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Abstract

The interaction of two near marginal tearing modes in the presence of shear flow is studied. To find the time asymptotic states, the resistive magnetohydrodynamic (MHD) equations are reduced to four amplitude equations, using center manifold reduction. These amplitude equations are subject to the constraints due to the symmetries of the physical problem. For the case without flow, the model which we adopted has translation and reflection symmetries. Presence of flow breaks the reflection symmetry, while the translation symmetry is preserved, and hence flow allows the coefficients of the amplitude equations to be complex. Bifurcation analysis is employed to find various possible time asymptotic states. In particular, the oscillating magnetic island states discovered numerically by Persson and Bondeson[17] are discussed. It is found that the flow introduced parameters (imaginary part of the coefficients) play an important role in driving these oscillating islands.

I. Introduction

Resistive tearing instability is important in many aspects; it concerns problems such as confinement in laboratory plasmas and conversion of magnetic energy to thermal and kinetic energy. The nonlinear evolution of a single tearing mode has received much attention.[1,2,3,4,5] Saramito and Maschke[6] applied bifurcation theory for compact operators to the general problem of the nonlinear solution of the 3D incompressible visco-resistive magnetohydrodynamic (MHD) equations, and they proposed that there exists a saturated tearing mode state when S_R (the magnetic Reynolds number) exceeds the critical value, where the original equilibrium loses stability. Recently, Grauer[7] has studied the nonlinear interactions of two tearing modes near marginal stability. Applying center manifold reduction, the resistive MHD equations were reduced to four amplitude equations, which are significantly easier to analyze. Compared with the usual small amplitude expansion, center manifold expansion has two advantages.[8] Firstly, the center manifold has been rigorously shown to be locally attractive,[9,10] i.e. any solutions which stay sufficiently close to the original equilibrium must eventually converge to the center manifold. Thus for local time-asymptotic states, such as steady state and periodic solutions, the center manifold reduced equations give a complete description. Here "local" means that the solution is close to the original equilibrium. Secondly, unlike the usual small amplitude expansion in which the dependence upon small parameters must be specified, in center manifold expansion the order of magnitude of all variables are naturally expressed in terms of the (small) distance from the marginal equilibrium state. However, the calculation of the coefficients in the center manifold reduction is as tedious as the small amplitude expansion and usually numerical evaluation is required.

If the model considered possesses certain symmetries, the reduced equations can be discussed in general terms without knowing the coefficients. Even though the presence of symmetry may complicate the problem by forcing the marginal modes to have a multiplicity larger than unity, it can greatly simplify the reduced equations by selecting only the terms satisfying symmetry constraints.[11,12] Recent studies of mode interactions for systems possessing symmetries have been very successful in explaining complicated behavior in some experiments; for example, Taylor-Couette flow,[13] and Faraday's experiment.[14] The model used by Grauer possesses $\mathcal{O}(2)$ symmetry: rotations, elements of $SO(2)$, act by translation of $X \rightarrow X+X_0$, and reflections, elements of $Z(2)$, act by flipping $X \rightarrow -X$, where X denotes a coordinate of the system. This $\mathcal{O}(2)$ symmetry is common for systems with circular or periodic slab geometry. Features of $\mathcal{O}(2)$ symmetry-constrained amplitude equations have been well studied by Dangelmayr,[15] Armbruster *et al.*[16] and many others.[11,12] In these references it is shown that such systems can saturate at various types of symmetry-broken states depending on the parameter domain. The predicted solutions of mixed mode, and traveling and standing waves have been observed in Grauer's simulations.[7]

In many circumstances the plasma equilibrium is not static. For example in recent Tokamak experiments shear flow plays an important role in the transition from Low to High confinement mode. When shear flow is present the reflection symmetry in Grauer's model will be broken, while the translation symmetry survives. Consequently shear flow is expected to affect the nonlinear evolution of tearing modes. In recent numerical simulations, based on straight cylinder reduced magnetohydrodynamic equations, Persson and Bondeson[17] have discovered nonlinear oscillating island states for the evolution of tearing modes, which are driven unstable by shear flow. They also found that the oscillating island states remain when the spectrum is limited to include only the modes $m/n = 2/1$ and $4/2$.

In the present paper, we also study the nonlinear evolution of tearing modes in the presence of shear flow. We treat a slab geometry model in which two modes with wave numbers α and 2α are near marginal, while other modes are stable. Thus the nonlinear evolution in this model is dominated by the interaction of modes α and 2α. The slab geometry is adopted for simplicity and to be consistent with our previous linear calculations.[18,19] Since magnetic reconnection occurs only in a very thin layer, slab geometry provides a physical picture for understanding more complicated geometries, such as cylindrical and toroidal. To find the asymptotic states of the nonlinear interaction, the dissipative MHD equations are reduced to four amplitude equations, using center manifold reduction. The model which we will use is similar to the one used in Ref. 7; however, the breaking of reflection by the presence of shear flow allows the coefficients of the reduced equations to be complex. Thus the dynamics of the reduced amplitude equations are more complicated. Employing bifurcation analysis, various structures in addition to the oscillating island states are discovered. Also the roles of the new parameters introduced by flow (imaginary parts of the coefficients) are identified.

In Sec. II, the basic equations are described, and the linear results are briefly reviewed. Section III is devoted to the center manifold reduction, in light of the constraints of symmetry on the reduced equations. Solutions of the reduced equations are discussed in Sec. V. Finally Sec. VI contains discussion and a summary.

II. Basic Equations and Linear Problem

We start from the incompressible dissipative MHD equations:

$$\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} = -\frac{V_A^2}{\rho} \nabla P + (\nabla \times \vec{B}) \times \vec{B} + S_V^{-1} \nabla^2 \vec{V}$$

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \left(S_R^{-1} (\nabla \times \vec{B}) \right) + \nabla \times (\vec{V} \times \vec{B}) \tag{1}$$

$$\nabla \cdot \vec{V} = 0$$

$$\nabla \cdot \vec{B} = 0 \ ,$$

where all quantities are dimensionless. Space is scaled with a, the width of the current sheet; time with the Alfvén time $\tau_A = \frac{B}{\sqrt{4\pi\rho}}$, where B is a characteristic measure of the magnetic field; and velocity is scaled with $V_A \equiv \frac{a}{\tau_A}$. The quantities $S_R \equiv \frac{4\pi a^2}{\eta\tau_A} \gg 1, S_V \equiv \frac{a^2}{\nu\tau_A} \gg 1$, where η and ν are respectively the resistivity and the kinematic viscosity.

Assuming slab geometry and independence of the z coordinate, the magnetic and velocity fields can be represented as follows: $\vec{B} = \hat{z} \times \nabla\psi(x,y), \vec{V} = \hat{z} \times \nabla\phi(x,y)$. Thus Eq. (1) becomes

$$\begin{aligned} &\frac{\partial\Omega}{\partial t} + \vec{V}\cdot\nabla\Omega = \vec{B}\cdot\nabla j + S_V^{-1}\nabla_\perp^2\Omega \\ &\frac{\partial\psi}{\partial t} + \hat{z}\cdot(\nabla\phi \times \nabla\psi) = S_R^{-1} j - E_z \ , \end{aligned} \tag{2}$$

where Ω and j are respectively the vorticity and current in z direction, i.e. $\Omega = \nabla_\perp^2\phi \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi$, $j = \nabla_\perp^2\psi$; E_z is the external electrical field, which is applied to compensate the magnetic diffusion in the equilibrium state. We assume that the equilibrium state depends only on y, and magnetic diffusion dominates the vorticity diffusion, i.e. $\frac{S_V^{-1}}{S_R^{-1}} \ll 1$. In the equilibrium state,

$$S_R^{-1} j_0(y) = E_z \ .$$

Let $\psi = \psi_0(y) + \psi_1(x,y,t), \phi = \phi_0(y) + \phi_1(x,y,t)$, where the subscripts 0 and 1 denote the equilibrium and perturbation states, respectively. Equations (2) become

$$\frac{\partial}{\partial t}\begin{pmatrix} \nabla_\perp^2\phi_1 \\ \psi_1 \end{pmatrix} = \mathbf{L}\begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix} + N(\phi_1,\psi_1) \ , \tag{3}$$

where

$$\mathbf{L} = \begin{pmatrix} S_V^{-1}\nabla_\perp^4 + \phi_0'(y)\frac{\partial}{\partial x}\nabla_\perp^2 - \phi_0'''(y)\frac{\partial}{\partial x} & -\psi_0'\frac{\partial}{\partial x}\nabla_\perp^2 + \psi_0'''(y)\frac{\partial}{\partial x} \\ -\psi_0'(y)\frac{\partial}{\partial x} & S_R^{-1}\nabla_\perp^2 + \phi_0'\frac{\partial}{\partial x} \end{pmatrix} ,$$

$$N(\phi_1,\psi_1) = \hat{z}\cdot\begin{pmatrix} \nabla\psi_1 \times \nabla j_1 - \nabla\phi_1 \times \nabla\Omega_1 \\ -\nabla\phi_1 \times \nabla\psi_1 \end{pmatrix} .$$

Here prime denotes differentiation with respect to y. We assume that the magnetic null plane is at $y = 0$, i.e. $B_0(0) = -\psi_0'(0) = 0$.

Assuming periodic boundary conditions in the x-direction, the perturbed stream functions can be expanded as

$$\psi_1(x,y,t) = \sum_n \psi_{1n}(y,t)e^{in\alpha x} + \text{c.c.}$$

$$\phi_1(x,y,t) = \sum_n \phi_{1n}(y,t)e^{in\alpha x} + \text{c.c.} \tag{4}$$

where $\frac{\alpha = 2\pi}{L}$, and L is the period in the x-direction.

The linear tearing mode problem with shear flow has been studied before.[17,18,19,20,21] In the region away from the magnetic null plane (external ideal region), the magnetic field is frozen into the flow; hence the global flow can dramatically change the matching quantity $\Delta' \equiv \frac{1}{\psi_1}\left(\frac{\partial\psi_1(0^+)}{\partial y} - \frac{\partial\psi_1(0^-)}{\partial y}\right)$. In the region around the magnetic null plane, the tearing mode is very sensitive to the flow shear. If $|\gamma/\alpha B_0'(0)\varepsilon| < \left|\frac{V_0'(0)}{B_0'(0)}\right|$ where γ is the growth rate and ε is the resistive layer scalelength, then convection dominates the inertia term and the scaling of the tearing mode growth rate will change.[18] Hence near marginal stability, even small flow shear will cause a significant change in the linear problem.[17] It has also been shown that the stable tearing mode can be driven unstable[18,19,22] by strong shear flow ($V_0'(0)/B_0'(0) \sim \mathcal{O}(1)$) provided $V_0'(0)V_0''(0) - B_0'(0)B_0''(0) \neq 0$ and $S_R \ll S_V \ll 1$. In this case the condition of $\Delta' > 0$ for instability is removed; such destabilized tearing modes have been found numerically.[17]

The magnetic field and shear flow are characterized by parameters, such as their magnitudes and shear lengths. Near the parameter values where two modes with wavenumbers α and 2α are marginally stable (and all other modes are stable), the nonlinear evolution can be studied analytically. There are many situations where we can find such critical parameter values. One example is a piecewise continuous magnetic field with a separated double jet flow

$$B_0(y) = \begin{cases} 1, & y > 1 \\ y, & |y| < 1 \\ -1, & y < -1 \end{cases} ,$$

$$V_0(y) = \begin{cases} 0, & |y| > b \\ V_0, & b > |y| > 1 \\ 0, & |y| < 1 \end{cases} .$$

Here flow only exists in the external ideal region and the tearing mode is unstable only if $\Delta' > 0$. With the assumed profiles, Δ' is equal to zero at the wavenumber α_0 where

$$\frac{(1 - \alpha_0 \tanh(\alpha_0))}{\alpha_0} = (1 - V_0^2) \frac{V_0^2 + (2 - V_0^2) e^{2\alpha_0(b-1)}}{-V_0^2 + (2 - V_0^2) e^{2\alpha_0(b-1)}} .$$

Assuming $1 < V_0^2 < 2$, there exist two solutions for α_0, and by choosing appropriate values of V_0 and b, we are able to get $\Delta_1 = 0$ and $\Delta_2 = 0$, while $\Delta_n < 0$ for $n \geq 3$. Thus the above profiles describe the desired multiple instability. Another example can be constructed from a magnetic profile where all the tearing modes are stable, i.e. $\Delta'_n < 0$ for all n, by including a shear flow that drives the tearing mode unstable. This is the case treated in Persson and Bondeson's numerical simulations. Again by choosing the parameters $V'(0)/B'(0), V''(0)/B''(0)$, and S_V^{-1}/S_R^{-1} appropriately, the modes α and 2α can be driven simultaneously unstable while the other modes remain stable.

III. Center Manifold Reduction

Center manifold reduction and related theorems can be found in many references[10,23,24] and and we give only a brief description. For simplicity and clarity, let us first look at a finite-dimensional dynamical system

$$\dot{X} = AX + N(X, Y) , \tag{5}$$

$$\dot{Y} = BY + M(X, Y) , \tag{6}$$

where matrices A and B describe the marginally stable and stable linear modes, respectively, M and N denote the nonlinear terms and $M(0,0) = 0, N(0,0) = 0$. Thus $(X, Y) = (0,0)$ is an equilibrium state (fixed point). For the equations linearized about $(X, Y) = (0,0)$, there exists an invariant "center eigenspace" spanned by the eigenvectors of A, i.e. $(X, Y) = (X, 0)$.

When the nonlinear terms are included, the center manifold theorem states that there still exists an invariant subspace called the center manifold. The center manifold is tangent to the center eigenspace at $(X, Y) = (0, 0)$, as shown in Fig. 1, and has the same dimension as the center eigenspace. Thus the center manifold can be expressed as a "graph of a function"; i.e. as

$$(X, Y) = (X, h(X)) \ , \tag{7}$$

with

$$h(0) = 0, \ \frac{\partial h}{\partial X}(0) = 0 \ . \tag{8}$$

Also, if M and N are differentiable to order r, then h is differentiable to order r^{-1}. As mentioned earlier in the Introduction, the center manifold is locally attractive, and so for the purpose of finding the local time-asymptotic states, the system can be reduced to lower dimension, the dimension of the center manifold. The dynamics on the center manifold are expressed as

$$\dot{X} = AX + N\left(X, h(X)\right) \ . \tag{9}$$

It now remains to calculate $h(X)$, which is achieved by plugging Eq. (7) into Eq. (6). We have

$$\frac{\partial h}{\partial X}(X)(AX + N(X, h(X)) = Bh(X) + M\left(X, h(X)\right) \ . \tag{10}$$

In most cases Eq. (10) cannot be solved exactly for $h(X)$ (otherwise an exact solution of the original equations would be found). However, $h(X)$ can be approximated as a Taylor series near $(X, Y) = (0, 0)$ satisfying the conditions of Eq. (8). Usually only a few terms are needed to describe all of the local asymptotic states.

The technique of center manifold reduction can also be extended to partial differential equations. However the main center manifold theorem cannot be applied directly to our present problem, since the spectra of the modes α and 2α are not exactly on the imaginary axis, but this difference can be overcome by shifting parameters. Let Z_0 denote the distance of the parameters from their critical values, discussed in the last section, and expand the dynamical system by adding a new equation

$$\dot{Z}_0 = 0 . \tag{11}$$

Since the modes α and 2α are near marginal, Z_0 is very small. Taking the equilibrium state of the enlarged system as $\begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix} = 0$, $Z_0 = 0$, the spectrum of this new equilibrium with wavenumber α and 2α lies on the imaginary axis.

Similar to Eq. (7), we have in the center manifold

$$\begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix} (x,y,t) = \sum_{n=1,2} Z_n(t) \begin{pmatrix} \phi_{nc} \\ \psi_{nc} \end{pmatrix} (y) e^{in\alpha x} + \text{c.c.} + h(x,y,Z_n,\overline{Z}_n,Z_0) , \tag{12}$$

where $\begin{pmatrix} \phi_{nc} \\ \psi_{nc} \end{pmatrix}$ with $n = 1,2$ correspond respectively to the critical linear marginal eigenfunctions of modes α and 2α, Z_n are their amplitudes, $\overline{Z}_n$ is the complex conjugate of Z_n, and function h is subject to the following constraints

$$h(x,y,0,0,0) = 0, \frac{\partial h}{\partial Z_n}(x,y,0,0,0) = 0 ,$$

$$\frac{\partial h}{\partial \overline{Z}_n}(x,y,0,0,0) = 0 , \frac{\partial h}{\partial Z_0}(x,y,0,0,0) = 0 .$$

Plugging Eq. (12) into Eq. (3), results in the following reduced amplitude equations (see the appendix for details):

$$\dot{Z}_1 = f_1(Z_1,\overline{Z}_1,Z_2,\overline{Z}_2,Z_0)$$

$$\dot{Z}_2 = f_2(Z_1,\overline{Z}_1,Z_2,\overline{Z}_2,Z_0) \tag{13}$$

with $f_1(0) = 0, f_2(0) = 0$, $\frac{\partial f_i}{\partial Z_0}(0) = 0$, $\frac{\partial f_i}{\partial Z_n}(0) = 0$, $\frac{\partial f_i}{\partial \overline{Z}_n} = 0$ ($n = 1,2$ and $i = 1,2$). For the nonlinear evolution with $|Z|$ small, the functions f_i in Eqs. (13) can be Taylor expanded. One needs only expand to some finite order to unfold the new branches of solutions. However, to just third order, the number of terms is very large, and calculation of all of the coefficients laborious. Fortunately in the present model, many terms in the expansion will vanish due to the constraint imposed by the symmetry of the system. Thus the reduced amplitude equations are greatly simplified and can be discussed as to the possible kind of solutions, even without knowing the coefficients. As noted in the case without flow ($\phi_0 \equiv 0$) this model possesses $\mathcal{O}(2)$ symmetry,[7] and Eqs. (3) are "equivariant" under the following transformations:

$$\mathcal{T}_{x0}\begin{pmatrix}\phi_1\\ \psi_1\end{pmatrix}(x,y,t) = \begin{pmatrix}\phi_1\\ \psi_1\end{pmatrix}(x+x_0,y,t)\ ,$$

$$\mathcal{Z}\begin{pmatrix}\phi_1\\ \psi_1\end{pmatrix}(x,y,t) = \begin{pmatrix}-\phi_1\\ \psi_1\end{pmatrix}(-x,\tilde{y},t)\ ,$$

where $\mathcal{T}_{x0}$ arises because of periodic boundary conditions. In words, the symmetries imply that if $\begin{pmatrix}\phi_1\\ \psi_1\end{pmatrix}(x,y,t)$ is a solution of Eqs. (3), then so are $\mathcal{T}_{x0}\begin{pmatrix}\phi_1\\ \psi_1\end{pmatrix}(x,y,t)$ and $\mathcal{Z}\begin{pmatrix}\phi_1\\ \psi_1\end{pmatrix}(x,y,t)$. Inclusion of shear flow breaks the reflection symmetry, however the translation symmetry $\mathcal{T}_{x0}$ is preserved. Thus the reduced amplitude equations (13) are equivariant under translation in x which acts on the amplitudes (Z_1, Z_2) by:

$$\mathcal{T}_{x_0}(Z_1, Z_2) = (e^{i\alpha x_0} Z_1, e^{2i\alpha x_0} Z_2)$$

(c.f. (12)). The basic monomial invariants for the above operations are $|Z_1|^2$, $|Z_2|^2$, $\overline{Z}_1^2 Z_2$, $Z_1^2\overline{Z}_2$, thus the expansion of Eq. (13) must have the form:

$$\dot{Z}_1 = (\lambda_1 + i\omega_{1c})Z_1 + a_1\overline{Z}_1 Z_2 + b_1 Z_1|Z_1|^2 + c_1 Z_1\,|Z_2|^2 + 0(Z^4)$$

$$\dot{Z}_2 = (\lambda_2 + i\omega_{2c})Z_2 + a_2 Z_1^2 + b_2 Z_2|Z_1|^2 + c_2 Z_2|Z_2|^2 + 0(Z^4) \qquad (14)$$

where $\lambda_i + i\omega_{ic}$ $(i = 1, 2)$ are the linear eigenvalues of the near marginal modes, ω_{ic} are the eigenfrequencies at the critical parameters values $Z_0 = 0$, and $\lambda_i = \vartheta(Z_0)$. Formulas for the coefficients of Eqs. (14) in terms of the (adjoint) eigenfunctions of the linear problem are calculated in the Appendix. Breaking reflection symmetry allows the coefficients a, b and c

to be complex. Similar equations have been discussed in Ref. 25 for studying the interactions of two oscillators with 2:1 resonant frequencies. However, these authors mainly discussed the equations near a special degenerate parameter regime, where the pure mode solution (c.f. Sec. IV) has double zero eigenvalues. Here we are interested in more general parameter regimes.

Furthermore, the unessential nonlinear term $Z_1|Z_1|^2$ can be removed by a near identity $SO(2)$ invariant coordinate transformation $Z_1 \to Z_1, Z_2 \to Z_2 - \frac{b_1}{a_1}Z_1^2$. Equations (14) become

$$\dot{Z}_1 = (\lambda_1 + i\omega_{1c})Z_1 + a_1\overline{Z}_1 Z_2 + c_1 Z_1|Z_2|^2 + \mathcal{O}(Z^4) \tag{15}$$

$$\dot{Z}_2 = (\lambda_2 + i\omega_{2c})Z_2 + a_2 Z_1^2 + bZ_2|Z_1|^2 + c_2 Z_2|Z_2|^2 + 0(Z^4) \ , \tag{16}$$

where $b = b_2 + 2b_1$, and the small modification of the coefficient a_2 is neglected. Now let $a_1 = \rho_3 e^{i\theta_3}$, $a_2 = \rho_4 e^{i\theta_4}$, $b = \rho_b e^{i\theta_b}$, $c_1 = \rho_1 e^{i\theta_1}$, $c_2 = \rho_2 e^{i\theta_2}$ and assume $|a_1 a_2 c_2| \neq 0$. We can reduce the number of parameters in Eqs. (15) and (16) by doing the following rescaling

$$Z_1 \to \frac{\rho_3}{\rho_2}\sqrt{\frac{\rho_3}{\rho_4}}Z_1, \ Z_2 \to \frac{\rho_3}{\rho_2}e^{-i\theta_3}Z_2, \ t \to \frac{\rho_2}{\rho_3^2}t \ .$$

The rescaled equations are

$$\dot{Z}_1 = (\tilde{\lambda}_1 + \tilde{\omega}_{1c})Z_1 + \overline{Z}_1 Z_2 + \tilde{\rho}_1 e^{i\theta_1} Z_1|Z_2|^2 + \mathcal{O}(Z^4) \tag{17}$$

$$\dot{Z}_2 = (\tilde{\lambda}_2 + \tilde{\omega}_{2c})Z_2 + e^{i\theta} Z_1^2 + \tilde{\rho}_b e^{i\theta_b} Z_2|Z|^2 + e^{i\theta_2} Z_2|Z_2|^2 + 0(Z^4) \ , \tag{18}$$

where $\theta = \theta_3 + \theta_4$, $\tilde{\lambda}_1 = \frac{\rho_2}{\rho_3^2}\lambda_1$, $\tilde{\lambda}_2 = \frac{\rho_2}{\rho_3^2}\lambda_2$, $\tilde{\omega}_{1c} = \frac{\rho_2}{\rho_3^2}\omega_{1c}$, $\tilde{\omega}_{2c} = \frac{\rho_2}{\rho_3^2}\omega_{2c}$, $\tilde{\rho}_1 = \frac{\rho_1}{\rho_2}$, $\tilde{\rho}_b = \frac{\rho_3\rho_b}{\rho_2\rho_4}$. For convenience we drop the tilde in the following discussion. In the next section, bifurcation analysis is employed to find the possible time asymptotic states, i.e. branches of nonlinear solutions of the reduced amplitude equations that result from the linear instability.

IV. Solutions of the Reduced Equations

Bifurcation analysis is the natural technique for finding the possible time asymptotic states when parameters Z_0 are away from but still near their critical values. There are two types of bifurcations[23]: "local" and "global." Local bifurcation is recognized by a change in the stability of a solution. Depending on how the stability is changed, local bifurcations are again divided into two types: "steady state" and "Hopf bifurcations." If stability is changed because an eigenvalue transverses zero, the bifurcation is of the steady state type; if the stability is changed because eigenvalues are pure imaginary at criticality, the bifurcation is the Hopf type and new branches of periodic solutions are often found. As for the global bifurcation, its existence is not revealed by local analysis, and will not be considered here.

We are interested in the local bifurcation near the original equilibrium $(Z_1, Z_2, Z_0 = 0)$. This problem has been reduced to the amplitude equations (17) and (18) which may be rewritten in polar coordinates $Z_1 = r_1 e^{i\varphi_1}, Z_2 = r_2 e^{i\varphi_2}$ as

$$\dot{r}_1 = (\mu_1 + r_2 \cos\varphi + \rho_1 \cos\theta_1 r_2^2) r_1 \tag{19}$$

$$\dot{r}_2 = \mu_2 r_2 + \cos(\varphi - \theta) r_1^2 + \rho_b \cos\theta_b r_1^2 r_2 + \cos\theta_2 r_2^3 \tag{20}$$

$$\dot{\varphi} = \delta - (\sin(\varphi - \theta)\frac{r_1^2}{r_2} + 2\sin\varphi r_2) + \rho_b \sin\theta_b r_1^2 + d r_2^2 \tag{21}$$

where $\mu_i = \mathrm{Re}\,\lambda_i, \beta_i = \mathrm{Im}\,\lambda_i$ $(i = 1, 2)$, $\delta = \omega_{2c} + \beta_2 - 2(\omega_{1c} + \beta_1)$, $\varphi = \varphi_2 - 2\varphi_1$, and $d = \sin\theta_2 - 2\rho_1 \sin\theta_1$. Since the frequencies ω_{ic} arises mostly from Doppler shifting, and such Doppler shifts are cancelled in the combination $\omega_{2c} - 2\omega_{1c}$, δ is a small parameter. In the case without flow, $\delta = \sin\theta = d = \sin\theta_b = 0$, and hence these are the parameters introduced by flow. Note that what matters in the nonlinear evolution is the phase difference, not the individual phases of each mode. Thus the original amplitude equations (17) and (18) are reduced to the three independent ones of Eqs. (19)–(21). (This happens because of the translation symmetry.) The variation of the individual phases of each mode are governed by the equations

$$\dot{\varphi} = \omega_{1c} + \beta_1 + \sin\varphi r_2 + \rho_1 \sin\theta_1 r_2^2 \ ,$$

$$\dot{\varphi} = \omega_{2c} + \beta_2 - \sin(\varphi - \theta)\frac{r_1^2}{r_2} + \rho_b \sin\theta_b r_1^2 + \sin\theta_2 r_2^2 \ .$$

Small amplitude solutions of Eqs. (19)–(21) have the following magnetic flux function near the magnetic null line (for the constant-ψ tearing mode):

$$\psi \approx -\frac{1}{2}B_x'(0)y^2 + r_1\cos(\alpha x + \omega_1 t + \varphi_1) + r_2\cos(2(\alpha x + \omega_1 t + \varphi_1) + \varphi) \ , \qquad (22)$$

where $\omega_1 = \omega_{1c} + \beta_1 + r_2\sin\varphi + \rho_1\sin\theta_1 r_2^2$, and φ_1 is the initial phase. Note in the above expression we haven't take into account the rescalings of coefficients, but this will not change the qualitative physical picture. If $\omega_1 = 0$, it is a steady island state, while if $\omega_1 \neq 0$, it is a traveling island state. It is interesting to note that a pure mode solution with $r_1 = 0, r_2 \neq 0$ still solves the nonlinear interaction equations, even when higher order terms are included. This is due to the symmetry which has forced many nonlinear interaction terms to vanish. However, a pure mode of the form $r_2 = 0, r_1 \neq 0$ is not a solution. Contour plots of Eq. (22) with $\omega_1 t + \varphi_1 = 0$ and $\varphi = 0$ is given in Fig. 2. If the stability of the bifurcated solution changes, a secondary bifurcation can happen. However, a secondary steady state bifurcation is not of much interest, since it does not change the magnetic field structure. What is of interest is a secondary Hopf bifurcation. When this happens, Eq. (22) becomes

$$\psi \approx -\frac{1}{2}B_x'(0)y^2 + (r_{10} + r_{11}\cos(\omega_0 t))\cos(\alpha x + \omega_1 t + \varphi_1) +$$

$$(r_{20} + r_{21}\cos(\omega_0 t))\cos(2(\alpha x + \omega_1 t + \varphi_1) + \varphi_0 + \varphi_{11}\cos(\omega_0 t)) \ , \qquad (23)$$

where ω_0 is the Hopf bifurcation frequency. In this new magnetic field structure, the amplitude and phase differences between the two modes oscillate. If it is not far away from the secondary bifurcation, then $|\varphi_1| \ll 1$, and Eq. (23) is close to a modulated traveling wave state. This is the form of the oscillating island state observed in simulations by Persson and Bondeson. Below we will discuss the parameter domains for the pure mode and the mixed mode solutions with their secondary Hopf bifurcations. Since only the stable time asymptotic states are practically observable, we also discuss stability of the solutions.

A. Pure mode solution ($r_1 = 0, r_2 \neq 0$)

In this case, the amplitude equations decouple from the phase equation and Eqs. (19)–(21) become

$$\dot{r}_1 = 0$$

$$\dot{r}_2 = \mu_2 r_2 + \cos\theta_2 r_2^3 \tag{24}$$

$$\dot{\varphi}_2 = \omega_{2c} + \beta_2 + \sin\theta_2 r_2^2 \ ,$$

which have the solution

$$r_2^2 = -\frac{\mu_2}{\cos\theta_2}, \quad \text{if } \frac{\mu_2}{\cos\theta_2} < 0 \ . \tag{25}$$

This solution is a traveling wave state with phase velocity $(\omega_{2c} + \beta_2 + \sin\theta_2 r_2^2)/2\alpha = (\omega_{2c} + \beta_2 - \frac{\sin\theta_2}{\cos\theta_2}\mu_2)/2\alpha$, which differs from the steady state case when there is no flow. The contour plot of magnetic flux corresponding to the pure mode is shown in Fig. 2a. There are two magnetic islands in one period length. From Eqs. (19) and (20) the stability of this solution is determined by eigenvalues $-2\mu_2$ and $\mu_1 + \sqrt{-\frac{\mu_2}{\cos\theta_2}}\cos\varphi - \frac{\rho_1\cos\theta_1}{\cos\theta_2}\mu_2$. Since the phase difference φ is arbitrary, the pure mode is stable when $\mu_2 > 0$, and $\mu_1 + \sqrt{-\frac{\mu_2}{\cos\theta_2}} < 0$. A secondary bifurcation occurs when the stability changes. The first stability change occurs at $\mu_2 = 0$, which represents the initial bifurcation of the pure mode. The second eigenvalue comes from the perturbations of the mode α, thus the secondary bifurcated solution is a mixed mode solution, which will be discussed next.

B. Mixed Mode Solution ($r_1 r_2 \neq 0$)

Equations (19)–(21) yield

$$\cos\varphi r_2 + \rho_1 \cos\theta_1 r_2^2 = -\mu_1 \tag{26}$$

$$\sigma\cos(\varphi-\theta) r_2 + (\rho_b \cos\theta_b \sigma + \cos\theta_2) r_2^2 = -\mu_2 \tag{27}$$

$$-(\sigma\sin(\varphi-\theta) + 2\sin\varphi) r_2 + (\rho_b \sin\theta_b \sigma + d) r_2^2 = -\delta \ , \tag{28}$$

where $\sigma = \frac{r_1^2}{r_2^2}$ denotes the ratio of amplitudes of the two modes. Let us first look at the stability of the solutions, which is determined by the solutions of a third order polynomial

$$\lambda^3 - d_1\lambda^2 + d_2\lambda - d_3 = 0 \ , \tag{29}$$

where

$$d_1 = -2(\sigma\cos(\varphi-\theta) + \cos\varphi) r_2 + \vartheta(r_2^2) \ ,$$

$$d_2 = \left[(\sigma - 4\sin\varphi\sin(\varphi-\theta)) - \frac{4r_2}{\sigma}\cos\varphi\cos\theta_2\right] r_1^2 + \vartheta(r_1^2 r_2) \ ,$$

$$d_3 = 2(\sigma\cos\varphi + 2\cos(\varphi-\theta)) r_1^2 r_2 + \vartheta(r_1^2 r_2^2) + \vartheta(\sigma r_1^2 r_2^2) \ .$$

A stable solution requires that $d_1 < 0$, $d_2 > 0$, $d_3 < 0$. If the three eigenvalues are real, the above conditions are also sufficient. If two eigenvalues are complex, then $d_1 d_2 - d_3 < 0$ guarantees the stability. For $d_2 > 0$, and $d_1 d_2 - d_3 = 0$, there exist pure imaginary eigenvalues $\lambda = \pm i\sqrt{d_2}$. Thus a secondary Hopf bifurcation could occur along $d_1 d_2 - d_3 = 0$. Due to the exchange of stability principle,[26] the Hopf bifurcated solution is stable on the side $d_1 d_2 - d_3 > 0$, while unstable on the side $d_1 d_2 - d_3 < 0$. The side on which the Hopf bifurcated solution appears depends on the sign of the coefficients of higher order terms.

Equations (26)–(28) are still very difficult to solve directly, so we consider several special cases: $\theta = 0$, $\theta = \pi$, and $\theta = \pi/2$. For the case $\theta = 0$ and $\theta = \pi$, we discuss the difference made by flow introduced parameter δ. Note $\theta = \pi/2$ is only possible with flow.

(i) $\theta = 0$. In this case, Eqs. (26)–(28) imply

$$
\begin{aligned}
r_2 &= \frac{\sqrt{(2\mu_1+\mu_2)^2+\delta^2}}{\sigma+2}(1+\vartheta(r_2)) \\
\cos\varphi &= -\frac{2\mu_1+\mu_2}{\sqrt{(2\mu_1+\mu_2)^2+\delta^2}} + \vartheta(r_2) \qquad (30) \\
\sin\varphi &= \frac{\delta}{\sqrt{(2\mu_1+\mu_2)^2+\delta^2}} + \vartheta(r_2)\ .
\end{aligned}
$$

In the case without flow, the solution requires that phase differences of the two modes φ must be 0 or π. Here φ can be any value depending on the ratio $(2\mu_1+\mu_2)/\delta$. The coefficients in the stability eigenvalue Eq. (29) become

$$
\begin{aligned}
d_1 &= -2(\sigma+1)\cos\varphi r_2 + \vartheta(r_2^2)\ , \\
d_2 &= \left(\sigma - 4\sin^2\varphi - \frac{4r_2}{\sigma}\cos\varphi\cos\theta_2\right) r_1^2 + \vartheta(r_1^2 r_2)\ , \\
d_3 &= 2(\sigma+2)\cos\varphi r_1^2 r_2 + \vartheta(r_1^2 r_2^2) + \vartheta(\sigma r_1^2 r_2^2)\ .
\end{aligned}
$$

Thus a stable solution is possible only if $\cos\varphi \sim \vartheta(r_2)$, and either $\sigma > 4$ or $\sigma \ll 1$ with $-4 - \frac{r_2}{\sigma}\cos\varphi\cos\theta_2 > 0$. The secondary Hopf bifurcation is possible also only if the above conditions are satisfied. From Eqs. (30), $\cos\varphi \sim \vartheta(r_2)$ requires $\left|\frac{2\mu_1+\mu_2}{\delta}\right| \sim \vartheta(r_2) \ll 1$. Since δ is the parameter introduced by flow, there exists no stable solution and secondary Hopf bifurcation in the case $\theta = 0$ without flow. From Eq. (27), the amplitude ratio σ of the two modes is

$$\sigma = -\frac{\mu_2 + \cos\theta_2 r_2^2}{(\cos\varphi/r_2 + \rho_b\cos\theta_b)r_2^2} > 0\ .$$

When $\sigma \ll 1$, it is required that $\mu_2 + \cos\theta_2 r_2^2 \to 0$. Obviously this is a solution bifurcated from the pure mode solution along

$$-\frac{\mu_2}{\cos\theta_2} = \frac{(2\mu_1+\mu_2)^2+\delta^2}{4}\ .$$

When $\sigma \gg 1$, it is required that either $|\mu_2/r_2^2| \gg 1$ or $|\cos\varphi/r_2 + \rho_b\cos\theta_b| \ll 1$.

(ii) $\theta = \pi$. In this case, there exists a special amplitude ratio $\sigma = 2$, at which the phase difference φ can be any value even in the case without flow. We first consider the special case where $\sigma = 2$. Equations (26)–(28) yield

$$r_2^2 = -\frac{2\mu_1 + \mu_2}{2\rho_1 \cos\theta_1 + 2\rho_b \cos\theta_b + \cos\theta_2} = -\frac{\delta}{2\rho_b \sin\theta_b + d} > 0$$

$$\cos\varphi = -\frac{\mu_1}{r_2 + \vartheta}(r_2) = -\frac{\mu_2}{2r_2} + \vartheta(r_2) ,$$

and the coefficients of the stability eigenvalue equations become

$$d_1 = 2(\sigma - 1)\cos\varphi r_2 + \vartheta(r_2^2) ,$$

$$d_2 = \left(\sigma + 4\sin^2\varphi - \frac{4r_2}{\sigma}\cos\varphi\cos\theta_2\right) r_1^2 + \vartheta(r_1^2 r_2) ,$$

$$d_3 = 2(\sigma - 2)\cos\varphi r_1^2 r_2 + \vartheta(r_1^2 r_2^2) + \vartheta(\sigma r_1^2 r_2^2) .$$

Thus a stable solution with $\sigma = 2$ requires $\cos\varphi < 0$. A secondary bifurcation is possible only when $\cos\varphi \sim \vartheta(r_2)$, which requires

$$\frac{2\mu_1}{2\mu_1 + \mu_2} \sim \frac{\mu_2}{2\mu_1 + \mu_2} \sim \vartheta(1) .$$

Now for the case $\sigma \neq 2$, Eqs. (26)–(28) yield

$$r_2^2 = \frac{(2\mu_1 + \mu_2)^2 + \delta^2}{(\sigma - 2)^2}(1 + \vartheta(r_2))$$

$$\cos\varphi = \pm\frac{2\mu_1 + \mu_2}{\sqrt{(2\mu_1 + \mu_2)^2 + \delta^2}} + \vartheta(r_2) \tag{31}$$

$$\sin\varphi = \frac{\delta}{\sqrt{(2\mu_1 + \mu_2)^2 + \delta^2}} + \vartheta(r_2) .$$

If $|\frac{r_2}{\cos\varphi}| \ll 1$, the solution is stable only if either $\sigma > 2, \cos\varphi < 0$ or $\sigma < 1, \cos\varphi > 0$. Equations (26)–(28) yield in this case

$$r_2 = -\frac{\mu_1}{\cos\varphi}\left(1 + \vartheta\left(\frac{r_2}{\cos\varphi}\right)\right) ,$$

$$\sigma = -\frac{\mu_2}{\mu_1}\left(1 + \vartheta\left(\frac{r_2}{\cos\varphi}\right)\right) + \vartheta\left(\frac{r_2}{\cos\varphi}\right) .$$

The assumption $|\frac{r_2}{\cos\varphi}| \ll 1$ requires that $|\frac{\mu_1\delta^2}{(2\mu_1+\mu_2)^2}| \ll 1$. A secondary Hopf bifurcation can only occur for $\sigma < 1$, which implies from the above

$$-\frac{\mu_2}{\mu_1} < 1 \ .$$

Obviously flow introduced parameters are not important in this limit.

If $\cos\varphi \sim \vartheta(r_2)$, similar to the case discussed in $\theta = 0$, it is required that $\frac{2\mu_1+\mu_2}{\delta} \sim \vartheta(r_2)$.

(iii) $\theta = \pi/2$. This case is only possible with flow. The coefficients of the stability eigenvalue equations become

$$d_1 = -2(\sigma\sin\varphi + \cos\varphi)r_2 + \vartheta(r_2^2) \ ,$$

$$d_2 = \left(\sigma + 2\sin 2\varphi - \frac{4r_2}{\sigma}\cos\varphi\cos\theta_2\right) r_1^2 + \vartheta(r_1^2 r_2) \ ,$$

$$d_3 = 2(\sigma\cos(\varphi + 2\sin\varphi)r_1^2 r_2 + \vartheta(r_1^2 r_2^2) + \vartheta(\sigma r_1^2 r_2^2) \ .$$

When $\sigma > \sqrt{2}$, the solution is stable only if $\sin\varphi > 0$, $\cos\varphi < 0$, $-\sigma/2 < \tan\varphi < -1/\sigma$, and $\sigma + 2\sin 2\varphi > 0$. When $\sigma < \sqrt{2}$, the solution is stable only if $\sin\varphi < 0$, $\cos\varphi > 0$, $-\sigma/2 > \tan\varphi > -1/\sigma$, and $\sigma + 2\sin 2\varphi - \frac{4r_2}{\sigma}\cos\varphi\cos\theta_2 > 0$.

Assuming $\sigma \gg 1$, Eqs. (26)–(28) give

$$\begin{aligned} r_2^2 &\approx \frac{\sqrt{(2\mu_1+\mu_2)^2+\delta^2}}{\sigma} \\ \cos\varphi &\approx -\frac{\delta}{\sqrt{(2\mu_1+\mu_2)^2+\delta^2}} \\ \sin\varphi &\approx -\frac{\mu_2}{\sqrt{(2\mu_1+\mu_2)^2+\delta^2}} \ . \end{aligned} \tag{32}$$

Equations (32) imply that $|\frac{\mu_1}{\mu_2}| \ll 1$, or $\left|\frac{\mu_1}{\delta}\right| \ll 1$. A secondary Hopf bifurcation occurs when $|\tan\varphi| \sim 1/\sigma \ll 1$, which implies that $|\frac{\mu_2}{\delta}| \ll 1$.

From the above calculations, we see that flow can significantly affect the nonlinear evolution of tearing modes. In the case without flow, a secondary Hopf bifurcation can only occur with $\theta = \pi, \sigma < 1$ or $\sigma = 2$. Thus shear flow plays an important role in driving the oscillating islands with $(\sigma \gg 1)$ in Persson and Bondeson's simulations.

V. Summary and Discussion

The nonlinear evolution of plasmas can saturate in time asymptotic states, or a transition to turbulence may occur. Generally the governing partial differential equations are analytically intractable and so we are unable to predict these asymptotic states. However, in recent years studies of nonlinear finite dimensional systems have been successful. Very complicated behavior, even chaotic states, have been found in finite dimensional systems. Since finite dimensional systems appear to possess solutions as complicated as those expected for the plasma evolution, it is natural to attempt to model the dynamics of plasmas by some finite system. This is also reasonable physically, since in many situations only a finite number of degrees-of-freedom are excited. For example, magnetic island coalescence[27] can be modeled by the interaction of two modes α and 2α. Thus the model equations are the same as Eqs. (14) with the restriction that the coefficients be real. Given the pure mode state, i.e. that there are two magnetic islands in one period length, the analysis in part A of Sec. III tell us that this pure mode state is stable only if $\mu_2 > 0$, $\mu_1 + \sqrt{-\frac{\mu_2}{\cos\theta_2}} < 0$. For the magnetic profiles chosen in Ref. 27, $\mu_1 > 0$ if $\mu_2 > 0$. Thus the given pure mode is unstable and will evolve to a mixed mode state, i.e. two islands in one period length will coalesce. In another example, Parker *et al.*[28] studied the nonlinear evolution of tearing modes without flow, using the period length as the bifurcation parameter. When the period length is short, only one tearing mode is excited, then the finite time asymptotic state is the usual saturation state. When the period length becomes longer and longer, more and more modes will be excited. This can be modeled as the interaction of two, three or more modes. The symmetry $(\mathcal{O}(2))$ will limit the nonlinear terms in the model amplitude equations, and enable us to discuss the solution in general terms. Even though the above suggested reduced models are not rigorously justified, they give some qualitative insight into the problem. For the interaction of near marginal modes, the model can be justified by using small amplitude or center manifold reduction. Strictly speaking these reductions are valid only close to the original marginal equilibrium, however very often results are valid well away from marginality.

In the present paper we have studied the interactions of two near marginal tearing modes with wavenumbers α and 2α, in the presence of shear flow. Employing the center manifold reduction method, the resistive MHD equations were reduced to amplitude equations. The model which we used is similar to the one in Ref. 7, however the presence of shear flow in our problem breaks the reflection symmetry, and allows the coefficients of the reduced equations to be complex. The most important parameters introduced by shear flow are δ and $\sin\theta \neq 0$. The bifurcation analysis was used to find possible time asymptotic states in different parameter regimes. Various states such as traveling and oscillating magnetic islands were found, and their observable parameter domains were also discussed. It was shown that shear flow plays an important role in driving this oscillating island state with $\sigma \gg 1$, i.e. the mode α dominates the mode 2α.

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Appendix — Calculation of Coefficients

Let $L = L_c + \Delta L$ where L_c corresponds to the linear operator at criticality, and

$$\Delta L = \left.\frac{\partial L}{\partial Z_0}\right|_{Z_0=0} Z_0 \ .$$

Here for convenience we denote $\overline{Z}_1$ and $\overline{Z}_2$ by Z_3 and Z_4, respectively. The function $h(x,y)$ in Eq. (12) is expanded in powers of the amplitudes

$$h(x,y) = \sum_{m,n=0,4}^{4} Z_m(t) Z_n(t) \begin{pmatrix} \phi_{mn} \\ \psi_{mn} \end{pmatrix} + \sum_{m,n,p=0,4}^{4} Z_m(t) Z_n(t) Z_p(t) \begin{pmatrix} \phi_{mnp} \\ \psi_{mnp} \end{pmatrix} + \cdots \ .$$

Inserting Eqs. (12) into Eqs. (3), and equating terms of order $\vartheta(|Z|)$ yields the linear problem

$$L_c \begin{pmatrix} \phi_{nc} \\ \psi_{nc} \end{pmatrix} - i\omega_{nc} \begin{pmatrix} \nabla_\perp^2 \phi_{nc} \\ \psi_{nc} \end{pmatrix} = 0 \ . \tag{A1}$$

The corresponding adjoint problem is

$$L_c^\perp \begin{pmatrix} \overline{\phi}_{nc}^\perp \\ \overline{\psi}_{nc}^\perp \end{pmatrix} - i\omega_{nc} \begin{pmatrix} \nabla_\perp^2 \overline{\phi}_{nc}^\perp \\ \overline{\psi}_{nc}^\perp \end{pmatrix} = 0 \ , \tag{A2}$$

with

$$L_c^\perp = \begin{pmatrix} S_V^{-1}\nabla_\perp^4 - \phi_0' \nabla_\perp^2 \frac{\partial}{\partial x} - 2\phi_0'' \frac{\partial}{\partial y}\frac{\partial}{\partial x} & \psi_0' \frac{\partial}{\partial x} \\ \psi_0' \nabla_\perp^2 \frac{\partial}{\partial x} + 2\psi_0'' \frac{\partial}{\partial y}\frac{\partial}{\partial x} & S_R^{-1}\nabla_\perp^2 - \phi_0' \frac{\partial}{\partial x} \end{pmatrix} .$$

The normalization is defined as

$$\left(\phi_{mc}^\perp, \nabla_\perp^2 \phi_{nc}\right) + \left(\psi_{mc}^\perp, \psi_{nc}\right) = \iint \left(\overline{\phi}_{mc}^\perp \nabla_\perp^2 \phi_{nc} + \overline{\psi}_{mc}^\perp \psi_{nc}\right) dx\, dy = \delta_{mn} \ .$$

To order $\vartheta(|Z|^2)$ we obtain

$$\sum_{0\le m\le n\le 4} Z_m Z_n \left[L_c \begin{pmatrix} \phi_{mn} \\ \psi_{mn} \end{pmatrix} - i(\omega_m + \omega_n) \begin{pmatrix} \nabla_\perp^2 \phi_{mn} \\ \psi_{mn} \end{pmatrix} \right] = \begin{pmatrix} \phi_{inh} \\ \psi_{inh} \end{pmatrix} \tag{A3}$$

with

$$\phi_{inh} = \left[\left(\frac{\lambda_1}{Z_0} Z_0 Z_1 + a_1 \overline{Z}_1 Z_2 \right) \nabla_\perp^2 \phi_{1c} + \left(\frac{\lambda_2}{Z_0} Z_0 Z_2 + a_2 Z_1^2 \right) \nabla_\perp^2 \phi_{2c} + \text{c.c.} \right]$$

$$+ \sum_{1\le m\le n\le 4} Z_m Z_n \frac{1}{1+\delta_{mn}} \left[\frac{\partial \phi_{mc}}{\partial x} \frac{\partial \Omega_{nc}}{\partial y} - \frac{\partial \phi_{mc}}{\partial y} \frac{\partial \Omega_{nc}}{\partial x} - \frac{\partial \psi_{mc}}{\partial x} \frac{\partial j_{nc}}{\partial y} \right.$$

$$+\frac{\partial\psi_{mc}}{\partial y}\frac{\partial j_{nc}}{\partial x}+\text{ terms interchanging } m \text{ and } n\Bigg]-\sum_{m=1}^{4}\frac{\partial L}{\partial Z_0}Z_m Z_0\phi_{mc}\ ,$$

$$\psi_{inh}=\left[\left(\frac{\lambda_1}{Z_0}Z_0Z_1+a_1\overline{Z}_1Z_2\right)\psi_{1c}+\left(\frac{\lambda_2}{Z_0}Z_0Z_2+a_2Z_1^2\right)\psi_{2c}+\text{c.c.}\right]$$

$$+\sum_{1\le m\le n\le 4}Z_mZ_n\frac{1}{1+\delta_{mn}}\left[\frac{\partial\phi_{mc}}{\partial x}\frac{\partial\psi_{nc}}{\partial y}-\frac{\partial\phi_{mc}}{\partial y}\frac{\partial\psi_{nc}}{\partial x}\right.$$

$$\left.+\text{ terms interchanging } m \text{ and } n\right]-\sum_{m=1}^{4}\frac{\partial L}{\partial Z_0}Z_0\psi_{mc}\ .$$

Considering, respectively, terms with Z_0Z_1 and Z_0Z_2, one can obtain the following through the Fredholm solvability condition:

$$\frac{\lambda_1}{Z_0}=\iint\left(\overline{\phi}_{1c}^{\perp}\frac{\partial L}{\partial Z_0}\phi_{1c}+\overline{\psi}_{1c}^{\perp}\frac{\partial L}{\partial Z_0}\psi_{1c}\right)\,dx\,dy$$

$$\frac{\lambda_2}{Z_0}=\iint\left(\overline{\phi}_{2c}^{\perp}\frac{\partial L}{\partial Z_0}\phi_{2c}+\overline{\psi}_{2c}^{\perp}\frac{\partial L}{\partial Z_0}\psi_{2c}\right)\,dx\,dy\ .$$

Actually if we had already solved the linear problem, λ_1 and λ_2 could be written down immediately. For terms with $\overline{Z}_1Z_2$, rewrite Eqs. (A3) as

$$\overline{Z}_1Z_2\left[L_c\begin{pmatrix}\phi_{32}\\ \psi_{32}\end{pmatrix}-i\omega_1\begin{pmatrix}\nabla_\perp^2\phi_{32}\\ \psi_{32}\end{pmatrix}\right]=\left[i(\omega_2-2\omega_1)\begin{pmatrix}\nabla_\perp^2\phi_{32}\\ \psi_{32}\end{pmatrix}\right.$$

$$\left.+\,a_1\begin{pmatrix}\nabla_\perp^2\phi_{1c}\\ \psi_{1c}\end{pmatrix}+\cdots\right]\overline{Z}_1Z_2\ .$$

In the linear problem, the eigenfrequencies arises mostly from Doppler shifting, so $\omega=\omega_2-2\omega_1$ is a small quantity. Since $\begin{pmatrix}\phi_{32}\\ \psi_{32}\end{pmatrix}+\begin{pmatrix}\phi_{1c}\\ \psi_{1c}\end{pmatrix}$ is also a solution of the above equations, the gauge is defined by $(\phi_{1c}^{\perp},\nabla_\perp^2\phi_{32})+(\psi_{1c}^{\perp},\psi_{32})=0$. Applying the Fredholm solvability condition to the above equation yields

$$a_1=\iint\left[\overline{\phi}_{1c}^{\perp}\left(-\frac{\partial\phi_{1c}}{\partial x}\frac{\partial\nabla_\perp^2\phi_{2c}}{\partial y}-\frac{\partial\phi_{2c}}{\partial x}\frac{\partial\nabla_\perp^2\phi_{1c}}{\partial y}+\frac{\partial\phi_{1c}}{\partial y}\frac{\partial\nabla_\perp^2\phi_{2c}}{\partial x}+\frac{\partial\phi_{2c}}{\partial y}\frac{\partial\nabla_\perp^2\phi_{1c}}{\partial x}\right.\right.$$

$$\left.+\frac{\partial\psi_{1c}}{\partial x}\frac{\partial\nabla_\perp^2\psi_{2c}}{\partial y}+\frac{\partial\psi_{2c}}{\partial x}\frac{\partial\nabla_\perp^2\psi_{1c}}{\partial y}+\frac{\partial\psi_{1c}}{\partial y}\frac{\partial\nabla_\perp^2\psi_{2c}}{\partial x}+\frac{\partial\psi_{2c}}{\partial y}\frac{\partial\nabla_\perp^2\psi_{1c}}{\partial x}\right)+\overline{\psi}_{1c}^{\perp}$$

$$\left(-\frac{\partial\phi_{1c}}{\partial x}\frac{\partial\psi_{2c}}{\partial y} + \frac{\partial\phi_{1c}}{\partial y}\frac{\partial\psi_{2c}}{\partial x} - \frac{\partial\phi_{2c}}{\partial x}\frac{\partial\psi_{1c}}{\partial y} + \frac{\partial\phi_{2c}}{\partial y}\frac{\partial\psi_{1c}}{\partial x} \right)\Bigg] \, dx\, dy + \vartheta(\omega) \; . \qquad \text{(A4)}$$

Similarly, we obtain

$$a_2 = \iint \Bigg[\overline{\phi}_{2c}^{\perp} \left(\frac{\partial\phi_{1c}}{\partial x}\frac{\partial\nabla_\perp^2\phi_{1c}}{\partial y} - \frac{\partial\phi_{1c}}{\partial y}\frac{\partial\nabla_\perp^2\phi_{1c}}{\partial x} - \frac{\partial\psi_{1c}}{\partial x}\frac{\partial\nabla_\perp^2\psi_{1c}}{\partial y} + \right.$$

$$\left. \frac{\partial\psi_{1c}}{\partial y}\frac{\partial\nabla_\perp^2\psi_{1c}}{\partial x} \right) + \overline{\psi}_{2c}^{\perp} \left(\frac{\partial\phi_{1c}}{\partial x}\frac{\partial\psi_{1c}}{\partial y} - \frac{\partial\phi_{1c}}{\partial y}\frac{\partial\psi_{1c}}{\partial x} \right) \Bigg) \, dx\, dy + \vartheta(\omega) \; . \qquad \text{(A5)}$$

So a_1 and a_2 are determined by linear marginal eigenfunctions to order $\vartheta(\omega)$.

Continuing the procedure, coefficients of order $\vartheta(|Z|^3)$ are calculated:

$$b_1 = \iint \Bigg[\overline{\phi}_{1c}^{\perp} \left(\frac{\partial\phi_{11}}{\partial x}\frac{\partial\Omega_{3c}}{\partial y} - \frac{\partial\phi_{11}}{\partial y}\frac{\partial\Omega_{3c}}{\partial x} - \frac{\partial\psi_{11}}{\partial x}\frac{\partial j_{3c}}{\partial y} + \frac{\partial\psi_{11}}{\partial y}\frac{\partial j_{3c}}{\partial x} \right.$$

$$\left. + \frac{\partial\phi_{13}}{\partial x}\frac{\partial\Omega_{1c}}{\partial y} - \frac{\partial\phi_{13}}{\partial y}\frac{\partial\Omega_{1c}}{\partial x} - \frac{\partial\psi_{13}}{\partial x}\frac{\partial j_{1c}}{\partial y} + \frac{\partial\psi_{13}}{\partial y}\frac{\partial j_{1c}}{\partial x} \right)$$

$$+ \overline{\psi}_{1c}^{\perp} \left(\frac{\partial\phi_{11}}{\partial x}\frac{\partial\psi_{3c}}{\partial y} - \frac{\partial\phi_{11}}{\partial y}\frac{\partial\psi_{3c}}{\partial x} + \frac{\partial\phi_{13}}{\partial x}\frac{\partial\psi_{1c}}{\partial y} - \frac{\partial\phi_{13}}{\partial y}\frac{\partial\psi_{1c}}{\partial x} \right) \Bigg] \, dx\, dy \qquad \text{(A6)}$$

$$b_2 = \iint \Bigg[\overline{\phi}_{2c}^{\perp} \left(\frac{\partial\phi_{12}}{\partial x}\frac{\partial\Omega_{3c}}{\partial y} - \frac{\partial\phi_{12}}{\partial y}\frac{\partial\Omega_{3c}}{\partial x} - \frac{\partial\psi_{12}}{\partial x}\frac{\partial j_{3c}}{\partial y} + \frac{\partial\psi_{12}}{\partial y}\frac{\partial j_{3c}}{\partial x} \right.$$

$$+ \frac{\partial\phi_{23}}{\partial x}\frac{\partial\Omega_{1c}}{\partial y} - \frac{\partial\phi_{23}}{\partial y}\frac{\partial\Omega_{1c}}{\partial x} - \frac{\partial\psi_{23}}{\partial x}\frac{\partial j_{1c}}{\partial y} + \frac{\partial\psi_{23}}{\partial y}\frac{\partial j_{1c}}{\partial x}$$

$$\left. + \frac{\partial\phi_{13}}{\partial x}\frac{\partial\Omega_{2c}}{\partial y} - \frac{\partial\phi_{13}}{\partial y}\frac{\partial\Omega_{2c}}{\partial x} - \frac{\partial\psi_{13}}{\partial x}\frac{\partial j_{2c}}{\partial y} + \frac{\partial\psi_{13}}{\partial y}\frac{\partial j_{2c}}{\partial x} \right)$$

$$+ \overline{\psi}_{2c}^{\perp} \left(\frac{\partial\phi_{12}}{\partial x}\frac{\partial\psi_{3c}}{\partial y} - \frac{\partial\phi_{12}}{\partial y}\frac{\partial\psi_{3c}}{\partial x} + \frac{\partial\phi_{23}}{\partial x}\frac{\partial\psi_{1c}}{\partial y} - \frac{\partial\phi_{23}}{\partial y}\frac{\partial\psi_{1c}}{\partial x} \right.$$

$$\left. + \frac{\partial\phi_{13}}{\partial x}\frac{\partial\psi_{2c}}{\partial y} - \frac{\partial\phi_{13}}{\partial y}\frac{\partial\psi_{2c}}{\partial x} \right) \Bigg] \, dx\, dy \qquad \text{(A7)}$$

$$c_1 = \iint \Bigg[\overline{\phi}_{1c}^{\perp} \left(\frac{\partial\phi_{12}}{\partial x}\frac{\partial\Omega_{4c}}{\partial y} - \frac{\partial\phi_{12}}{\partial y}\frac{\partial\Omega_{4c}}{\partial x} - \frac{\partial\psi_{12}}{\partial x}\frac{\partial j_{4c}}{\partial y} + \frac{\partial\psi_{12}}{\partial y}\frac{\partial j_{4c}}{\partial x} \right.$$

$$+\frac{\partial \phi_{24}}{\partial x}\frac{\partial \Omega_{1c}}{\partial y} - \frac{\partial \phi_{24}}{\partial y}\frac{\partial \Omega_{1c}}{\partial x} - \frac{\partial \psi_{24}}{\partial x}\frac{\partial j_{1c}}{\partial y} + \frac{\partial \psi_{24}}{\partial y}\frac{\partial j_{1c}}{\partial x}$$

$$+\frac{\partial \phi_{14}}{\partial x}\frac{\partial \Omega_{2c}}{\partial y} - \frac{\partial \phi_{14}}{\partial y}\frac{\partial \Omega_{2c}}{\partial x} - \frac{\partial \psi_{14}}{\partial x}\frac{\partial j_{2c}}{\partial y} + \frac{\partial \psi_{14}}{\partial y}\frac{\partial j_{2c}}{\partial x}\Bigg)$$

$$+\overline{\psi}_{1c}^{\perp}\Bigg(\frac{\partial \phi_{12}}{\partial x}\frac{\partial \psi_{4c}}{\partial y} - \frac{\partial \phi_{12}}{\partial y}\frac{\partial \psi_{4c}}{\partial x} + \frac{\partial \phi_{24}}{\partial x}\frac{\partial \psi_{1c}}{\partial y} - \frac{\partial \phi_{24}}{\partial y}\frac{\partial \psi_{1c}}{\partial x}$$

$$+\frac{\partial \phi_{14}}{\partial x}\frac{\partial \psi_{2c}}{\partial y} - \frac{\partial \phi_{14}}{\partial y}\frac{\partial \psi_{2c}}{\partial x}\Bigg)\Bigg]\, dx\, dy \qquad \text{(A8)}$$

$$c_2 = \iint \Bigg[\overline{\phi}_{2c}^{\perp}\Bigg(\frac{\partial \phi_{22}}{\partial x}\frac{\partial \Omega_{4c}}{\partial y} - \frac{\partial \phi_{22}}{\partial y}\frac{\partial \Omega_{4c}}{\partial x} - \frac{\partial \psi_{22}}{\partial x}\frac{\partial j_{4c}}{\partial y} + \frac{\partial \psi_{22}}{\partial y}\frac{\partial j_{4c}}{\partial x}$$

$$+\frac{\partial \phi_{24}}{\partial x}\frac{\partial \Omega_{2c}}{\partial y} - \frac{\partial \phi_{24}}{\partial y}\frac{\partial \Omega_{2c}}{\partial x} - \frac{\partial \psi_{24}}{\partial x}\frac{\partial j_{2c}}{\partial y} + \frac{\partial \psi_{24}}{\partial y}\frac{\partial j_{2c}}{\partial x}\Bigg)$$

$$+\overline{\psi}_{1c}^{\perp}\Bigg(\frac{\partial \phi_{22}}{\partial x}\frac{\partial \psi_{4c}}{\partial y} - \frac{\partial \phi_{22}}{\partial y}\frac{\partial \psi_{4c}}{\partial x} + \frac{\partial \phi_{24}}{\partial x}\frac{\partial \psi_{2c}}{\partial y} - \frac{\partial \phi_{24}}{\partial y}\frac{\partial \psi_{2c}}{\partial x}\Bigg)\Bigg]\, dx\, dy\ . \qquad \text{(A9)}$$

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Figure Captions

1. Depiction of the center manifold.

2. Contour plots of the magnetic flux. (a) $r_1 = 0$, $r_2 = 0.02$, (b) $r_1 = 0.02$, $r_2 = 0.02$, (c) $r_1 = 0.02$, $r_2 = 0.01$.

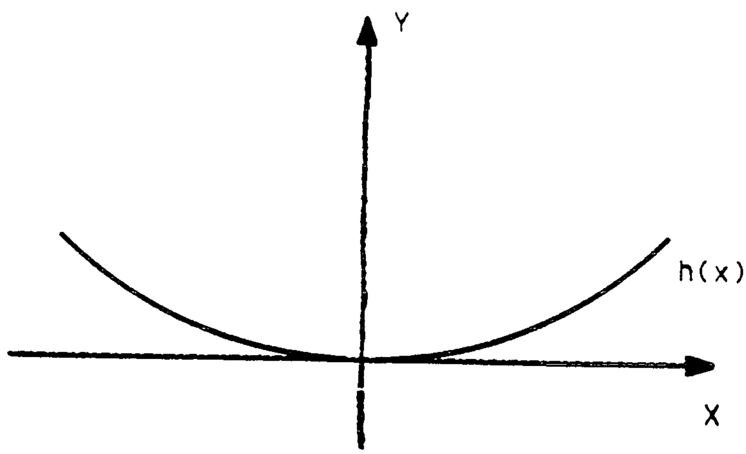

Fig. 1

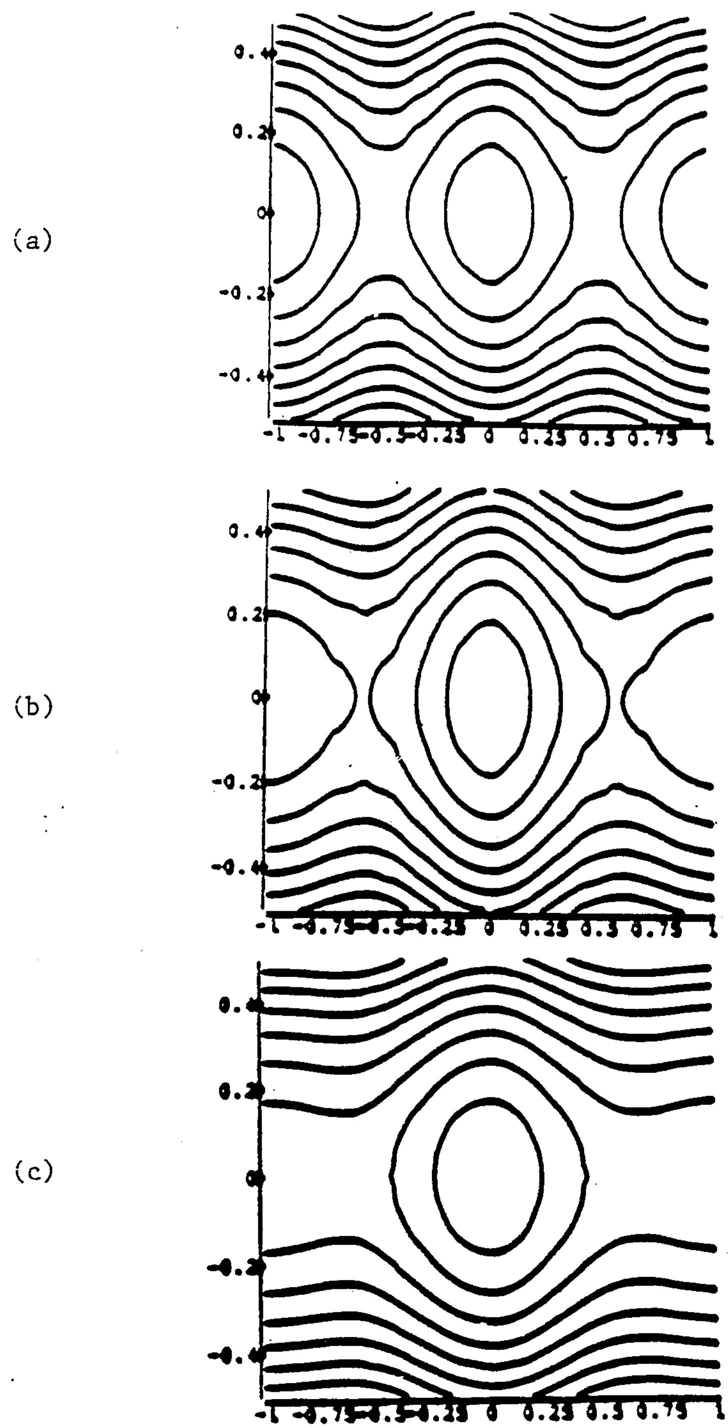

Fig. 2