INSTITUTE FOR FUSION STUDIES

DOE/ET-53088-509 IFSR #509

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July 1991

THE UNIVERSITY OF TEXAS

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(Received July 1991)

The rotational transform associated with the magnetic surfaces of a toroidal magnetic field with a non-planar axis is an example of the angle anholonomy recently much discussed in quantum and classical dynamics (the Berry phase and Hannay angle). The same anholonomic angle appears in the phase of a charged particle spiraling around its guiding center in a strong magnetic field. This accounts for a contribution to the longitudinal invariant, associated with the guiding-center motion, which is different for guiding-center orbits that circulate in opposite directions and is absent for orbits that are reflected between mirrors.

I. INTRODUCTION

There has been much interest, for both quantum and classical dynamics, in the so-called topological, geometric, or anholonomic-angle (the Berry phase [1,2] and Hannay angle [1,3]). The simplest classical example of this [3] arises when the external parameters $\mathbf{X}(t)$ of a Hamiltonian system $H(p,q;\mathbf{X})$ are carried slowly around a closed cycle. If the "instantaneous" action-angle variables are $I(p,q;\mathbf{X})$ and $\theta(p,q;\mathbf{X})$, then I is an adiabatic invariant and does not change during the cycle. The change in the conjugate angle θ is given by

$$\Delta\theta = \oint \frac{\partial H}{\partial I}\, dt + \oint \left\langle \frac{\partial \theta}{\partial \mathbf{X}} \right\rangle \cdot d\mathbf{X} \ , \tag{1}$$

where $\langle \cdot \rangle$ denotes average over θ at fixed $\mathbf{X}$. The first term is simply the integral of the local rate of change and may be termed the "dynamic" angle. The second is the geometric angle (Hannay's angle [3]); it is an anholonomy that depends on the path $\mathbf{X}(t)$ in the external parameter space. (Note that the coordinate θ may be chosen somewhat arbitrarily at each $\mathbf{X}$; consequently the angle $\Delta\theta$ is well defined only for a cyclic change in which $\mathbf{X}$ returns to its initial value.)

If the cycle of parameters is not carried out slowly, but the motion remains on some invariant surface I, then equation (1) is replaced by

$$\Delta\theta = \oint \left\langle \frac{\partial H}{\partial I} \right\rangle + \oint \left\langle \frac{\partial \theta}{\partial \mathbf{X}} \right\rangle \cdot d\mathbf{X} \tag{2}$$

and interpreted as the average of $\Delta\theta$ for particles distributed over the invariant surface (the Aharonov-Anandan angle [4]).

Much of the interest in the geometric angle, or phase, concerns its mathematical attributes in an abstract space, or its gauge invariance and group structure. It is worth noting, therefore, that a practical application of the geometric angle has been exploited by plasma physicists for many years — in the rotational transform of the magnetic field lines in the "figure-eight" stellarator introduced by Spitzer in 1951 [5].

Apart from its interest as an anticipation of the geometric phase (for others see Berry [6]), this problem of magnetic field lines in a torus with spatial (i.e., nonplanar) axis exhibits the anholonomy arising from parallel transport of a vector around a twisted axis in its most basic form. This anholonomy is given by the purely geometric expression $i_g = \oint \tau ds$, where s is the path length along the magnetic axis and τ is the torsion of the axis.

The same geometric angle i_g also appears in the guiding-center motion of a charged particle in a magnetic field — specifically as part of the "gyrophase" (i.e., the phase of the particle in its gyration around its guiding center), which has been studied extensively by Littlejohn [7,8]. This geometric increment in gyrophase is undetectable in the face of the much larger dynamic increment $\oint \omega_c \, dt$ (where ω_c is the gyrofrequency) but, as we point out below, it nevertheless has an effect on the dynamics of the guiding-center motion.

II. ROTATIONAL TRANSFORM

Here we follow closely Solov'ev and Shafranov [9]. Consider a magnetic field that generates magnetic surfaces surrounding a magnetic axis $\mathbf{X}(s)$. We introduce local coordinates ρ, θ, s, where ρ, θ are polar coordinates in the plane orthogonal to the local direction $\mathbf{e}_1 = d\mathbf{X}/ds$ of the magnetic axis. As with the Hamiltonian angle variable, the angle θ can be measured arbitrarily at different points around the axis [8,10], but we choose to measure it from the direction of the principal normal $\mathbf{e}_2 = (1/k)(d\mathbf{e}_1/ds)$, where $k \equiv |d\mathbf{e}_1/ds|$ is the curvature of the axis. Then the binormal is $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$ and the torsion of the axis is $\tau \equiv -\mathbf{e}_2 \cdot (d\mathbf{e}_3/ds) = \mathbf{e}_3 \cdot (d\mathbf{e}_2/ds)$. The metric is

$$dl^2 = d\rho^2 + \rho^2(d\theta + \tau ds)^2 + (1 - k\rho\cos\theta)^2 ds^2 \ . \tag{3}$$

With these definitions $\nabla\theta$ is not orthogonal to ∇s. The orthogonal direction is

$$\nabla\omega = \nabla\theta + \tau\nabla s \ , \tag{4}$$

so, using the field-line equation

$$\rho \frac{d\omega}{B_\omega} = (1 - k\rho\cos\theta)\frac{ds}{B_s} , \tag{5}$$

we may write

$$\frac{d\theta}{ds} = \frac{B_\omega(1 - k\rho\cos\theta)}{\rho B_s} - \tau . \tag{6}$$

The rotational transform is the weighted average of $d\theta/ds$, given by

$$\iota \equiv \oint \left\langle B_s \frac{d\theta}{d_s} \right\rangle \frac{ds}{\langle B_s \rangle} \tag{7}$$

so that

$$\iota = \oint \left\langle \frac{B_\omega(1 - k\rho\cos\theta)}{\rho} \right\rangle \frac{ds}{\langle B_s \rangle} - \oint \tau \, ds \quad \equiv \iota_m + \iota_g \tag{8}$$

where $\langle\cdot\rangle$ here denotes the flux-surface average defined in Ref. [9].

The first term in equation (8), ι_m, represents the "dynamic" (or in this case "magnetic") angle depending on $\mathbf{B}$. The second is the geometric angle, ι_g, depending *only* on the path taken by the magnetic axis (specifically its torsion: there is no geometric contribution for a planar axis). If the torsion is not continuous (i.e., there are points along the axis at which the normal vector rotates discontinuously), then the sum of the angles through which the normal vector discontinuously rotates must be added to the integral in ι_g. We see from (8) that, even if there is no poloidal component of magnetic field ($B_\omega = 0$), the geometric rotation remains. This geometric angle is the basis of the figure-eight and other stellarators with nonplanar axes (with the geometric angle coming solely from discontinuities in Spitzer's original description [5]).

To show that the geometric part of the rotational transform has the anholonomic form of equation (1), note that

$$d\mathbf{e}_2 = \frac{\partial \mathbf{e}_2}{\partial X_j} dX_j , \tag{9}$$

so

$$d\theta = -dX_j \left(\frac{\partial e_{2i}}{dX_j}\right) e_{3i} = -\mathbf{R}\cdot d\mathbf{X} = -\tau ds \ , \tag{10}$$

with $R_j \equiv (\partial e_2/\partial X_j)e_{3i} = (\partial\theta/\partial X_j)$. Since no adiabatic assumption has been made, the geometric rotational transform corresponds to the Aharonov-Anandan angle rather than the Hannay angle.

It should be noted that there can be a magnetic contribution to the rotational transform, in addition to the geometric one, even from a vacuum magnetic field. For example, if the vacuum field were confined within a conducting shell of elliptic or triangular cross-section, which itself rotated to form a boundary with a helical deformation, then there would be a contribution to the rotational transform from the first term in equation (8) even if the magnetic axis was planar. This is the principle of magnetic molding [11].

III. ADIABATIC INVARIANTS

It should be clear that the rotational transform discussed above arises from nothing more than the transport (in the Fermi-Walker sense) of a triad of vectors along the axis. It is not surprising then that the same angle appears in the gyrophase of a charged particle spiraling along a magnetic field line. (It also appears in the rotation of the plane of polarization of light traveling along a twisted optical waveguide [12-16].) Littlejohn [8] has shown that for strong magnetic fields the gyrophase angle evolves according to

$$\frac{d\theta}{dt} = \omega_c + \mathbf{R}\cdot\frac{d\mathbf{X}}{dt} + \cdots \ , \tag{11}$$

where $\mathbf{R}$ is the vector introduced in equation (10) and the ellipsis indicate higher-order terms in the ratio of Larmor radius ρ to scale length of magnetic field. This can readily be interpreted as gyration at frequency ω_c plus rotation arising from transport of the $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ system. It follows from equation (6) that if the guiding center were to travel around a closed path, the change in gyrophase would be

$$\Delta\Omega = \oint \omega_c \, dt + \oint \tau \, ds \equiv \Delta\Omega_d + \Delta\Omega_g \tag{12}$$

again illustrating the dynamic and geometric angles. However, in this case the geometric angle would seem undetectable in the face of the large dynamic phase (unlike the rotation of the magnetic-field line or of the optical polarization, where the geometric angle is readily detectable). Nevertheless the geometric angle *does* have an effect on the guiding-center motion. This arises because, in addition to the magnetic-moment invariant, conjugate to the gyrophase, there is a second invariant J conjugate to the periodic motion *of the guiding center itself.* (This periodic motion may be reflection between magnetic mirrors or circulation around a torus). To lowest order in Larmor radius, the invariant $J = m \oint \nu_{gc} \, ds$, but in higher order there are additional contributions (which influence the long-term average of the guiding-center motion.) One such contribution is remarkable in that it depends on the direction of circulation of the guiding center and is absent when the guiding center oscillates between mirrors [10]. It is given by

$$J_1 = \frac{mv_\perp^2}{eB} \oint \tau \, ds \ . \tag{13}$$

We can now interpret this in the light of the geometric phase in the following manner. We should really define J over a closed particle orbit. To a first approximation this is just the integral over the guiding-center path, but closure of the guiding-center path does not ensure closure of the particle orbit, since the initial and final gyrophase angles may be different. Part of this difference is the rapidly changing dynamic phase which averages out. However, there remains the geometric angle $\Delta\Omega_g$. This changes the path of integration by $\rho\Delta\Omega_g$ and so makes a contribution $mv\rho\Delta\Omega_g$ to the invariant J. This is identical with Eq. (13).

In conclusion, we have shown that the rotational-transform angle in a torus with a twisted axis is a particularly direct example of the anholonomic, geometric angle. The same angle arises in the gyrophase of a charged particle where, despite the fact that it is negligible compared to the total change in gyrophase, it has an effect on the guiding-center motion. This accounts for an otherwise puzzling term in the longitudinal invariant J, which is different for guiding-center orbits which circulate in opposite directions and is absent for guiding-center orbits which are reflected.

ACKNOWLEDGMENTS

This research is supported by the U.S. Department of Energy contracts #DEFG02-86ER-53222 and DEFG05-80ET-53088.

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