0}{4\pi n e}$, $\tau = t/\tau_0$, $\rho = r/a$, and $\hat{\mathbf{B}} = \mathbf{B}/B_0$. $\hat{b}_r = \partial A_z / \rho \partial \theta$, $\hat{b}_\theta = -\partial A_z / \partial \rho$, $\hat{\mathbf{B}}_r = \cos \theta \Theta(t) + \hat{b}_r$ and $\hat{B}_\theta = -\sin \theta \Theta(t) + \hat{b}_\theta$.

$\hat{B}_z = \hat{b}_z$. Previously these equations have been solved⁵ with $\eta_H = 0$, $\hat{B}_z = 0$. We assume solutions of the form

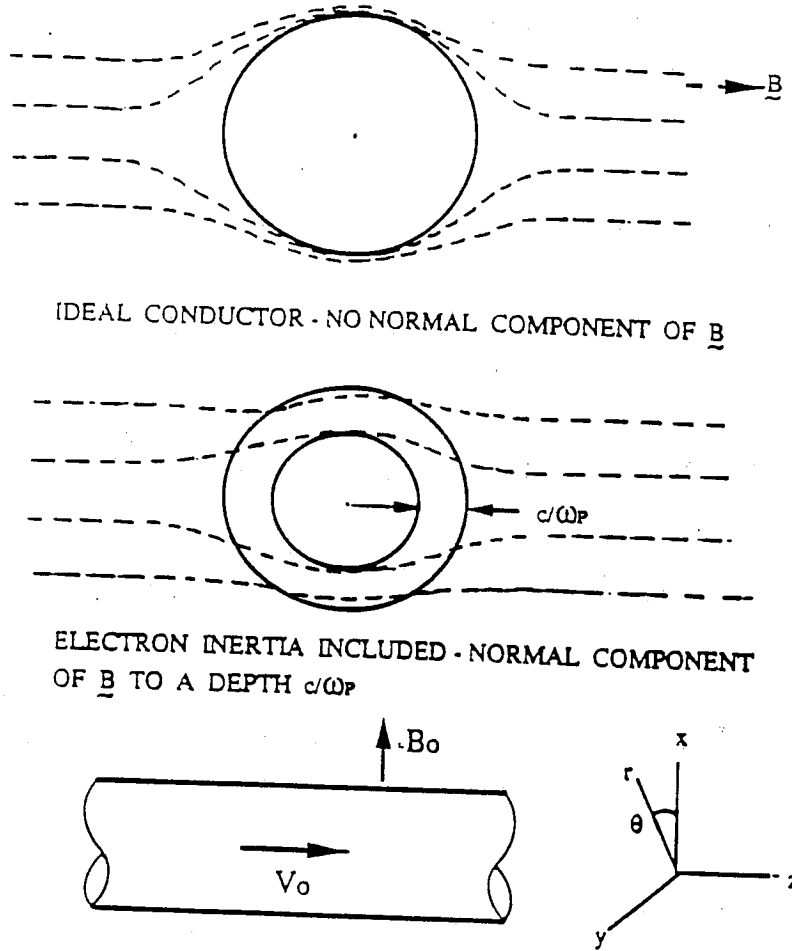


Fig. 2 Cylindrical Model of a Neutralized Ion Beam

$$A_z = \sum_{\ell=1}^{\infty} f_{\ell}(\rho, \tau) \sin \ell \theta \sim f_1 \sin \theta \quad (12)$$

$$\hat{B}_z = \sum_{\ell=1}^{\infty} g_{\ell}(\rho, \tau) \sin \ell \theta \sim g_2 \sin 2\theta. \quad (13)$$

The leading term of Eq. (12) is suggested by the source term of Eq. (10). Equations (10) and (11) are linearized by the approximation $\hat{\mathbf{B}} \cdot \nabla \cong \cos \theta (\partial/\partial \rho) - (\sin \theta/\rho) \partial/\partial \theta$ in which case $(\hat{\mathbf{B}} \cdot \nabla) \nabla^2 A_z \sim \sin 2\theta$. Equations (10) and (11) are solved in terms of the eigenfunctions

and eigenvalues of

$$\nabla_{\ell}^2 \Phi_n^{\ell}(\rho) = -(\lambda_n^{\ell})^2 \Phi_n^{\ell}(\rho) \quad (14)$$

with the boundary condition

$$\left\{ \frac{d}{d\rho} \Phi_n^{\ell} + \ell \Phi_n^{\ell} \right\}_{\rho=1} = 0. \quad (15)$$

The normalized eigenfunctions are

$$\Phi_n^{\ell}(\rho) = \frac{\sqrt{2}}{J_{\ell}(\alpha_{\ell-1,n})} J_{\ell}(\alpha_{\ell-1,n} \rho) = |n, \ell\rangle \quad (16)$$

and $\lambda_n^{\ell} = \alpha_{\ell-1,n}$, $n = 1, 2 \dots$ etc. the roots of the Bessel function of order $\ell - 1$. Assuming $f_1(\rho, \tau) = \sum_n a_n^{(1)} |n, 1\rangle$ and $g_2(\rho, \tau) = \sum_n a_n^{(2)} |n, 2\rangle$ Eqs. (10) and (11) are transformed to

$$\frac{da_n^{(1)}}{d\tau} + \alpha_{0n}^2 a_n^{(1)} + \frac{\eta_H}{2\eta} \sum_m a_m^{(2)}(\tau) \left\langle n, 1 \left| \frac{d}{d\rho} + \frac{2}{\rho} \right| m, 2 \right\rangle = -\frac{2\sqrt{2}}{\alpha_{0n}^2} \delta(\tau) \quad (17)$$

$$\frac{da_n^{(2)}}{d\tau} + \alpha_{1n}^2 a_n^{(2)} + \frac{\eta_H}{2\eta} \sum_m \alpha_{0n}^2 a_m^{(1)}(\tau) \left\langle n, 2 \left| \frac{d}{d\rho} - \frac{1}{\rho} \right| m, 1 \right\rangle = 0. \quad (18)$$

From previous analysis⁵ it is apparent that after a short time only the lowest $n = 1$ mode survives. Based on this fact we retain only the lowest terms $n = 1$ and $m = 1$ as a first approximation. In this case the electron inertia term can also be neglected. The two coupling terms are

$$\left\langle 1, 1 \left| \frac{d}{d\rho} + \frac{2}{\rho} \right| 1, 2 \right\rangle = 3.3 \quad \text{and} \quad \left\langle 1, 2 \left| \frac{d}{d\rho} - \frac{1}{\rho} \right| 1, 1 \right\rangle = -1.3.$$

The solution for A_z is

$$A_z = -\frac{4}{\alpha_{01}^2} \frac{J_1(\alpha_{01}\rho)}{J_1(\alpha_{01})} e^{-(\alpha_{11}^2 + \alpha_{01}^2)(\tau/2)} \left[\cos \beta\tau + \left(\frac{\alpha_{11}^2 - \alpha_{01}^2}{2\beta} \right) \sin \beta\tau \right] \sin \theta \quad (19)$$

where $\beta^2 = 25 (\eta_H/2\eta)^2$ and $\beta\tau \cong \Omega_{\perp}/\omega_p^2 (c/a)^2 t$ so that the time behavior of Eq. (19) is similar to the lowest modes of Eqs. (8) or (9) of the slab model. Thus we have shown that for a cylindrical beam crossing a magnetic field the fast penetration of the magnetic field is due to the whistler mode time scale that dominates the usual diffusion time scale.

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