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Abstract

Drift wave turbulence in weakly driven or decaying states possesses strong correlations requiring the concept of a weakly correlated vortex gas. Recent progress on the effects of inhomogeneities on the structure, stability and life-time of the vortices is reviewed. In particular, two cases (i) of a finite temperature gradient, and (ii) of a shearing of the magnetic field across the vortex structure are analyzed. A new formulation of drift wave turbulence proposed by Zakharov (1991) in terms of the separation of short and long scales is applied.

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I. Rossby-Drift Wave Equations

In rotating fluids and magnetized plasmas the vortex state is an important structure that naturally arises. An example of such a long-lived structure is shown in Fig. 1, which is a weather map taken from the April 1, 1985 Los Angeles Times. The figure shows a cyclonic vortex with radius of 1000 km that produced severe cold weather for many days. In plasmas such vortex states are thought to be responsible for part of the anomalous transport measured in confinement systems. The fundamental equations for the dynamics of the slow neutral fluid flows on rotating planets and the $\mathbf{E} \times \mathbf{B}$ drift flows in inhomogeneous magnetized plasmas are isomorphic problems governed by what is now known as the Charney-Hasegawa-Mima equation (CHM). This important result was established by Hasegawa, McLennan and Kodama (1979) and Petviashvili (1980).

The reason for the coincidence of the two different systems and the conditions for the breakdown of the CHM are seen by considering the conservation laws

$$\frac{\partial n}{\partial t} + \mathbf{v} \cdot \nabla n + n \nabla \cdot \mathbf{v} = 0 \tag{1}$$

and

$$\frac{d\mathbf{v}}{dt} = -\nabla U + \mathbf{v} \times \boldsymbol{\Omega} \tag{2}$$

where for shallow water flows the conserved field $n \rightarrow H(x,y,t)$ — the depth of fluid layer and $U = gH(x,y,t)$ from the hydrostatic pressure $p = \rho g H$, while for plasmas, n is the ion density and $U = e\Phi/m_i$ where Φ is the electrostatic potential. On the rotating planet $\boldsymbol{\Omega}$ is the Coriolis force parameter $\Omega = f = 2\Omega_p \sin\theta$, where θ is the latitude and $\Omega_p = 2\pi/T$ with T the period (day) for rotation, while for plasmas $\Omega = eB/m_i$ is the cyclotron frequency.

Taking the rotational part of Eq. (2) and defining the vorticity as $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ we obtain

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)(\boldsymbol{\Omega}+\boldsymbol{\omega}) = -(\boldsymbol{\Omega}+\boldsymbol{\omega})(\nabla\cdot\mathbf{v}) + (\boldsymbol{\Omega}+\boldsymbol{\omega})\cdot\nabla\mathbf{v}$$

and using Eq. (2) yields

$$= (\boldsymbol{\Omega}+\boldsymbol{\omega})\frac{1}{n}\frac{dn}{dt} + (\boldsymbol{\Omega}+\boldsymbol{\omega})\cdot\nabla\mathbf{v}\,. \tag{3}$$

The last term in Eq. (3) is the "vortex stretching term" that gives rise to the kinking and the reconnection of vortex filaments in the non-rotating or unmagnetized hydrodynamics. In the presence of large $|\Omega/\omega|$, however, the vortex filaments are forced to remain nearly straight (Taylor-Proudman Theorem) from the x, y-components of Eq. (3), and thus it is often a good approximation to drop this last term in Eq. (3), which describes the parallel compression of the fluid. Assuming $(\boldsymbol{\Omega}+\boldsymbol{\omega})\cdot\nabla v_z \simeq \Omega\,\partial_z\, v_z$ is small, the parallel component of Eq. (3) gives Ertel's theorem

$$\frac{d}{dt}\left(\frac{\Omega+\omega_z}{n}\right) = 0 \tag{4}$$

where $\omega_z = \partial_x\, v_y - \partial_y\, v_x$. The conservation law (4) gives the CHM equation when ω_z is evaluated in the geostrophic or $\mathbf{E}\times\mathbf{B}$ drift approximation and only linear gradients are taken into account.

A. Ordering for slow flow motions

For flows that evolve slowly compared with Ω we introduce the small ordering parameter

$$\varepsilon = \frac{1}{\Omega}\frac{\partial}{\partial t} \sim \frac{\mathbf{v}\cdot\nabla}{\Omega} \ll 1\,. \tag{5}$$

In the first order Eq. (2) yields

$$\mathbf{v} = \frac{\hat{\mathbf{z}}\times\nabla U(x,y,t)}{\Omega} \tag{6}$$

and the inertial acceleration $-d\mathbf{v}/dt$ correction gives

$$\mathbf{v} = \frac{\hat{\mathbf{z}}\times\nabla U}{\Omega} - \frac{1}{\Omega^2}\frac{d}{dt}\nabla U\,. \tag{7}$$

Using Eq. (6) to calculate ω_z and the convective derivative, Eq. (4), leads to the Charney equation

$$\frac{dq}{dt} = \left(\partial_t + \frac{gH_0}{\Omega}\{h,\}\right)\left[\frac{\left(\Omega + \frac{gH_0}{\Omega}\nabla^2 h\right)}{H_0(1+h)}\right] = 0 \tag{8}$$

where $h = (H - H_0)/H_0$ is the relative depth of the shallow fluid. In the case of the plasma we assume that the electrons quickly thermalize in the potential Φ giving $n = N(x)\exp(e\Phi/T_e)$ and the calculation of $\omega_z = (\Omega\,\rho_s^2\,e/T_e)\nabla^2\Phi$ so that the corresponding nonlinear p.d.e. is

$$\frac{dq}{dt} = \left(\partial_t + \frac{1}{B}\{\Phi,\}\right)\left[\frac{\Omega\left[1 + (\rho_s^2/T_e)\nabla^2(e\Phi)\right]}{N(x)\exp(e\Phi/T_e)}\right] = 0\,. \tag{9}$$

Thus with the identification of $\rho_s^2 = m_i T_e/e^2 B^2$ with $L_d^2 = gH_0/\Omega^2$ and $e\Phi/T_e$ with h and the expansion of the Boltzman distribution as $\exp(e\Phi/T_e) \simeq 1 + e\Phi/T_e$ the two equations (8) and (9) are the same and can be written in units of $\rho_s \leftrightarrow L_d$ and $1/\Omega$ as

$$(\partial_t + \{\varphi, \quad\})(\nabla^2\varphi - \varphi + v_d x) = 0 \tag{10}$$

where $q \cong \nabla^2\varphi - \varphi + v_d x$, and $v_d = c_s^2/\Omega L_n \leftrightarrow gH_0/\Omega R_p$ is the linear long wave phase velocity. Here x is in the direction of the gradient (northward) and y in the direction of the linear wave propagation (westward).

Equation (10) is the CHM equation which has the following properties:

(1) conserves mass, energy, enstrophy and integrals of arbitrary functions of q

(2) translationally invariant in x, y

(3) possesses the exact Larichev-Reznick (1976) dipole vortex solutions.

The dipole vortex solution is a two-parameter family (r_0 = radius, u = speed) of solitary vortex solutions. The speed c of the wave components of Eq. (10) are in the range $0 < c < v_d$ and the speed u of the vortex solutions is either $u > v_d$ or $uv_d < 0$. Soliton-like or weakly inelastic collisions with zero impact parameter are given by Makino *et al.* (1982) and McWilliams and Zabusky (1982), and the strongly inelastic collisions that occur when the impact parameter $b \simeq r_0$ are shown in Horton (1989).

B. Inhomogeneous systems

Now when the size of the vortex r_0 is taken comparable with the variation of the inhomogeneous background of the medium, the structure of the Rossby-Drift Wave equation changes. Petviashvili (1977) and Tasso (1967) have shown that in a plasma the inhomogeneity introduces the KdV nonlinearity $\varphi\,\partial_y\varphi$ into Eq. (10). This change alone, however, spoils the conservation from (9) of the equation. The correct treatment of the nonlinear-inhomogeneous systems expands Eq. (9) to obtain the generalized q-conserving equation (Su *et al.* (1991))

$$\left(\frac{1}{T(x)} - \nabla^2\right)\frac{\partial\varphi}{\partial t} + (v_{d0} + v_d' x - K_T\varphi)\frac{\partial\varphi}{\partial y} - [\varphi, \nabla^2\varphi] = 0 \tag{11}$$

where $v_{d0} + v_d' x$ is the inhomogeneous drift-Rossby speed and $K_T = (d/dx)(1/T) = -T'/T^2$ is the inhomogeneity of the dispersion scale $\rho_s^2 = T/\Omega^2$ or $L_d^2 = gH_0/\Omega^2$. Here $T(x)$ is the dimensionless temperature profile normalized to unity at the center of the vortex. Equation (11) has the following important properties:

(1) mass, energy and moments of q are conserved, but not the usual enstrophy

(2) the dipole vortex is split into monopole vortices with only one sign (cyclone or anticyclone depending on the sign of v_d') being a long-lived vortex

(3) the equation is not translationally invariant in the x (north-south) direction — there is now a preferred direction for wave propagation which is toward the equatorial zone or to the hot plasma region.

We have shown in Su *et al.* (1991) that Eq. (11) has monopole vortex solutions given by

$$\nabla^2\varphi = k^2(u,x)\varphi + \frac{v_{d0}'}{2u^2}\varphi^2 \tag{12}$$

where $k^2 = (1/T(x) - v_d(x)/u) \sim \varepsilon$. The expansion of $k^2(u,x)$ about the location of the vortex leads to

$$k^2 = k_0^2 + \alpha x$$

where

$$\alpha = (K_T - v'_{d0}/u) \sim \varepsilon^2 \tag{13}$$

and the monopole vortex is given by

$$\varphi(x,y,t) \cong -\frac{4.8u^2\,k_0^2}{v'_{d0}} \cosh^{-4/3}\left(\frac{3}{4}\,k_0\sqrt{x^2+(y-ut)^2}\right) . \tag{14}$$

The speed u of the vortex depends on the amplitude φ_m with

$$u = \frac{1}{2}\left[v_{d0} + (v_{d0}^2 + 0.83\,v'_{d0}\,\varphi_m)^{1/2}\right] \approx v_{d0}(1 + 0.21 v'_{d0}\,\varphi_m) \tag{15}$$

for small, positive $v'_{d0}\,\varphi_m$. Thus the shear in the drift-Rossby speed $v_d(x)$ acts to change the solitary solution from a dipole to a monopole. One recalls that the monopole vortex is the natural solution in sheared flows (Horton *et al.*, 1987).

II. Propagation and Collisions of the Monopole Vortices

A. Vortex-vortex interactions

We have established that the monopole vortices in Eq. (14) can behave under collisions either as

(1) soliton-like collisions with the stronger vortex overtaking and passing through the weaker vortex

or

(2) point vortex-like interactions where two strong monopole vortices, which by Eq. (14) must be of the same sign, rotate about one another.

In Fig. 2 we show an example of the soliton-like pass-through collision. In Fig. 3 we show an example of the second case where two nearly equal strength monopole vortices interact like point-vortices rotating around one another.

B. Wave radiation

As the amplitude φ_m of the vortices becomes small, the speed of propagation in Eq. (15) approaches the linear wave speed and the coupling to the wave field radiates energy from the vortex. Su *et al.* (1991) calculate this radiative decay of the vortex. The local energy conservation equation is

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \mathbf{S} = 0 \tag{16}$$

where

$$\mathcal{E}(x,y,t) = \frac{1}{2}\left[\frac{\varphi^2}{T(x)} + (\nabla\varphi)^2\right] \tag{17}$$

and the "Poynting" flux is

$$\mathbf{S} = \left(\frac{1}{2}\, v_d(x)\, \varphi^2 - \frac{1}{3}\, K_T\, \varphi^3\right)\hat{\mathbf{y}} - \varphi\nabla\frac{\partial\varphi}{\partial t} - \nabla^2\varphi\hat{\mathbf{z}} \times \nabla(\varphi^2/2)\,. \tag{18}$$

The results of a lengthy calculation are that the decay of the vortex energy

$$E_v = \int \mathcal{E}\, d^2x = \frac{8.2\pi\, u^4\, k_0^2}{(v'_{d0})^2}\left(\frac{4}{3} + k_0^2\right) \tag{19}$$

is given by

$$\frac{dE_v}{d\tau} \cong -\frac{u|\alpha|\varphi_m^2}{16\pi\, k_0^3}\exp\left(-\frac{4(1-v_d/u)^{3/2}}{3|\alpha|}\right). \tag{20}$$

The exponential decay factor is controlled by the strength of the inhomogeneity through α (Eq. (13)) and the closeness of the speed of propagation u to the drift speed v_d at the center of the vortex through $k_0^3 = (1 - v_{d0}/u)^{3/2}$. Simulations for $k_0^3 \gg \alpha$ and $k_0^3 \ll \alpha$ are shown in Su *et al.* (1991).

The decay of the vortex amplitude in Eq. (20) via the coupling to the radiation field will cause the speed u to decrease through Eq. (15), and as the speed $u(t)$ decreases the vortex decay rate increases exponentially through Eq. (20). Thus, the vortex will decay slowly initially and then suffer an abrupt death.

In the case of magnetic shear, which is another form of inhomogeneity that gives rise to a coupling to vertical v_z oscillations, Meiss and Horton (1983) show that the decay rate of the dipole vortex soliton is given by

$$\frac{dE_{dp}}{dt} = -2S\varphi_m^2 \left(1 - \frac{v_d}{u}\right)^{-1/2} \exp\left(-\frac{\pi L_s}{2L_n}\left(1 - \frac{v_d}{u}\right)\right) \quad . \tag{21}$$

where L_s/L_n is magnetic shear (inhomogeneity) length over the density gradient scale length L_n. We have also performed simulations for the shear induced decay of the vortex structures.

These vortex structures appear to be a natural or "self-organized" way in which the plasma can feed upon the free energy available in the density gradient and limit the radiation damping inherent in small amplitude waves. Recent simulations indicate that the vortex localization process in systems with rather different linear growth rates (due to damping caused by magnetic or velocity shear) can end up in similar final turbulent states when enough energy is fed into the system. This is because the localization to vortex structures essentially eliminates the shear damping mechanisms. This nonlinear dynamics and the shear damping introduces a form of hysteresis into the system, due to the slow decay rate of the vortices once they are formed.

III. Driving of the Large Scale Vortex Structures by the Small Scale Rossby-Drift Wave Turbulence

Finally, following the suggestion of Zakharov (1991), we consider the interaction of the small scale, weakly correlated Rossby-drift wave fluctuations φ (small scale) $\rightarrow \psi = \sum_{\mathbf{k}} \psi_{\mathbf{k}}(\varepsilon x, \varepsilon t) e^{i\mathbf{k}\cdot\mathbf{x} - i\omega_{\mathbf{k}} t}$ with

$$\langle \psi_{\mathbf{k}}(\varepsilon \mathbf{x}, t) \psi_{\mathbf{k}'}(\varepsilon \mathbf{x}, t) \rangle = \frac{\delta_{\mathbf{k},\mathbf{k}'} N_{\mathbf{k}}(\mathbf{X}, T)}{1 + k_\perp^2 \rho_s^2} \ , \tag{22}$$

where the wave density $N_{\mathbf{k}}(\mathbf{X}, T)$ satisfies

$$\frac{\partial N_{\mathbf{k}}}{\partial T} + \frac{\partial \tilde{\omega}_{\mathbf{k}}}{\partial \mathbf{k}} \cdot \frac{\partial N_{\mathbf{k}}}{\partial \mathbf{X}} - \frac{\partial \tilde{\omega}_{\mathbf{k}}}{\partial \mathbf{X}} \cdot \frac{\partial N_{\mathbf{k}}}{\partial \mathbf{k}} = 2\gamma_k N_{\mathbf{k}} + T_{\mathbf{k}}^{n\ell}(N_{\mathbf{k}}, N_{\mathbf{k}}) \tag{23}$$

as in Horton (1986), and the local drift wave frequency depends on the large scale $\varphi_L, \tilde{n}_L$ variations through

$$\tilde{\omega}_{\mathbf{k}} = \mathbf{k} \cdot \hat{\mathbf{z}} \times \nabla \varphi_L + \frac{\mathbf{k} \cdot \hat{\mathbf{z}} \times \nabla (n_0 + \tilde{n}_L)}{1 + k_\perp^2 \rho_s^2} \,,$$

with large scale motions governed by φ (large scale) $\rightarrow \varphi(X, Y, T)$ that satisfies the CHM equation containing the average of $\langle \{\psi, \nabla^2 \psi\} \rangle$ over the small scale turbulence. The driven large scale CHM equation is given by

$$(1 - \nabla^2)\partial_T \varphi + v_d \, \partial_Y \varphi - \{\varphi, \nabla^2 \varphi\} = (\partial_X^2 - \partial_Y^2) A + \partial_{XY}^2 B \tag{24}$$

where

$$\begin{bmatrix} A(X, Y, T) \\ B(X, Y,) \end{bmatrix} = \sum_{\mathbf{k}} \begin{bmatrix} (k_x \, k_y) \\ (k_x^2 - k_y^2) \end{bmatrix} \frac{N_{\mathbf{k}}(X, Y, T)}{1 + k_\perp^2 \rho_s^2} \,. \tag{25}$$

This system of equations leads to the modulational growth of large scale structures from the inhomogeneity of the distribution of the small scale fluctuations. In a tokamak the small scale turbulence is known to have a strongly increasing strength toward the low density side and, at a given radius, an increase in strength toward the outside of the torus compared with the inside. In planetary atmospheric turbulence it may be expected that the intensity of the small scale turbulence is stronger in the equatorial zones than in the high latitude regions. It is clear from the structure of Eqs. (23)–(25) that when the basic assumptions of the scale separations are satisfied, that anisotropy and inhomogeneity in the small scale turbulence is a driving force on the large scale structures. We are in the process of investigating the driven CHM equation (24) and the propagation of the small scale turbulence by the nonlinear wave kinetic equation (23) for various systems. This separation of space-time scales appears to be an effective method for extending the study of Rossby-Drift Wave turbulence to more realistic inhomogeneous turbulent states compared with the previously studied homogeneous turbulent states, as for example, in Horton (1986).

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Figure Captions

1. Large cyclonic vortex structure bringing heavy snow in late spring over the New York-Great Lakes area. The radius is approximately 1000 km and $h = \delta p/p \simeq 1/30$.

2. Nearly elastic overtaking collision of a strong monopole vortex with a weaker monopole. The profile of $1/T(x) = 1/L_d^2 = \exp(.046x)$ gives a variation of 1.6 over the core of the vortex. The gradient in the Rossby speed is $v_D(x) = 1 - x/20$. The solution conserves $q = \nabla^2\varphi - \varphi/T(x) + \int^x v_D\,dx'$. The speeds are $u_1 = 1.1$ and $u_2 = 1.7$ giving the expected collision time $\Delta\ell/\Delta u = 20/0.6 = 33$ compared with observed overlapping at $t = 22$ in frame (b). After separation (c)–(d) the weaker vortex still has 5 closed contours.

3. Point vortex type of interaction of two strong monopole vortices with speeds $u_1 = 1.30$ and $u_2 = 1.35$. Although merging might be expected here, instead the vortices rotate around one another after pulling together from the initial separation of $20\,L_d$.

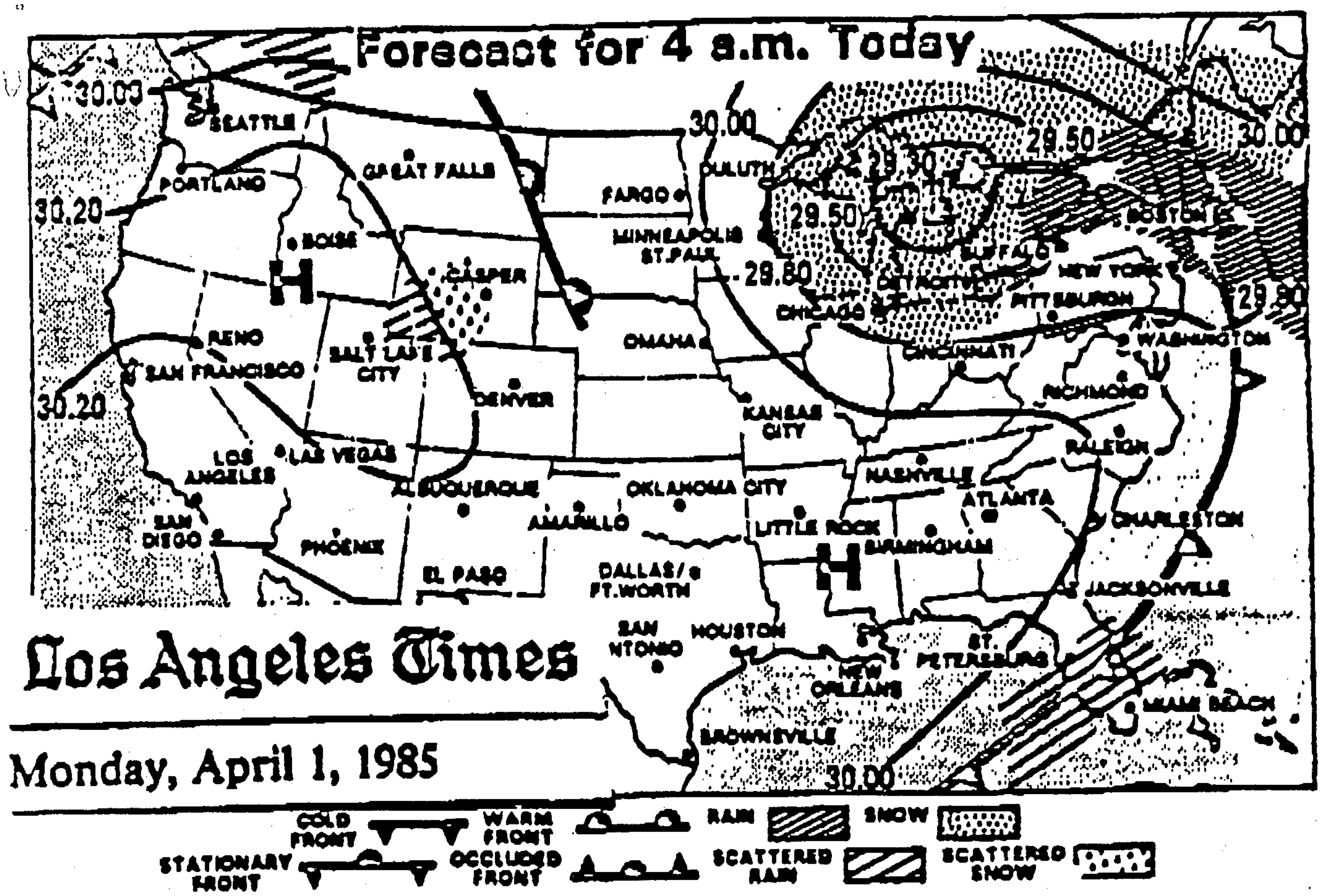

Forecast for 4 a.m. Today
Los Angeles Times
Monday, April 1, 1985
COLD FRONT
WARM FRONT
RAIN
SNOW
STATIONARY FRONT
OCCLUDED FRONT
SCATTERED RAIN
SCATTERED SNOW

Fig. 1

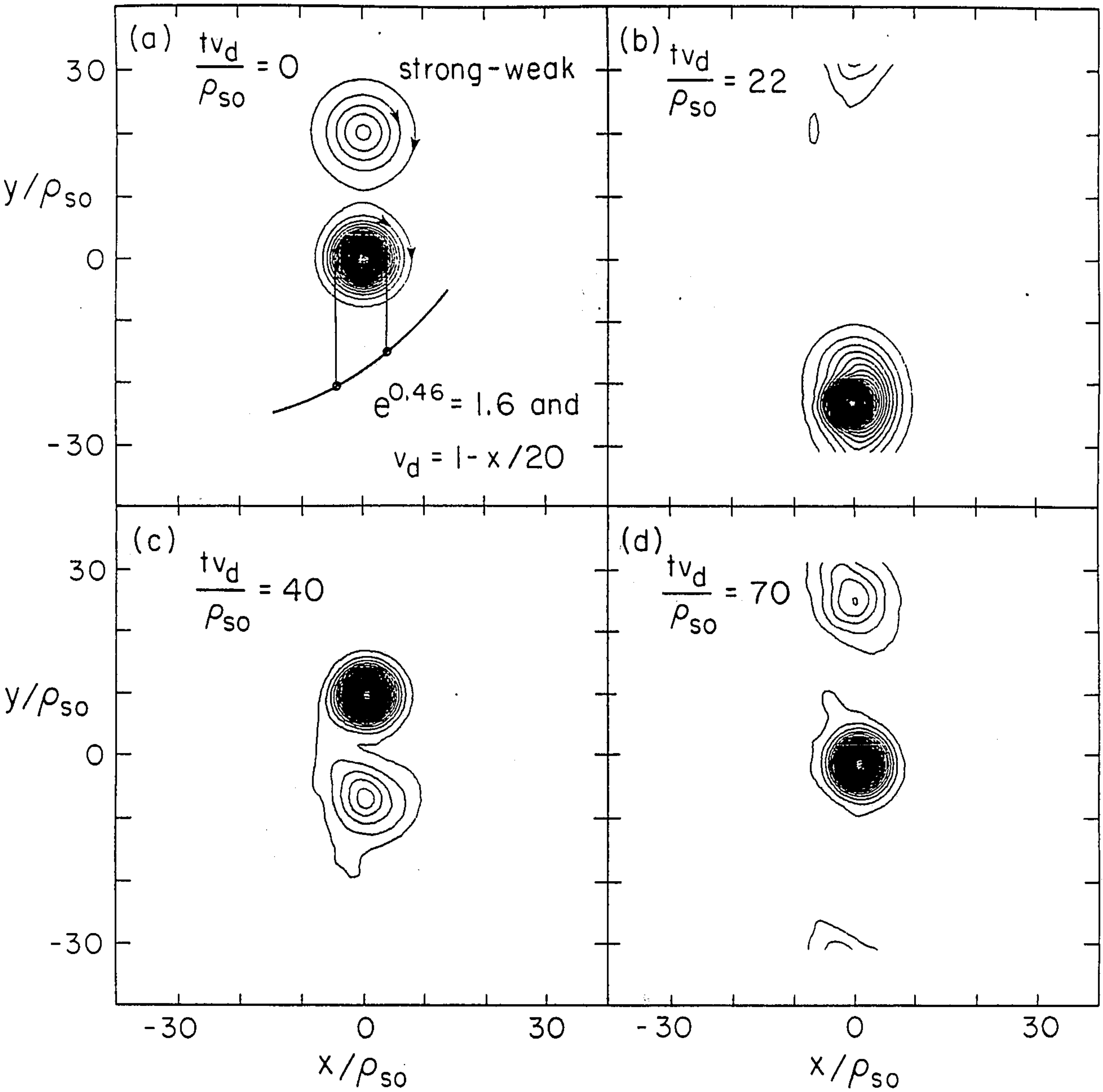
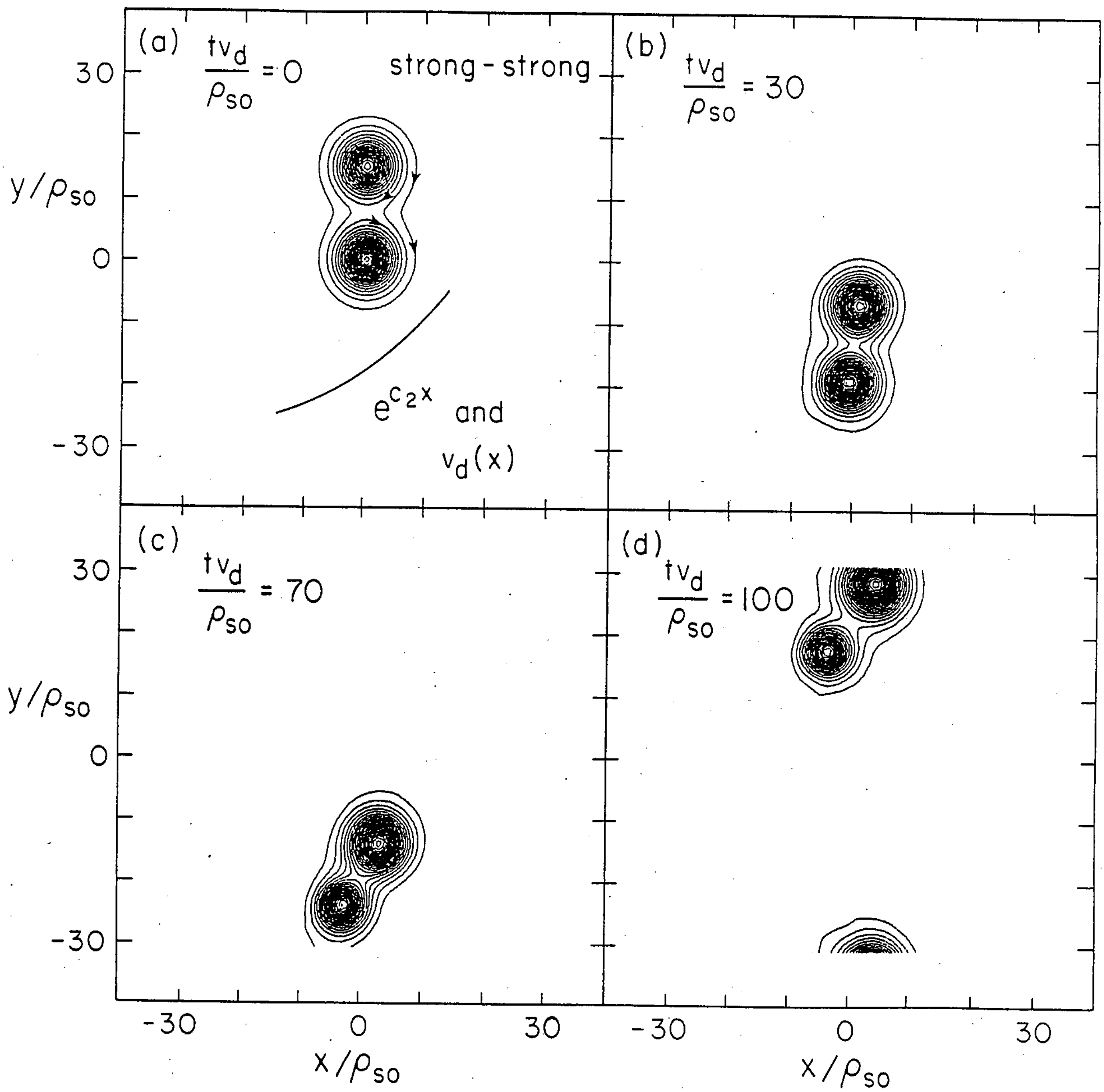

(a)
$\frac{tv_d}{\rho_{so}} = 0$
strong-weak
$e^{0.46} = 1.6$ and
$v_d = 1 - x/20$
(b)
$\frac{tv_d}{\rho_{so}} = 22$
(c)
$\frac{tv_d}{\rho_{so}} = 40$
(d)
$\frac{tv_d}{\rho_{so}} = 70$
30
0
-30
y/ρ_{so}
-30
0
30
x/ρ_{so}

Fig. 2(a-d)



(a)
$\frac{tv_d}{\rho_{so}} = 0$
strong - strong
$e^{c_2 x}$ and
$v_d(x)$
(b)
$\frac{tv_d}{\rho_{so}} = 30$
(c)
$\frac{tv_d}{\rho_{so}} = 70$
(d)
$\frac{tv_d}{\rho_{so}} = 100$
y/ρ_{so}
x/ρ_{so}
30
0
-30

Fig. 3(a-d)