INSTITUTE FOR FUSION STUDIES

DOE/ET-53088-520 IFSR #520

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Y.Z. ZHANG
Institute for Fusion Studies
The University of Texas at Austin
Austin, Texas 78712

September 1991

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Y.Z. Zhang
Institute for Fusion Studies
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Abstract

The ballooning symmetry, referred to a translational invariance in an axisymmetric toroidal plasma pinch, is shown to be spontaneously broken for non-ideal systems, i.e., the lowest order mode amplitude varies (exponentially) in radial direction. The ballooning equation has thus to be modified according to the solution of the solvability condition in higher order ballooning theory. Derived in this letter is a new set of equations suitable for non-ideal systems. It may yield significant modifications to plasma stability described by the conventional ballooning theory for such systems.

In an axisymmetric toroidal pinch, like tokamak, the turbulence composed of various types of high toroidal number (n) modes is generally considered to be responsible for the anomalous transport that deteriorates confinement of the system. In order to understand the origin of the turbulence, one generally starts with investigating linear instabilities of the high n modes, which very often necessitates solving a two-dimensional (2-D) eigenvalue problem. However, the solution seems to be unsuccessful until an elaborate scheme is devised, namely the ballooning transform,[1–3] which manifestly displays a translational invariance of the system at the lowest order of an $1/\sqrt{n}$ expansion. It is this translational invariance which we call the ballooning symmetry that reduces the intrinsic 2-D equation to an one-dimensional (1-D) form.

The 1-D equation, referred collectively as the ballooning equation, is only associated with plasma parameters at one specific magnetic surface r_0. It approximately yields the eigenvalue of the 2-D system for a localized central Fourier mode, characterized by the poloidal number $m = nq(r_0)$ ($q(r)$ is the safety factor), coupled to sidebands with the same n, and $m \pm l$ ($l = 1, 2, 3 \ldots$) due to toroidicity. The typical l should be much smaller than m for the $1/\sqrt{n}$ expansion to be convergent, otherwise the eigenvalue of the 1-D equation would be in error of $O(1)$, even if n is infinitely large.[4,5] This condition is called the solvability condition of ballooning theory, which is precisely the condition for determining the specific r_0, with which the ballooning equation is associated. For a given 2-D eigenmode equation it is found that only at a few r_0 the solvability condition can be satisfied.[4,5] Recent studies on the solvability condition in a fluid drift wave model indicate that for a non-ideal system the solvability condition is generally composed of two equations due to complexity of the system.[5] As a result, it is difficult to make the solvability condition satisfied by merely adjusting one parameter r_0.

The clue to overcome this difficulty lies in disclosing a hidden over-constraint imposed by the previous approaches. Making use of the 2-D ballooning transform[5]

$$\phi(x,l) = \oint d\lambda dk \exp[ik(x-l) - i\lambda l]\widehat{\varphi}(k,\lambda) \ , \tag{1}$$

where $\phi(x,l)$ is defined by physical mode $\Phi(x,\theta,\zeta) = \exp(in\zeta - im\theta)\sum_l \exp(-il\theta)\phi(x,l)$ with $x = n[q(r) - q(r_0)]$, one can see that there is indeed lack of rationale to confine λ on real axis, as being treated previously,[1-5] i.e., the variable λ may have a parametric imaginary part λ_I. This additional parameter λ_I is found not only to be helpful to resolve the difficulty in satisfying the solvability condition, but also to yield an important modification to the ballooning equation.

The non-zero λ_I (henceforth, the analytical continuation $\lambda \to \lambda_r + i\lambda_I$ with a parametric λ_I is understood) immediately destroys the translational invariance of the lowest order mode amplitude in the radial direction. Taking $\widehat{\varphi}(k,\lambda) \to \widehat{\varphi}_0(k,\lambda) \sim \delta(\lambda - \lambda^*)$ at the lowest order,[5] where λ^* stands for the localization in λ space, we readily find that $\Phi_0(x+\overline{m},\theta,\zeta) = \exp[-i\overline{m}(\lambda^* + \theta)]\Phi_0(x,\theta,\zeta)$ with an arbitrary integer $\overline{m}$, where the subscript 0 refers to the lowest order, i.e., the radial variation of the (lowest order) mode amplitude is $\exp(\overline{m}\lambda_I)$. Notice that $\lambda_I (\equiv \mathrm{Im}\,\lambda^*)$ as well as $\mathrm{Re}\,\lambda^*$ should be determined *a postpriori* by the higher order equations of ballooning theory. When λ_I is solved to be non-zero, the spontaneous breaking of the ballooning symmetry takes place.

Let us consider a 2-D eigenmode equation which resembles, without loss of generality, to the fluid drift wave equation described in Ref. 5. The 2-D ballooning transform Eq. (1) converts the eigenmode equation in real space into an equation in the $k - \lambda$ representation[5]

$$\left[L^{(0)} + L^{(1)}\frac{\partial}{\partial\lambda} + L^{(2)}\frac{\partial^2}{\partial\lambda^2} + L^{(\bar{1})} + \text{ higher orders}\right]\widehat{\varphi}(k,\lambda) = 0 \ , \tag{2}$$

where $L^{(i)} = \Pi_1^{(i)}\partial^2/\partial k^2 + \Pi_2^{(i)}k^2 + \Pi_3^{(i)} + \cos(k+\lambda)\Pi_4^{(i)} + \sin(k+\lambda)\Pi_5^{(i)}k$ $(i = 0,1,2,\bar{1})$, and $\Pi_j^{(0)} \sim O(1)$, $\Pi_j^{(1)} \sim O(1/n)$, $\Pi_j^{(2)} \sim O(1/n^2)$ are independent of k,λ, determined completely by the local parameters, and $\Pi_j^{(\bar{1})} = f(k) + g(k)\partial/\partial k \sim O(1/n)$. Expressions for all Π's can be derived in a straightforward manner for a given equilibrium. The non-zero $L^{(1)}, L^{(2)} \ldots$ reflect the fact that the translational invariance is not exact for the entire system.

To illustrate our mechanism, we consider a toy model, wherein both $iL^{(1)}$ and $L^{(2)}$ are real numbers independent of k, denoted by L_1/n and L_2/n^2 respectively (we have spelled out the order n explicitly, so that $L_1 \sim L_2 \sim O(1)$), and $L^{(\bar{1})} = 0$. This toy model is equivalent to $\Pi_j^{(1)} = \Pi_j^{(2)} = 0$ for $j = 1, 2, 4, 5$, and $L_1 = in\Pi_3^{(1)}, L_2 = n^2\Pi_3^{(2)}$. In this limiting case Eq. (2) is separable, and $\hat{\varphi}(k,\lambda)$ can be written as $\chi(k,\lambda)\Psi(\lambda)$ with $\chi(k,\lambda)$ being the solution of the ballooning equation

$$\hat{L}^{(0)}(\lambda)\chi \equiv \Pi_1^{(0)}\frac{\partial^2\chi}{\partial k^2} + \left[\Pi_2^{(0)}k^2 + \cos(k+\lambda)\Pi_4^{(0)} + \sin(k+\lambda)\Pi_5^{(0)}k\right]\chi(k,\lambda) = -\Pi(\lambda)\chi(k,\lambda), \quad (3)$$

where $\Pi(\lambda)$ is the eigenvalue of the ballooning equation, and parametrically dependent on λ via $\cos\lambda$ and $\sin\lambda$. The reflection symmetry of Eq. (3) in $k \to -k, \lambda \to -\lambda$ indicates that $\Pi(\lambda) = \Pi_0 + \Pi_2\cos\lambda + \Pi_2\cos 2\lambda + \cdots$. Assuming a not very strong toroidal coupling and a wave $\Psi(\lambda)$ localized by the potential well $\cos\lambda$, we approximately have $\Pi(\lambda) = \Pi_1(ch\lambda_I\cos\lambda_r - ish\lambda_I\sin\lambda_r)$, where λ_r is the variable. Substituting Eq. (3) into Eq. (2) yields the equation for $\Psi(\lambda)$:

$$\frac{L_2}{n^2}\frac{d^2\Psi}{d\lambda^2} - i\frac{L_1}{n}\frac{d\Psi}{d\lambda} + \left[\Pi^{(0)} - \Pi_1\cos(\lambda_r + i\lambda_I)\right]\Psi(\lambda) = 0\,, \quad (4)$$

where $\Pi^{(0)} = \Pi_3^{(0)} + O(1/n)$ is the eigenvalue of the 2-D system. The most localized solution of Eq. (4) for $\lambda_r \sim 0$ is $\Psi(\lambda) \sim \exp[i(L_1/2L_2)n\lambda_r - (n\sqrt{p}/2)(\lambda_r + i\tanh\lambda_I)^2]$ with $p \equiv -\Pi_1 ch\lambda_I/2L_2$. If λ_I were taken to be zero *a priori* for a non-zero L_1, a too fast variation of $\Psi(\lambda)$ would be superimposed on the appropriate ballooning ordering $d\ln\Psi/d\ln\lambda \sim \sqrt{n}$, resulting in a divergence of the expansion in power series of $(1/n)\partial/\partial\lambda$ for Eq. (2), thus leading to a difficulty of the ballooning theory.[5] Therefore, λ_I should be determined by eliminating this too fast variation of the wave function $\Psi(\lambda)$, i.e., the solvability condition can be satisfied by choosing $sh\lambda_I = L_1\sqrt{(-ch\lambda_I/2\Pi_1 L_2)}$, so that the valid ballooning ordering is restored.

With full operator form of $L^{(1)}$ and $L^{(2)}$, Eq. (2) can be solved perturbatively.[5] For the most localized $\Psi \sim \exp(-n\sqrt{p}\lambda_r^2/2)$ the solvability condition consists of the equation

$$i\Pi_1 sh\lambda_I = n\sqrt{p}\overline{\langle \chi L^{(1)} \chi \rangle} \tag{5}$$

with $p \equiv -\Pi_1 ch\lambda_I / 2n^2[\overline{\langle \chi L^{(2)} \chi \rangle} + \overline{\langle \chi L^{(1)} \overline{\varphi}_1 \rangle}]$, where $\overline{\langle \cdots \rangle} \equiv \int dk \cdots / \int dk \chi^2$, and $\overline{\varphi}_1 \sim O(1/n)$ is the inhomogeneous solution of the equation $L^{(0)}\overline{\varphi}_1 + (L^{(1)} - \overline{\langle \chi L^{(1)} \chi \rangle})\chi = 0$.

Provided *a priori* $\lambda_I = 0$, the solvability condition Eq. (5) reduces to $\overline{\langle \chi L^{(1)} \chi \rangle} = 0$ [Eq. (8) of Ref. 5]. For an ideal system, e.g. the ideal ballooning mode, where the quantities associated with the ballooning equation are purely real, and also $i\overline{\langle \chi L^{(1)} \chi \rangle}$ is real, the solvability condition is likely to be satisfied by merely adjusting r_0 with $\lambda_I = 0$, and the lowest order mode amplitude is radially invariant.[4] In this case there is no symmetry breaking of the lowest order mode.

Generally, one has to solve Eq. (5) with the ballooning equation [Eq. (3)] to determine r_0 and λ_I simultaneously, so that the solvability condition is satisfied self-consistently. Then, the eigenvalue determined by the equation for $\Psi(\lambda)$ just gives an $O(1/n)$ correction to the eigenvalue of the ballooning equation: $\Pi_1 \cos(\mathrm{Re}\,\lambda^*) ch\lambda_I$ (for the potential well $\cos\lambda_r$, $\mathrm{Re}\,\lambda^*$ can only be 0 or π). Notice that there is a multiplier $ch\lambda_I$ along with Π_1. This does not mean that the multiplier is the correction to eigenvalues of the ballooning theory with $\lambda_I = 0$ (the conventional ballooning theory), because Π_1 implicitly depends on λ_I. However, one still may expect an $O(1)$ correction to the eigenvalues of the conventional ballooning theory, if λ_I is not small. Radial envelopes of physical mode $\Phi(x, \theta, \zeta)$ are also modified by the finite λ_I. It is estimated that the peak of radial envelope is shifted from r_0 to a radial position $r_0^{\lambda_I}$, which is defined by $q(r_0^{\lambda_I}) = q(r_0) + \sqrt{p}\lambda_I$. If $q(r_0^{\lambda_I})$ is well beyond the range of the safety factor within the plasma, the corresponding mode is not physically interesting.

In conclusion, the solution of the ballooning equation is the authentic lowest order one, only if the contribution from higher orders is shown to be negligible. This is indeed a problem for non-ideal systems, whereupon a novel quantity λ_I is thus introduced into the ballooning equation to make it possible to validate the ballooning ordering, if it would otherwise be violated. Therefore, the ballooning equation [Eq. (3)], the lowest order one, must be solved along with the solvability condition [Eq. (5)], the formal first order equation, simultaneously to determine the parameters pertaining to the ballooning equation: r_0, representing the equilibrium, and λ_I, describing the spontaneous breaking of the ballooning symmetry.

Acknowledgments

The author is indebted to Professor H.L. Berk for useful discussions related to this work. The work was supported by the U.S. Department of Energy contract No. DE-FG05-80ET-53088 and No. DE-FG05-80ER-53266.

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