INSTITUTE FOR FUSION STUDIES

DOE/ET-53088-517 IFSR #517

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August 1991

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July 31, 1991

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Many characteristic features of Alfvén waves and related instabilities are strongly dependent on the inhomogeneity of the background density and the magnetic field. On the other hand, these waves also have an influence on the inhomogeneity, which is caused by the enhancement of the cross-field transport through wave-distortion of flux surfaces. This problem is addressed here within the framework of the single-fluid reduced MHD model and generalized Lagrangian representation of motion. The new effect of transport enhancement is identified as a consequence of the local squeezing of adjacent flux surfaces, which results in increased radial gradients and cross-field fluxes. This effect is found to be proportional to the second power of the ratio of the magnetic field perturbation to the normal field component. The result is applied to several problems related to $m = 1$ equilibrium relaxation and Alfvén resonance broadening.

PACS numbers 52.30.Jb, 52.35.Bj, 52.35.Mw, 52.35.Py

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I. Introduction

In toroidal confinement devices and in other systems with closed magnetic surfaces, the cross-field transport can usually be described as a one-dimensional (1-D) process, since the temperature and density of the plasma are approximately constant on each flux surface. Evaluation of the transport coefficients, though, requires a 2-D or 3-D analysis of the geometry of these flux surfaces. An Alfvén wave, introduced in such a system, may change the diffusion in two different ways: by causing additional scattering of plasma particles, thus changing the local confinement properties, or by distorting the equilibrium configuration of the flux surfaces. In this paper we examine only the second mechanism, which is assumed to be the dominant one at least for the thermal plasma component and the low-frequency kink and Alfvén modes.

These modes are studied here within the single-fluid MHD model. Because of the mathematical difficulties, we consider only the case of a 2-D wave, although the same physical effects should be present in more general situations as well. In our model the transverse fluid motion is incompressible, and no coupling to sound waves occurs. All ideal motions are much faster than any dissipative ones and have reasonably large (nonlinear) amplitudes. However, the interesting process is the cross-field transport, which, of course, is slow. This type of model has many limitations, but it is relatively simple and physically clear.

The advantage and the main limitation of the model lies in its artificial preservation of topology of the magnetic surfaces, which eliminates all effects related to reconnection. In this case it is relatively simple to describe the plasma motion in Lagrangian-like coordinates, by introducing the slow "velocity of slip" between the fluid and the "frozen in" magnetic field. As pointed out by *Moffatt*,[1] this procedure can be unreliable for complicated field structures. However, it is adequate for simple initial equilibria either in the absence of a rational flux surface (where the pitch angle of the perturbation is equal to that of the magnetic field lines) or sufficiently far from a rational surface. This crucial point will be discussed in more detail in the Conclusion.

In Section II we consider the resistive diffusion of the magnetic flux. The related resistive damping of the nonlinear wave is discussed in Sec. III. Modifications of the cross-field diffusion of a passive scalar that is somewhat different from the transport enhancement of the magnetic flux, due to the role that the magnetic field plays in the propagation of an Alfvén wave, are described in Sec. IV. In Sec. V we consider some specific cases of application: namely, the evolution of the q-profile in a tokamak, the broadening of Alfvén resonances during intensive radio-frequency (RF) heating, and the self-consistent relaxation of a nonlinearly saturated $m = 1$ kink mode. In the Conclusion, we mention other possible applications and discuss the inherent limitations of this approach.

II. Current Diffusion

The only effect that is able to cause current profile to evolve in flux-surface related coordinates is the finite conductivity. Therefore, we have to discern this slow diffusion in the background of large-amplitude, high-frequency oscillations. It is fortunate that the fast motion conserves magnetic surfaces, and hence the Lagrangian approach may be used. In an ideal system this approach simplifies the description of nonlinear Alfvén waves and , in particular, the description of the internal $m = 1$ kink mode in tokamaks.[2]

To use such a method, one has at least to assume the conservation of regular structure of the magnetic surfaces. This restriction is not a very stringent one if, for a given wave, there is no rational surface in the plasma volume under investigation. Otherwise, reconnection will occur in the vicinity of this rational surface, leading to eventual stochastization of the magnetic structure. In some cases (e.g. tearing stability and high magnetic Reynolds number), one can neglect the thin resistive layer and exclude it from consideration. The rest of the motion will not destroy the structure of the magnetic surfaces, although it may change the current profile.

In this Section, our discussion will be based on the sheared slab model of the equilibrium magnetic structure. It is chosen for the sake of simplicity of the intermediate algebraic expressions, although the same logical steps can be performed in a cylinder configuration as well. Specific differences relevant to cylindrical geometry of the magnetic surfaces are listed in Appendix.

The first equation here is the Ohm's law, expressed as a transport equation for the modified magnetic flux ψ^*:

$$\frac{d\psi^*}{dt} = D_o \nabla_\perp^2 \psi^* - E_o(t). \tag{1}$$

Following Ref. 3, we have assumed the magnetic field to be of the form $B_z \gg B_\perp$, where $\mathbf{B}_\perp = \mathbf{e}_z \times \nabla\psi^*$. The z-axis is chosen along the symmetry axis of the wave, so that $B_\perp = 0$ at the resonant rational surface. Plasma motion is incompressible, $\nabla \cdot \mathbf{v} = 0$. Also, in Eq. (1), $D_o = c^2/4\pi\sigma$ is the resistive diffusion coefficient, and E_o is the z-component of the external electric field.

In the ideal MHD case, viz., $D_o = 0$ and $E_o = 0$, Eq. (1) is reduced to $\psi^* = \psi_o(x_o)$, where x_o is the Lagrangian coordinate (initial position of fluid element). Because of flux conservation, viz., $d\psi^*/dt = 0$, this position also labels a magnetic surface throughout the motion.

In the presence of resistive diffusion, it is useful to represent the magnetic flux as $\psi^* = \psi_o(x_o, t)$. In this situation, the formal definition of x_o must be changed in order to conserve the feature that x_o labels the magnetic surface, where the given fluid element resides at the current time t. This procedure of labeling is straightforward only in an equilibrium, so we must follow certain steps to define it.

First, we 'freeze' the system and assume ideal conductivity. Then, we relax the system to equilibrium, taking this constraint into account. Measured in the new equilibrium state, x_o is the physical radius (or other metric characteristic) of a magnetic surface.

This description can be also rephrased as follows. Each intermediate state, reached by the resistive system during its motion under the topological constraint, is equivalent to some other perturbed state of the ideal system with the same initial topology but with some different initial ψ_o. We define the independent coordinates (x_o, y_o) for our dissipative problem as the Lagrangian coordinates in this adjoint ideal system. Such an operation is, of course, time dependent on the slow resistive scale. Moreover, the most interesting parameter that can be found as a result of the entire study is the time dependence of $\psi_o(x_o, t)$, i.e., the time sequence of ψ profiles for equivalent ideal systems.

In an initially ideal system, viz., $D_o \to 0$, the operational definition of coordinates is equivalent to the Lagrangian one, which simplifies the description of ideal motion. Hence the Lagrangian derivative, $\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \mathbf{v}\cdot\nabla\psi$, will be represented as $\frac{d\psi}{dt} = \left(\frac{\partial\psi}{\partial t}\right)_{\rm qL} + V_x\left(\frac{\partial\psi}{\partial x_o}\right)_{qL}$, where $V_x \propto D_o$ is the slow cross-field velocity of the fluid, and $\left(\frac{\partial\psi}{\partial t}\right)_{\rm qL}$ is the partial derivative in the new quasi-Lagrangian system of coordinates (x_o, y_o, t). (Henceforth all partial derivatives can be taken to be in this quasi-Lagrangian system, unless otherwise stated.) The motion is now described by the time-dependent plane transformation $x = x(x_o, y_o, t)$, $y = y(x_o, y_o, t)$. Assuming transversal incompressibility of flow, we have the Jacobian of the transformation,

$$\mathcal{J} = \frac{\partial x}{\partial x_o}\frac{\partial y}{\partial y_o} - \frac{\partial y}{\partial x_o}\frac{\partial x}{\partial y_o} = 1, \tag{2}$$

or, using the notation of Poisson brackets, $[x, y] = 1$. In these coordinates we have $\nabla_\perp^2\psi^* = [[\psi_o, y], y] + [[\psi_o, x], x]$, so that the equation of magnetic flux transport, Eq. (1), takes the form

$$\frac{\partial\psi_o}{\partial t} + V_x\psi_o' = D_o(\psi_o''\left(\frac{\partial y}{\partial y_o}\right)^2 + \psi_o'\left[\frac{\partial y}{\partial y_o}, y\right] + \psi_o''\left(\frac{\partial x}{\partial y_o}\right)^2 + \psi_o'\left[\frac{\partial x}{\partial y_o}, x\right]) - E_o. \tag{3}$$

Here V_x is the velocity of fluid displacement relative to the flux surface $x_o =$ const. (and hence $V_x \propto D_o$ and $V_x \ll c_A$); also, $\psi_o' = \partial\psi_o/\partial x_o$ and $\psi_o'' = \partial^2\psi_o/\partial x_o^2$.

Our objective is to derive the y-independent equation for the evolution of flux. This is possible only with certain types of boundary conditions in y (otherwise the result could be y-dependent). In the following, we assume that the boundary conditions are defined as either the periodicity of perturbations with period L_y, or as $y = y_o$ and $x = x_o$ at $y_\pm = \pm L_y/2$ (which is equivalent to having impenetrable limiters at $y_\pm$). Note that if any function A is periodic in y, it is also periodic in y_o with the same period and remains so throughout the motion; the distance between boundaries $y_\pm$ in y_o also remains constant, viz., $\Delta y_o = L_y$. Bearing this in mind, we define the surface average of quantity A as $< A > \equiv \frac{1}{L_y} \oint A \, dy_o$.

The quasi-Lagrangian coordinates are defined with the volume conservation requirement; i.e., the definition of x_o is such that the total flux of fluid through the surface $x_o =$ const. is zero, or $< V_x >= 0$. This property allows us to separate the flux surface-averaged part of Eq. (3) from the y_o-dependent part. The first one governs the evolution of $\psi_o(x_o, t)$, whereas the latter only defines the value of the convective velocity V_x (which will be used in the next section but is not necessary for the description of the ψ^* transport).

One can use the following identity

$$\langle \left[\frac{\partial a}{\partial y_o}, a\right] \rangle = \langle \frac{\partial}{\partial x_o} \left(\frac{\partial a}{\partial y_o}\right)^2 - \frac{\partial}{\partial y_o} \left(\frac{\partial a}{\partial y_o} \frac{\partial a}{\partial x_o}\right) \rangle = \frac{\partial}{\partial x_o} \langle \left(\frac{\partial a}{\partial y_o}\right)^2 \rangle \tag{4}$$

for any function a that satisfies the boundary conditions, including y and x, and so express the averaged equation for the magnetic flux transport as

$$\frac{\partial \psi_o}{\partial t} = D_o \frac{\partial}{\partial x_o} \left(K \frac{\partial \psi_o}{\partial x_o} \right) - E_o. \tag{5}$$

Here

$$K \equiv 1 + \langle \left(\frac{\partial \lambda}{\partial y_o}\right)^2 + \left(\frac{\partial \xi}{\partial y_o}\right)^2 \rangle \tag{6}$$

is the transport enhancement factor, with $\lambda = y - y_o$ and $\xi = x - x_o$. This is an exact expression, obtained without quasilinear assumptions and valid for nonlinear perturbations of magnetic surfaces.

Displacements λ and ξ can be expressed via the perturbation of the magnetic field in the ideal limit ($D_o \to 0$). From

$$B_x = -\frac{\partial \psi}{\partial y} = [\psi_o, x] = \psi_o' \frac{\partial \xi}{\partial y_o},$$

$$B_y = \frac{\partial \psi}{\partial x} = [\psi_o, y] = \psi_o' \left(1 + \frac{\partial \lambda}{\partial y_o}\right),$$

it follows that

$$\left(\frac{\partial \xi}{\partial y_o}\right)^2 + \left(1 + \frac{\partial \lambda}{\partial y_o}\right)^2 = \frac{B_\perp^2}{B_{\perp 0}^2},$$

and hence

$$K = \langle B_\perp^2 / B_{\perp 0}^2 \rangle = 1 + \langle (\delta B_\perp / B_{y0})^2 \rangle. \tag{7}$$

While using the above equations for the description of the magnetic flux transport, one must bear in mind that the enhancement factor K can have components which are rapidly oscillating with time. If the wave is stationary, such as a saturated kink mode, or if it is a coherent wave moving with a fixed group velocity, then the averaging procedure in y_o is sufficient to remove all such oscillations from $K(x_o, t)$. In general, however, the enhancement factor should be averaged over these fast oscillations:

$$\bar{K}(x_o, t) = 1 + \sum_\omega \langle (\delta B_\perp^\omega / B_{\perp 0})^2 \rangle. \tag{8}$$

Thus, for example, the influence of a traveling wave on the diffusion will be two times greater than that of a standing wave of the same amplitude with nonzero frequency.

Equations (7) and (8) for the enhancement factor only appear to be simple. Complications are hidden in the simple notation $< ... >= (1/L_y) \oint (...) \, dy_o$, which denotes the averaging procedure on a magnetic surface. If the perturbation is large enough, this surface can be distorted in an number of different ways, including those shown in Fig. 1. This figure represents sample cross sections of perturbed flux surfaces, shown in pairs to illustrate the density of the y_o-grid, which is proportional to the relative distance h_y between adjacent flux surfaces. Let us define the distance along $y =$ const. as $H_y \equiv h_y \Delta x_o \equiv (\partial x / \partial x_o)_y \Delta x_o$, where the partial derivative is taken at constant y and Δx_o is the difference between the surface coordinates in equilibrium. The density of the y_o grid is $J = (\partial y_o/\partial y)_{x_o}$, where the partial derivative is taken at constant x_o, i.e., on the surface. Between these two quantities we have the direct relationship

$$h_y \equiv \left(\frac{\partial x}{\partial x_o}\right)_y = \frac{[x,y]}{[x_o,y]} = \left(\frac{\partial y}{\partial y_o}\right)_{x_o}^{-1},$$

and thus for any A

$$\langle A\rangle = \frac{1}{L_y}\oint A(x_o,y)\left(1+\left(\frac{\partial \xi}{\partial x_o}\right)_y\right)dy. \qquad (9)$$

However, this particular expression is not too convenient, because the transformation from y_o to y may not be single-valued, as shown in Fig. 1(b).

It is instructive to examine how the averaging procedure works in the example cases (b) and (c) shown in Fig. 1. The relative distance along the normal to the perturbed surface is

$$h = \frac{h_y h_x}{\sqrt{h_x^2+h_y^2}} = \left(\left(\frac{\partial y}{\partial y_o}\right)^2 + \left(\frac{\partial x}{\partial y_o}\right)^2\right)^{-1/2}, \qquad (10)$$

so that $K =< h^{-2} > \propto < B_\perp^2 >$ and $B_\perp \propto 1/h$. Here $B_\perp$ is the perturbed field. In cases (b) and (c), the minimum distance between the flux surfaces is approximately the same, i.e., the maximum perturbed magnetic field is approximately the same. Also, the fraction of the surface area where the field is thus perturbed, is of the order of unity in both cases. However, the resulting scalings of the enhancement factor K with $B_{\perp\max}$ are quite different: $K \propto {B_\perp}^2$ in case (b), and $K \propto {B_\perp}^1$ in case (c). The reason for this is quite evident in physical terms: in case (b), the increase of the flux surface area is inversely proportional to h, so that the flux (and K) should increase as the inverse second power of h, whereas in the case (c), this area is approximately independent of h and the flux increases only as h^{-1}. In terms of the mathematical averaging procedure, this means that most of the integral in y_o is accumulated over the area where most of the inter-surface fluid is situated. In case (c) this is the area of large h and hence small $B_\perp$.

Equation (5) governs the evolution of the current in the presence of a given large-amplitude Alfvén wave. To make our approach self-consistent, we next consider the evolution of the wave itself. In the following section, we study this problem using the modification of the ideal nonlinear description presented in Ref. 2 and show that the damping is predominantly caused by factors other than those responsible for the enhancement of diffusion.

III. Resistive Damping

As we have seen, the mean component of the Ohm's law does not contain any information about the damping of the wave, at least in the leading order in D_o. Such information can be obtained from the convective flow (V_x, V_y), defined by the y_o-dependent component of Eq. (5). It can be conveniently described by an effective potential U as follows: $V_x = \partial U/\partial y_o$, $V_y = -\partial U/\partial x_o$, where U can be found as

$$U = \int V_x dy_o = D_o \int dy_o \left(\frac{\psi_o''}{\psi_o'} \left(\frac{\partial y}{\partial y_o} \right)^2 + \left[\frac{\partial y}{\partial y_o}, y \right] \right)^{\sim}, \tag{11}$$

so that for any a, we have

$$da/dt = (da/dt)_{\mathrm{qL}} - [U, a] \equiv \hat{R}a, \tag{12}$$

where the index 'qL' means the quasi-Lagrangian coordinates.

To describe the propagation and the damping of Alfvén waves, we start with the dynamic (vorticity) equation

$$\nabla \times \left(\rho \frac{d\mathbf{v}}{dt} \right) = \frac{1}{c} \nabla \times [\mathbf{J} \times \mathbf{B}] \tag{13}$$

and rewrite it using true Lagrangian coordinates (x_o, y_o) in the slab model:

$$\left[\rho \frac{\partial^2 \lambda}{\partial t^2}, y \right] + \left[\rho \frac{\partial^2 \xi}{\partial t^2}, x \right] = \frac{1}{4\pi} \left[\psi, \nabla_\perp^2 \psi \right]. \tag{14}$$

(The corresponding expression for the cylindrical case is derived in Ref. 2.) In the ideal limit, viz., $\psi = \psi_o(x_o)$, we can further transform Eq. (14) to obtain

$$\left[\rho \frac{\partial^2 y}{\partial t^2}, y \right] + \left[\rho \frac{\partial^2 x}{\partial t^2}, x \right] = \frac{1}{4\pi} \left[(\psi_o')^2 \frac{\partial^2 \lambda}{\partial y_o^2}, y \right] + \frac{1}{4\pi} \left[(\psi_o')^2 \frac{\partial^2 \xi}{\partial y_o^2}, x \right], \tag{15}$$

or

$$\left[\hat{M}y, y\right] + \left[\hat{M}x, x\right] = 0, \tag{16}$$

where $\hat{M}a \equiv 4\pi\rho\partial^2 a/\partial t^2 - (\psi_o')^2\partial^2 a/\partial y_o^2$.

Equation (15) demonstrates the well-known fact that in a slab geometry, the ideal linear wave solutions are at the same time the exact solutions of the nonlinear equation.[4] Indeed, for any λ, for ξ defined by the incompressibility condition $[x_o + \xi, y_o + \lambda] = 1$ and $v_A \equiv \psi_o'/(4\pi\rho)^{1/2}$, the pair

$$\lambda = \lambda(x_o, y_o \pm v_A t), \qquad \xi = \xi(x_o, y_o \pm v_A t), \tag{17}$$

satisfies Eq. (16). Since $v_A \equiv \psi_o'/(4\pi\rho)^{1/2} = k_{\|}c_A$ depends on x_o, such waves, in general, experience spatial phase mixing, which leads to fast damping rates.

To include the influence of resistivity, let us transform the expression of Eq. (15) to our quasi-Lagrangian coordinates. We note that the right-hand side of this equation remains unchanged in the presence of resistivity. This is a consequence of the fact that in the newly chosen coordinates, defined in Sec. II, the magnetic line bending force is exactly the same as in the original Lagrangian coordinates. The only modification is in the inertial terms, where the time derivative is no longer a partial derivative and must be replaced by $\hat{R}$:

$$\left[4\pi\rho\hat{R}^2 y, y\right] + \left[4\pi\rho\hat{R}^2 x, x\right] = \left[(\psi_o')^2\frac{\partial^2\lambda}{\partial y_o^2}, y\right] + \left[(\psi_o')^2\frac{\partial^2\xi}{\partial y_o^2}, x\right]. \tag{18}$$

The form of the operator $\hat{R}$ is given by Eq. (12). This description may be closed by the inclusion of Eqs. (2) and (5) in the system and, if necessary, a similar equation for the diffusion of the density ρ.

As a next step, we argue that the influence of the resistivity becomes important only at large times $t \sim t_d \propto D_o^{-1/3}$. In this limit we have $v_A'(x_o)t \gg 1$, and the radial derivatives of a shear Alfvén perturbation become larger than any initial gradient, while the y_o derivative remains of the same order. Indeed, we find that

$$\frac{\partial}{\partial x_o} \sim \left.\frac{\partial}{\partial x_o}\right|_{\text{at } t=0} + v'_A t \frac{\partial}{\partial y_o} \gg \frac{\partial}{\partial y_o}. \tag{19}$$

Thus, we have

$$\frac{\partial}{\partial x_o} \approx v'_A t \frac{\partial}{\partial y_o}. \tag{20}$$

Evaluating U in this approximation yields

$$U \approx D_o(v'_A t)\lambda_y \tag{21}$$

where all significant nonlinear corrections have cancelled each other. The same thing happens with the approximate form of $\hat{R}$:

$$\hat{R}\lambda \approx \frac{\partial\lambda}{\partial t} - D_o(v'_a t)^2 \lambda_{yy}. \tag{22}$$

This result leads to the same conclusions as the linear model[5] in the sense that it recovers the fast exponential decay of the perturbations $\sim \exp(-\nu t^3)$. Indeed, with the lowest order correction [in $D_o(v'_a t)^2$], the equation of motion looks like

$$\frac{\partial^2\lambda}{\partial t^2} - v_A^2 \frac{\partial^2\lambda}{\partial y_o^2} - 2D_o(v'_A t)^2 \frac{\partial}{\partial t}\frac{\partial^2\lambda}{\partial y_o^2} = 0. \tag{23}$$

In our ordering, where $t \sim D_o^{-1/3}$, the correction term is still small, so that it does not significantly modify the conditions of wave propagation. However, the amplitude of the wave decreases as

$$\lambda_k(t) \propto \exp\left(-\frac{D_o}{3} v_A'^2 k_y^2 t^3\right), \tag{24}$$

which justifies the initial ordering.

We have shown that the damping of the large amplitude shear Alfvén waves does not differ significantly from that in the linear model, and thus the damping process can be decoupled from the essentially nonlinear effects responsible for the enhancement of the cross-field flux diffusion. This is not always true; for example, the case of the Global Alfvén Eigenmode[6] requires special treatment.

IV. Diffusion of a Passive Scalar

A number of extensive studies has been conducted to investigate the changes of diffusion in a perturbed magnetic field.[7–9] The main channel of the diffusion enhancement discussed in these papers is the stochastization of magnetic field lines. However, with our set of initial assumptions, no such stochastization occurs, and the increase in transport will be caused by effects similar to those that influence the diffusion of current (Sec. II). Only these effects are discussed in what follows.

The transport equation for a scalar T in incompressible plasma flow looks like

$$\frac{dT}{dt} = \nabla_\perp \left(\chi_\perp \nabla_\perp T\right) + \nabla_\| \left(\chi_\| \nabla_\| T\right), \tag{25}$$

where $d/dt \equiv \hat{R} = \partial/\partial t + V_x \partial/\partial x_o + V_y \partial/\partial y_o$ in our model coordinates. This differs from the flux diffusion equation, Eq. (1), in that it has an additional term on the right-hand side, describing the transport of the scalar T along the field lines, while ψ^* is constant on each flux surface by definition. Also, the perpendicular transport coefficient, $\chi_\perp$, is positioned in between the radial derivatives, rather than in front of them.

For a hot plasma in a strong magnetic field, parallel diffusion is usually much faster then perpendicular diffusion, viz., $\chi_\| \gg \chi_\perp$, so that there is a fast smearing out of T along each field line. Moreover, in a general tokamak-like sheared magnetic field, this parallel transport keeps the distribution of T homogeneous on each flux surface. The question of whether this property is retained in the presence of fast Alfvén -type fluctuations determines the relative importance of the convective transport (caused by the y-dependent part of the cross-field flow, V_x) as compared to the mean change in the effective diffusion due to the "squeezing" of flux surfaces (Sec. II). Here we show that the convective transport is not important. The reason is similar to that in the case of a stochastic field.[7–9]

An estimate of the T perturbation on the flux surface can be found from the linearized version of Eq. (25):

$$\frac{\partial \tilde{T}}{\partial t} - \nabla_\perp \left(\chi_\perp \nabla_\perp \tilde{T}\right) - \nabla_\parallel \left(\chi_\parallel \nabla_\parallel \tilde{T}\right) = V_x T_o' + \left(\nabla_\perp \chi_\perp \nabla_\perp\right)^\sim T_o. \tag{26}$$

The second term on the right-hand side of this equation is necessary because of the distortion of the magnetic field by the Alfvén wave. However, we have $V_x \sim D_o$, and if $\chi_\perp$ is also small (which is usually true), we get

$$\frac{\tilde{T}}{T_o} \sim \epsilon = \max\{D_o, \chi_\perp\} \left(\frac{1}{\omega_A L_T^2}\right) \ll 1, \tag{27}$$

and the convective contribution to the mean diffusion is second order in this parameter. Here $L_T = |\nabla_x \ln T_o|^{-1}$ and ω_A is the typical wave frequency. For stationary perturbations, ω_A should be exchanged for $\chi_\parallel k_\parallel^2$.

If we completely neglect the perturbation, $\tilde{T} = 0$, then the y-independent part of Eq. (25) looks like

$$\frac{\partial T_o}{\partial t} = \langle \nabla_\perp \chi_\perp \nabla_\perp \rangle T_o. \tag{28}$$

Evaluation of the right-hand side of this equation can be done along the same lines as in Sec. II, and we arrive at

$$\frac{\partial T_o}{\partial t} = \frac{\partial}{\partial x_o} \left(\chi_\perp K \frac{\partial T_o}{\partial x_o}\right), \tag{29}$$

or

$$\frac{\partial T_o}{\partial t} = \frac{1}{r_o} \frac{\partial}{\partial r_o} \left(r_o \chi_\perp K \frac{\partial T_o}{\partial r_o}\right), \tag{30}$$

where $\bar{K}(x_o, t) = 1 + \sum_\omega \langle (\delta B_\perp^\omega / B_{\perp 0})^2 \rangle$ as in Eq. (8). The diffusion corrections in these expressions are first order in $\epsilon \sim \max(D_o, \chi_\perp)$, and therefore the neglect of the convective contributions is justified.

We have found that the influence of a large-amplitude perturbation of the poloidal magnetic field on the transport can be described simply as an enhancement of the transport coefficient proportional to the second power of the perturbation amplitude. This result is only slightly different from a similar correction to the flux diffusion equation, which was obtained in Sec. II.

V. Examples

The conditions for the derivation of the flux diffusion equation (A5) are compatible with the so-called 'cylindrical tokamak' approximation. The requirement that the reconnection be absent or insignificant is definitely satisfied in such problems as the nonlinear saturation of the $m = 1$ internal kink mode in a tokamak with a nonmonotonic q profile,[10] or the long-term behaviour of the Alfvén resonances. In these cases there is no rational surface within the plasma column, so that reconnection cannot develop.

Before addressing these particular problems, it is convenient to rewrite the flux diffusion equation in a more useful form, as an equation governing the evolution of the rotational transform of the magnetic field lines, μ. In a cylindrical configuration the rotational transform can be expressed as $\mu(r_o) \propto \psi_o'/r_o$, and thus

$$\frac{\partial \mu}{\partial t} = \frac{1}{r_o}\frac{\partial}{\partial r_o}\left(\frac{D_o}{r_o}\frac{\partial}{\partial r_o} r_o^2 K \mu\right) \tag{31}$$

is the equation describing the resistive diffusion of μ, and, consequently, that of the safety factor $q(r_o) = 1/\mu$.

A. Relaxation of $m = 1$ helical equilibria

The general outline of the problem can be described as follows. The q profile in tokamaks is believed to experience periodic perturbations related to the sawtooth crashes and then to return to its initial form due to the resistive diffusion of current. During these returns, various transient forms of the q profile can be formed, for example, nonmonotonic functions of minor radius with minima close to but above $q = 1$ at some radius $r = r_e$. It has been shown that a plasma equilibrium with such a q profile can become ideally unstable with respect to $m = 1$ perturbations. Such instabilities and their subsequent saturation can be interpreted as the cause of the so-called partial sawteeth. These phenomena are seen in experiments as fast perturbations of the magnetic field and the temperature, centered close to the inversion radius of the usual sawteeth and causing perceptible $m = 1$ displacements of the plasma core. The significant feature of this process is that in a relatively short time the discharge can recover its original symmetric state without massive losses of energy from the central region, in variance with what is seen during full sawtooth reconnections. In this section, the return relaxation of a helical $m = 1$ equilibrium is described in terms of the fast changes in the q profile that are induced by the saturated mode itself.

The full asymptotic solution of the ideal saturation problem for the $m = 1$ kink mode in a tokamak with nonmonotonic q profile has been published in Ref. 10. This result will be used as a source of information about the form and intensity of the field perturbations expressed in terms of the fluid displacements. The radial displacement in cylindrical coordinates is given by Eq. (8) of this paper as

$$\xi(x_o, \theta) = \int_0^x \left[\frac{a^2 + x_o^2}{(f(x_o) + g(\theta))^{1/2}} - 1 \right] dx + h(\theta), \tag{32}$$

where

$$f \approx (a^2 + x_o^2)^2 + 24a^8(\nu/\pi)^2/(a^2 + x_o^2)^2;$$

$$a^{-4}g(\theta) \approx (8\nu/\pi)\cos\theta + 15(\nu/\pi)^2\cos 2\theta + (\nu/2\pi)^3[639\cos\theta + 135\cos 3\theta];$$

$a^2 \equiv 2\Delta q/q''$; $\nu^2 \equiv \xi^2 q''/(8\Delta q) \ll 1$; Δq is the difference between the q value at the minimum point and $q = 1$; q'' is the second radial derivative of q at the minimum; Δq_c is the threshold value of Δq; and the saturation condition is

$$\frac{\xi^2 q''}{\Delta q} = \frac{8}{71}\left(\frac{8\pi}{3}\right)^2 \left[\left(\frac{\Delta q_c}{\Delta q}\right)^{3/2} - 1 \right]. \tag{33}$$

The radial coordinate x_o is defined as a flux surface coordinate $x_o = r_o - r_e$, where r_e is the radius of the minimum-q point.

Equation (32) defines ξ as a function of a mixed, nonorthogonal pair of coordinates, so we need to recalculate it for use in formulae (31) and (A6). The fluid incompressibility can be expressed as $[r^2,\theta] = 2r_o$, and $\xi = r - r_o$, $\lambda = \theta - \theta_o$ by definition, so that

$$\left(\frac{\partial r}{\partial r_o}\right)_\theta = 1 + \frac{\partial \xi}{\partial r_o} - \frac{\partial \lambda}{\partial r_o}\frac{\partial \xi}{\partial \theta_o}\left(1 + \frac{\partial \lambda}{\partial \theta_o}\right)^{-1} = \frac{r_o}{r}\left(1 + \frac{\partial \lambda}{\partial \theta_o}\right)^{-1}. \qquad (34)$$

The partial derivative on the left-hand side of Eq. (34) is given in terms of the (r_o, θ) coordinate system. The left-hand side of this expression can be evaluated from Eq. (32), and we find

$$\frac{r^2}{r_o^2}\left(1 + \frac{\partial \lambda}{\partial \theta_o}\right)^2 = \frac{f+g}{(a^2+x_o^2)^2}. \qquad (35)$$

Conditions of the validity for Eq. (32) include the smallness of the radial displacement, viz., $\xi \ll r_o$, and the fast radial variation of the perturbation around $r = r_e$, or $x_o = 0$. In this approximation the expression for the enhancement factor K is simply $K \approx \langle (1 + \partial\lambda/\partial\theta_o)^2 \rangle$, and thus $K = < (f+g) > /(a^2+x_o^2)^2$. Substituting the known parameters into the cylindrical analog of Eq. (9) for the averaging procedure, we obtain

$$\langle \frac{f(r_o)+g(\theta)}{(a^2+x_o^2)^2} \rangle = \frac{1}{2\pi}\oint \frac{f+g}{(a^2+x_o^2)^2}\left(\frac{\partial r}{\partial r_o}\right)_\theta d\theta \approx \frac{1}{2\pi}\oint \frac{(f+g)^{1/2}}{a^2+x_o^2} d\theta, \qquad (36)$$

and, in the same approximation as in Ref.10, we find

$$K(x_o) \approx 1 + \frac{8^3}{639}\frac{(\Delta q_c/\Delta q)^{3/2} - 1}{(1+(x_o/a)^2)^4}, \qquad (37)$$

where $8^3/639 \approx 0.8$. The region of enhanced transport is localized around the q minimum point, $r_o = r_e$, and its width is determined by the parameter a. Equations (31) and (37) constitute a full system, capable of describing the self-consistent profile relaxation in detail. The physical assumptions for such a description would be: (1) that the resistive evolution of q is slow enough so that the system is always close to the ideal helical equilibrium; and (2) that the relaxation of the plasma pressure profile and the current profile around $r = r_e$ do not induce significant changes in the instability drive, as described by the parameter Δq_c.

Because of the subordinate role of this section, the details of the relaxation behaviour are not essential, whereas the estimate of the characteristic time scale involved in this process can be interesting as an insight into the possibility of experimental compatibility and into the relative importance of the nonlinear enhancement mechanism. Following this idea, we simplify the problem assuming $\Delta q_c - \Delta q < \Delta q \ll 1$ and also assuming q'' to be approximately constant during relaxation. Then the right-hand side of Eq. (32) can be evaluated at $x_o = 0$ to yield

$$\frac{\partial}{\partial t}(1 - \Delta q/\Delta q_c) \approx -\frac{3.2 D_o}{\Delta q_c^2} q''(1 - \Delta q/\Delta q_c), \tag{38}$$

and

$$1 - \frac{\Delta q}{\Delta q_c} = \left(1 - \frac{\Delta q}{\Delta q_c}\right)_{t=0} \exp\left\{-\frac{3.2 D_o}{\Delta q_c^2} q'' t\right\}. \tag{39}$$

Here $D_o = c^2/(4\pi\sigma)$ is the resistive diffusion coefficient, so that $\tau_R = 1/(q'' D_o)$ is the characteristic time of the q-profile change without magnetic perturbations. As can be seen from Eq. (39), the relaxation time of the helical equilibrium is

$$\tau_h \approx 0.3 \Delta q_c^2 \tau_R, \tag{40}$$

which is typically 10^2 times faster than the normal time scale associated with the change in the equilibrium, viz., $\tau \sim \Delta q_c \tau_R$, and is compatible with the observed time scales for partial sawteeth.

B. Broadening of the Alfvén wave resonances

In this subsection we qualitatively describe the process of radial resonance broadening, which may occur during Alfvén wave heating experiments. The importance of this effect is linked to the rate of wave energy absorption, which depends on the radial wavelength and will decrease if the gradient of the local Alfvén frequency becomes small at the resonance point.

An exact description of the wave structure in the vicinity of the Alfvén resonance can be achieved only in kinetic theory, which is beyond the scope of this paper. However, we may consider this wave to be externally given and assume the width of the resonance layer to be of the order of ρ_i and smaller than the hydromagnetic scale length, with which we are concerned.

From the standpoint of the rotational transform distribution, described by Eq. (31), the existence of a narrow resonance layer implies the generation of a local 'dent' in the overall profile. Indeed, in a stationary state, we have

$$\frac{D_o}{r_o}\frac{\partial}{\partial r_o}(\mu K r_o^2) = \text{const.}\ , \tag{41}$$

where K has a local maximum, and hence

$$\mu = \mu_o(r_o)/K(r_o). \tag{42}$$

Here $\mu_o(r_o)$ is the equilibrium profile of the rotational transform in the absence of the wave perturbation. If the resonance is sharp enough, this process can reverse the local gradient of μ even if the wave amplitude is small. The threshold for this effect is

$$L_\mu \left| \nabla_\perp \left(\frac{\tilde{B}}{B_\theta} \right)^2 \right| > 1, \tag{43}$$

where $L_\mu = |\nabla \ln \mu_o|^{-1} = r_o/S$, and $S = rq'/q$ is the usual definition of the magnetic shear.

The influence of the wave on the density profile is less significant (at least in our hydrodynamic approximation without heating and scattering effects). Since the enhancement factor K for passive scalar transport enters Eq. (30) only between the derivatives, a change is induced only in the value of the local density gradient and not in the value of the density itself. This effect can then be disregarded as long as the resonance is local and the wave amplitude is not too high.

The radial position of the hydromagnetic Alfvén resonance is determined by the condition

$$\omega = \omega_{loc} \ \propto \ \frac{|\mu - \alpha|}{\rho^{1/2}}, \tag{44}$$

where $\alpha = n/m = \text{const.}$ The usual stationary μ_o and ρ profiles correspond to a decrease of these quantities with radius, and for definiteness we assume that $\mu > \ \alpha$ and $|\nabla\mu| \ > \ 0.5|\nabla\rho|$, so that the local frequency $\omega_{loc}(r_o)$ decreases with radius. This means that any additional decrease in μ, such as that induced by the wave, shifts the resonance point inward. This inward shift of the resonance can be repeated in the new position by the same mechanism, and so forth, until a balance is achieved between the wave-induced perturbation and the regularizing background diffusion. It is also obvious that the system should maintain a certain significant level of fluctuations at all radii where the resonance point has been during the transition period. Otherwise a new resonance will arise there and increase the wave amplitude. The whole process amounts to the broadening of the resonance area in radius from the initial resonance point inward. This also can be rephrased as follows: the resonant Alfvén wave flattens the gradient of the right-hand side of equation (44), and thus causes the resonance condition $\omega = \omega_{loc}$ to be satisfied over a sizable radial interval.

The width of the new resonant area can be estimated from the following considerations. The depth of the drop in the μ value due to fluctuations should compensate the decrease of the equilibrium ω_{loc}^o with radius, so that the local Alfvén frequency at the innermost resonant point is the same as at the initial, outermost point where the process started. This means

$$K - 1 \approx \left| \frac{\Delta\mu_o}{\mu_o} - \frac{1}{2}\frac{\Delta\rho}{\rho} \right| = \left| \frac{S}{r_o} - \frac{1}{2L_n} \right| \Delta r_o, \tag{45}$$

and thus

$$\Delta r_o \approx \frac{\langle (\tilde{B}/B_\theta)^2 \rangle}{|S/r_o - 1/2L_n|}. \tag{46}$$

Here Δr_o is the resonance width, and $L_n = |\nabla \ln \rho|^{-1}$ is the density scale length.

Some rough numerical work, involving the nonlinear diffusion equation (31), has been done to check the largely qualitative discussions in this section. No significant discrepancies have been found.

VI. Conclusion

In this paper we have derived the 1-D equations describing modifications of the cross-field transport of the magnetic flux and of a passive scalar caused by a propagating Alfvén wave, an MHD instability, or other perturbations of the flux surfaces. The importance of the underlying physical process is in the induced local changes in the profile of the safety factor, which influences the MHD stability and the dispersion properties of the plasma column. The most interesting feature is the possibility of self-consistent adjustments between the q-profile and the driving MHD perturbation. This may account for some features of the penetration of the magnetic field into the discharge.

It is necessary to emphasize that the results obtained in this paper are not quasilinear and are valid for finite amplitudes, when $K \gg 1$. However, in practice such a situation is probable in the neighbourhood of a resonant magnetic surface, where either $B_\theta \rightarrow 0$ (for low frequency oscillations and internal kink instabilities) or the perturbation amplitude, $\tilde{B}$, is significant (for Alfvén wave heating experiments). In each case the impact on the transport properties is local, and the most affected characteristic of the equilibrium is the current profile or the profile of the safety factor q.

There are two important limitations of the applicability for the results derived in this paper. The first one is the absence of reconnection. This limitation is partially due to the mathematical difficulty of describing arbitrarily reconnected surfaces, but also is related to the real physical problem of Alfvén wave propagation in a medium with magnetic islands. However, the application of the formulae (5) or (A5) is possible for description of the perturbed flux surfaces just outside of a tearing island, and such application may be valuable for the problem of the nonlinear growth of tearing islands.

Another inherent limitation of our approach is the 2-D approximation of motion. This is certainly not very adequate for tokamaks and ballooning modes, but this limitation is not related to any major physical difference of the 2-D situation. The problem lies with the mathematical difficulty of describing all ideal motions and perturbations of magnetic surfaces in a full 3-D case. Indeed, the final result will probably contain information about the equilibrium configuration of the flux surfaces, which are not concentric and not even circular. An effort to overcome this difficulty can be launched as soon as interest in the overall problem has been established.

Acknowledgments The author wishes to express his gratitude to Dr. J.W. Van Dam of IFS for his comments and assistance in preparing this paper. This work was supported in part by the U. S. Department of Energy under grant DE-FGO5-80ET-53088.

Appendix. Flux diffusion in a cylindrical configuration

The definition of the modified poloidal magnetic flux in a cylinder is

$$\psi^* = \Psi - \nu r, \tag{A1}$$

where Ψ is the full poloidal flux, r is the radial coordinate, and the constant $\nu \ll 1$ is a characteristic of the helical axis of symmetry. The formal definition of ν follows from

$$\mathbf{e}_\zeta = \mathbf{e}_z - \nu \mathbf{e}_z \times \nabla r, \tag{A2}$$

where $\mathbf{e}_\zeta$ is a vector along the axis of symmetry of perturbations, so that $\mathbf{e}_\zeta \cdot \nabla = 0$. Equation (A1) is quite similar to that used in the derivation of Eq. (1) for slab geometry, but now we have $\nabla_\perp^2 r = 1/r \neq 0$, and , consequently, there is an additional contribution in E_o:

$$E_o = E_z + \nu D_o \nabla_\perp^2 r. \tag{A3}$$

However, this contribution is almost never important and cancels out exactly if we are dealing with the full flux Ψ, the safety factor q, or the rotational transform of the magnetic field $\mu = 1/q$, instead of the somewhat artificial modified flux ψ^*.

In the cylindrical quasi-Lagrangian system of coordinates, (r_o, θ_o, t), the incompressibility condition is $[r^2, \theta] = 2r_o$ and

$$\nabla_\perp^2 \Psi_o = \frac{1}{r[r,\theta]}\left[r\frac{[\Psi_o,\theta]}{[r,\theta]}, \theta\right] + \frac{1}{r[\theta,r]}\left[\frac{[\Psi_o,r]}{r[\theta,r]}, r\right]. \tag{A4}$$

We can define the surface average as $< ... > \equiv 1/(2\pi)\oint(...)\, d\theta_o$ and perform this operation over the parallel component of the Ohm's law in Eq. (1). As a result, we get the analog of Eq. (5) with small modifications:

$$\frac{\partial \Psi_o}{\partial t} = \frac{D_o}{r_o}\frac{\partial}{\partial r_o}\left(r_o K(r_o)\frac{\partial \Psi_o}{\partial r_o}\right) - E_z. \tag{A5}$$

Here Ψ_o is the poloidal magnetic flux, E_z is the component of the external magnetic field along the magnetic axis, and

$$K(r_o) = \left\langle \left(\frac{r}{r_o} \frac{\partial \theta}{\partial \theta_o} \right)^2 + \left(\frac{1}{r_o} \frac{\partial r}{\partial \theta_o} \right)^2 \right\rangle . \tag{A6}$$

The enhancement factor K can be similarly expressed through the perturbation of the magnetic field:

$$K = 1 + \langle \tilde{B}^2 \rangle / (B_o^*)^2 \tag{A7}$$

where $B^*(r_o) = \partial \psi_o / \partial r_o$ is the "reduced" magnetic field.[3]

References

[1]H.K. Moffatt, in *Magnetic Field Generation in Electrically Conducting Fluids* (Cambridge University Press, Cambridge, 1978), p. 45.

[2]A.D. Beklemishev and O.P. Pogutse, Sov. J. Plasma Phys. **15**, 650 (1989).

[3]B.B. Kadomtsev and O.P. Pogutse, Sov. Phys. JETP **65**, 575 (1973).

[4]H.K. Moffatt, in *Magnetic Field Generation in Electrically Conducting Fluids* (Cambridge University Press, Cambridge, 1978), p. 245.

[5]A.V. Timofeev, in *Reviews of Plasma Physics*, edited by M.A.Leontovich (Consultants Bureau, New York, 1986), Vol. 9, p. 265.

[6]S.M. Mahajan, D.W. Ross and G.-L. Chen, Phys. Fluids **26**, 2195 (1983).

[7]A.B. Rechester and M.N. Rosenbluth, Phys. Rev. Lett. **40**, 38 (1978).

[8]B.B. Kadomtsev and O.P. Pogutse, in *Plasma Physics and Controlled Nuclear Fusion Research 1978* (IAEA, Vienna, 1979), Vol. 1, p. 649.

[9]M.B. Isichenko, Plasma Physics and Controlled Fusion, **33**, 795 (1991).

[10]Avinash, R.J. Hastie, J.B. Taylor, and S.C. Cowley, Phys. Rev. Lett., **59**, 2647 (1987).

Figures

FIG. 1. Possible types of flux surface perturbations. Flux surface cross sections are plotted in pairs to represent the relative distance between adjacent surfaces. The area between each pair of surface cross sections is conserved by the incompressibility of the fluid. Case (a) is the unperturbed configuration; in case (b) the area of the flux surface increases roughly in inverse proportion to the relative distance; in case (c) the cross-field transport is determined predominantly by the "squeezed" areas of flux surfaces, which do not depend on the effective relative distance.

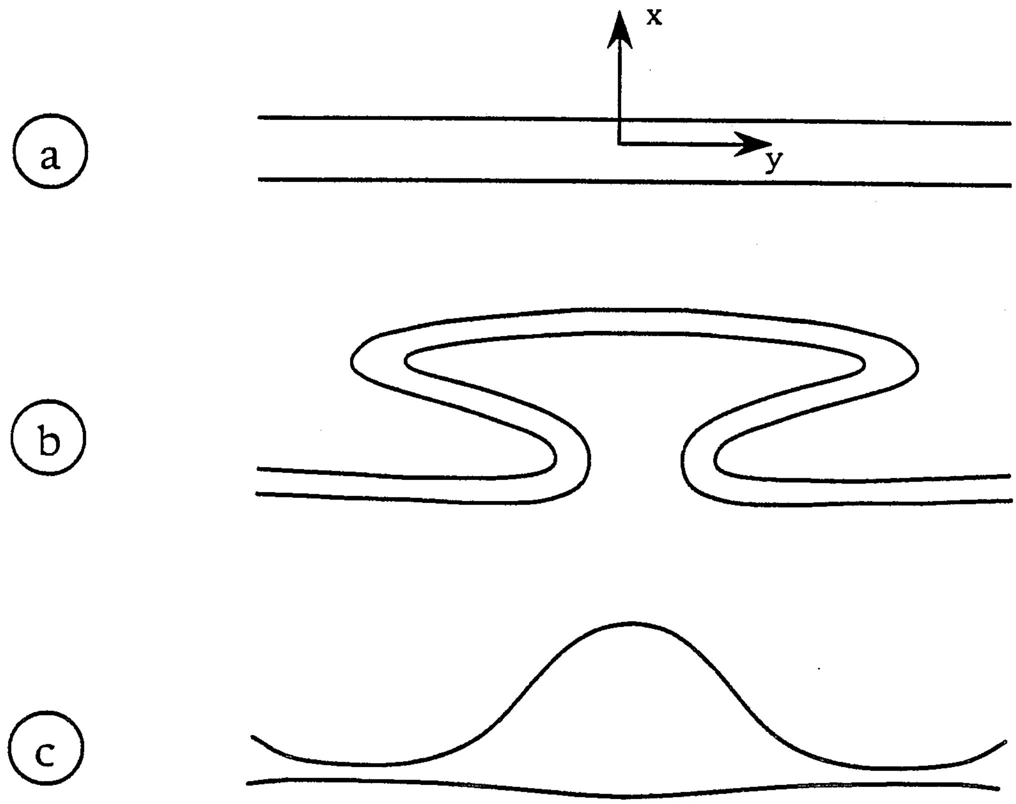

Fig. 1