INSTITUTE FOR FUSION STUDIES

DOE/ET-53088-507 IFSR #507

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July 1991

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ABSTRACT

The linear response of energetic particles to the TAE modes is calculated taking into account their finite orbit excursion from the flux surfaces. The general expression reproduces the previously derived theory for small banana width; when the banana width Δ_b is much larger than the mode thickness Δ_m, we obtain a new compact expression for the linear power transfer. When $\Delta_m/\Delta_b \ll 1$, the banana orbit effect reduces the power transfer by a factor of Δ_m/Δ_b from that predicted by the narrow orbit theory. A comparison is made of the contribution to the TAE growth rate of energetic particles with a slowing-down distribution arising from an isotropic source, and a balanced-injected beam source when the source speed is close to the Alfven speed. For the same stored energy density, the contribution from the principal resonances ($|v_{\|}| = v_A$) is substantially enhanced in the beam case compared to the isotropic case, while the contribution at the higher sidebands ($|v_{\|}| = v_A/(2\ell - 1)$ with $\ell \geq 2$) is substantially reduced.

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The confinement of alpha particles in an ignited plasma is of major concern to fusion research. In particular a potential mechanism of particle loss can arise from the excitation of toroidal Alfven eigenmodes (TAE) [1,2]. Recent high energy beam injection experiments carried out on TFTR [3] and DIII-D [4] demonstrated that the TAE modes can indeed be excited by energetic particles. Therefore it is highly desirable to have a proper theoretical estimate of the TAE instability drive.

Existing theories have so far assumed that the excursion length Δ_b of particle orbits from the flux surfaces (the so-called "banana width") is small compared to the TAE scale length, which we denote by Δ_m. However this assumption is generally violated, as can be seen from the following estimates. Roughly, we have the energetic particle diamagnetic frequency $\omega_{*\alpha} \approx m\rho v/r_m L_\alpha$, the mode frequency $\omega \approx v_A/2Rq$, the banana width $\Delta_b \approx q\rho$, and the mode scale length $\Delta_m \approx r_m^2/msR$. Here m is the poloidal mode number (the simplest structure of a TAE mode in large aspect ratio tokamak consists of two poloidal components m and $m+1$); v is the typical particle speed; ρ is the corresponding gyroradius; r_m is the mode location; L_α is the energetic particle density gradient scale length; v_A is the Alfven speed; q is the safety factor; and $s \equiv rq'/q$ is the local magnetic shear. Hence we find that with $v \approx v_A$,

$$\frac{\Delta_b}{\Delta_m} \approx \frac{sL_\alpha}{r_m}\frac{\omega_{*\alpha}}{\omega}. \tag{1}$$

From previous studies [5] we know that $\omega_{*\alpha} \gtrsim \omega$ is required for instability. If $\omega_{*\alpha}/\omega \gtrsim 1$ for $m = 1$, $L_\alpha/r_m \approx 1$, and s has a modest value, then $\Delta_b/\Delta_m \gtrsim 1$ will be fulfilled for higher m values. Therefore to interpret the experiments in which one observes moderately high m modes, and to make a reasonable prediction about the behavior of alpha particles in a fusion plasma, the linear theory of TAE excitation has to be generalized to arbitrary ratio of Δ_b/Δ_m. We will show that when this ratio is larger than unity, the instability drive from energetic particles is reduced by a scale factor of Δ_m/Δ_b compared to what the previous expression (Eq. (9) in Ref. [6]) would predict. This result follows from the finite orbit excursion that arises from the guiding center equations. We will indicate in the text a heuristic argument why the finite Larmor radius (FLR) effects are not essential to the result. A calculation with the FLR effects is deferred to a future paper.

In addition, past theories have only analyzed the situation where energetic particles form a slowing-down distribution from an isotropic source. With neutral beam injection, the distribution is highly anisotropic near the injection speed and only isotropizes at lower speeds. We will calculate the contributions to the TAE growth rate from energetic particles for both isotropic and beam sources. In particular we show that for injection near the Alfven speed, the contribution of the principal resonances with $|v_{||}| \approx v_A$ (which occur at speeds in a band that brackets v_A) is enhanced in the beam injection case compared to the isotropic case, and that the contribution from resonances at lower speeds, e.g., $|v_{||}| \approx v_A/3$ is less in the beam case than in the isotropic case. From this observation it appears likely that, in recent experiments that claim to have observed the TAE modes for injection speeds slightly below the Alfven speed, a substantial contribution to the instability drive comes from the principal resonances.

The TAE growth rate is given by

$$\gamma = \frac{P_\alpha - P_d}{2WE} \equiv \gamma_\alpha - \gamma_d, \tag{2}$$

where P_α is the power transfer from the energetic particles to the wave, P_d is the wave power absorbed by the background plasma (which needs a separate self-consistent calculation [7–9]), and WE denotes the wave energy. In this study we take P_d as given and only calculate P_α and WE.

We consider large aspect ratio tokamak with circular flux surfaces, for which the following mode representation is appropriate:

$$\begin{aligned} \delta E_{\|} &= 0; & \delta B_{\|} &= 0; \\ \delta E_r &= -\frac{\partial \phi_m}{\partial r}\cos\Psi_m; & \delta B_r &= \frac{k_{\|}c}{\omega}\frac{m}{r}\phi_m \sin\Psi_m; \\ \delta E_\theta &= -\frac{m}{r}\phi_m \sin\Psi_m; & \delta B_\theta &= -\frac{k_{\|}c}{\omega}\frac{\partial\phi_m}{\partial r}\cos\Psi_m; \end{aligned} \tag{3}$$

where $\phi_m(r)$ is the perturbed scalar potential, $\Psi_m \equiv n\varphi - m\theta - \omega t$, and $k_{\|} = R^{-1}[n - m/q(r)]$. In Eq. (3) (and Eqs. (8),(9),(12) and (14) below) a sum over poloidal mode numbers m and $m+1$ is implied. Typically the relevant radial structure of a TAE mode in the vicinity of an Alfven resonance radius r_m is of the form [7]

$$\phi_m(r) = \int_{r_m}^{r} dr' \frac{\Phi_m}{\pi}\left[\frac{\Delta_m + \alpha_m(r' - r_m)}{(r' - r_m)^2 + \Delta_m{}^2}\right], \tag{4}$$

where $\alpha_m = O(1)$, and $\Delta_m \approx r_m^2/msR$. (Though Φ_m and α_m are functions of r, they are slowly varying on the scale of Δ_m so we treat them as real constants.) We shall also assume that the mode width Δ_m satisfies $\Delta_m \ll r_m/ms$.

The wave energy is given by

$$WE = \int d\varphi d\theta dr Rr \left[\frac{\delta B_\theta^2}{8\pi} + \frac{\omega^2}{k_{\|}^2 v_A^2}\frac{\delta B_\theta^2}{8\pi}\right], \tag{5}$$

where the first term in the bracket is the field energy, and the second term is the kinetic energy associated with the $\delta\mathbf{E}\times\mathbf{B}$ drift motion. For TAE modes we have $\omega^2 \approx k_{\|}^2 v_A^2$, so the field energy and the kinetic energy are approximately equal. Using Eqs. (3) and (4) we find

$$WE = \frac{r_m R c^2}{4 v_A^2}\left[\frac{\hat{\Phi}_m^2}{\Delta_m} + \frac{\hat{\Phi}_{m+1}^2}{\Delta_{m+1}}\right], \tag{6}$$

with $\hat{\Phi}_m^2 = (1 + \alpha_m^2)\Phi_m^2$, and similarly for the $m+1$ component.

Within the guiding center approximation the general expression for the particle-to-wave power transfer, for waves with $\delta E_{\|} = 0$, is given by

$$P_\alpha = \int d\varphi d\theta dr Rr \int d^3v\,(-e_\alpha \mathbf{v}_d \cdot \delta\mathbf{E}) f, \tag{7}$$

where $\mathbf{v}_d = -v_d(\sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\boldsymbol{\theta}})$ is the unperturbed guiding center drift velocity ($v_d \equiv (v_\perp^2/2 + v_\parallel^2)/\Omega R$, $\Omega \equiv e_\alpha B/mc$), and f is the linear perturbation of the guiding center distribution function. For simplicity we consider the limiting case $\omega_{*\alpha} \gg \omega$.

Since the majority of energetic particles in a large aspect ratio tokamak are passing particles, we neglect the contributions from the trapped particles in the present study, and consider all the passing particles to have nearly constant $v_\parallel$ (this simplification is also relevant to the neutral beam injection). Since the magnetic moment is also conserved and B is nearly constant, to a good approximation we can put $v_\parallel$ and $v_\perp$ constant separately. In the limit when $\omega_{*\alpha} \gg \omega$, the perturbed distribution function f in Eq. (7) can be calculated by neglecting the perturbation in particle energy. With these simplifications, and using the mode representation (3), we can write the linearized kinetic equation for the energetic particles in the form (see Ref. [6])

$$\frac{\partial f}{\partial t} - v_d \sin\theta \frac{\partial f}{\partial r} + \dot{\varphi}\frac{\partial f}{\partial \varphi} + \dot{\theta}\frac{\partial f}{\partial \theta} = \frac{c}{B}\left(1 - \frac{k_\parallel v_\parallel}{\omega}\right)\left[\frac{m}{r}\phi_m \sin\Psi_m \frac{\partial f_0}{\partial r} - \frac{\partial \phi_m}{\partial r}\cos\Psi_m \frac{1}{r}\frac{\partial f_0}{\partial \theta}\right], \tag{8}$$

where $\dot{\varphi} \equiv v_\parallel/R$, and $\dot{\theta} \equiv v_\parallel/Rq$. The unperturbed distribution f_0 is an arbitrary function of $(v_\parallel, v_\perp, \bar{r})$, where $\bar{r} = r - \Delta_b \cos\theta$ ($\Delta_b \equiv Rqv_d/v_\parallel$ is the maximum orbit excursion from a flux surface). The particle orbit can be described with good accuracy by setting $\bar{r}$ = constant. We can intuitively justify neglecting the FLR effect as follows. Imagine a gyrating particle passing completely through a surface with a step-function potential change $\Delta\phi\exp(-i\omega t)$. If $\omega = 0$, the energy transfer to the particle is $e_\alpha\Delta\phi$, independent of FLR; if $\omega \ll |\mathbf{v}_d \cdot \mathbf{n}|/\rho$, where $\mathbf{n}$ is the unit vector normal to the potential surface, the energy transfer is $e_\alpha\Delta\phi\exp(-i\omega t)$, insensitive to the gyrophase; when $\rho \ll \Delta_b$, and $\partial\phi/\partial r \gg m\phi/r$, a detailed FLR theory (to be presented elsewhere) shows that gyro-averaging produces an additional factor $J_0(\chi)$ which multiplies the scalar potential ϕ, where J_0 is the Bessel function, and $\chi = \omega\rho/v_d \sin\theta$. Using $\omega = v_A/2qR$ we have $\chi = v_A/(vq|\lambda + \lambda^{-1}|\sin\theta)$, where $\lambda \equiv v_\parallel/v$.

For $v_A/v_0 \approx 1$ we have $\chi < 1$ except when $\sin\theta$ is small. To simplify our theory we therefore treat $J_0 \doteq 1$.

Changing the variable from r to $\bar{r}$, we can rewrite Eq. (8) as

$$\frac{\partial f}{\partial t} + \dot{\varphi}\frac{\partial f}{\partial \varphi} + \dot{\theta}\frac{\partial f}{\partial \theta} = \frac{c}{B}\left(1 - \frac{k_{\|}v_{\|}}{\omega}\right)\frac{1}{\bar{r}}\frac{\partial f_0}{\partial \bar{r}}\left[m\phi_m \sin\Psi_m + \frac{\partial \phi_m}{\partial \theta}\cos\Psi_m\right]. \qquad (9)$$

Note that ϕ_m is now a function of θ through $r = \bar{r} + \Delta_b \cos\theta$, thus it can be expanded into a Fourier series

$$\phi_m(\bar{r} + \Delta_b\cos\theta) = \sum_{\ell=0}^{\infty} \phi_{m,\ell}\cos\ell\theta. \qquad (10)$$

Using Eq. (4), the Fourier components $\phi_{m,\ell}$ can be evaluated explicitly, and one finds (the $\ell = 0$ component can be shown not to contribute in the final result)

$$\phi_{m,\ell} = -\frac{\Phi_m}{\pi\ell}(i + \alpha_m)z^\ell + c.c., \qquad \text{for } \ell > 0, \qquad (11)$$

where $z = -(x+iy) + [(x+iy)^2 - 1]^{1/2}$, $x \equiv (\bar{r} - r_m)/\Delta_b$, $y \equiv \Delta_m/\Delta_b$, and the branch of the square root must be chosen so that $|z| < 1$. Substituting Eq. (10) into (9) we obtain

$$\frac{\partial f}{\partial t} + \dot{\varphi}\frac{\partial f}{\partial \varphi} + \dot{\theta}\frac{\partial f}{\partial \theta} = \frac{c}{2B}\left(1 - \frac{k_{\|}v_{\|}}{\omega}\right)\frac{1}{\bar{r}}\frac{\partial f_0}{\partial \bar{r}}\sum_\ell (m+\ell)\phi_{m,\ell}\sin(\Psi_m - \ell\theta) \qquad (12)$$

where ℓ runs over all integers with the rule that $\phi_{m,-\ell} = \phi_{m,\ell}$. Therefore the solution is

$$\begin{aligned} f = &-\frac{c}{2B\omega}\frac{1}{\bar{r}}\frac{\partial f_0}{\partial \bar{r}}\sum_\ell [\dot{\Psi}_m\phi_{m,\ell} + \dot{\Psi}_{m+1}\phi_{m+1,\ell-1}](m+\ell) \\ &\times\left[\frac{\cos(\Psi_m - \ell\theta)}{\omega - n\dot{\varphi} + (m+\ell)\dot{\theta}} + \pi\sin(\Psi_m - \ell\theta)\delta(\omega - n\dot{\varphi} + (m+\ell)\dot{\theta})\right]. \end{aligned} \qquad (13)$$

Notice that both m and $m+1$ components have been included explicitly. We also have the power transfer from a single particle

$$-e_\alpha \mathbf{v}_d \cdot \delta\mathbf{E} = -e_\alpha v_d\left(\sin\theta\cos\Psi_m\frac{\partial\phi_m}{\partial r} + \frac{m}{r}\phi_m\cos\theta\sin\Psi_m\right). \qquad (14)$$

The second term on the right-hand side is negligible compared to the first, which yields

$$-e_\alpha \mathbf{v}_d \cdot \delta\mathbf{E} \doteq \frac{e_\alpha v_{\|}}{2Rq}\sum_\ell \left[\ell\phi_{m,\ell} + (\ell-1)\phi_{m+1,\ell-1}\right]\sin(\Psi_m - \ell\theta). \qquad (15)$$

Upon substituting Eqs. (13) and (15) into (7) we obtain the linear power transfer from the energetic particles

$$P_\alpha = -\frac{\pi^3}{2}\frac{c}{B}\frac{e_\alpha}{\omega}\int d\bar{r}d^3v\frac{v_\parallel^2}{Rq^2}\frac{\partial f_0}{\partial\bar{r}} \times \sum_\ell (m+\ell)\delta(\omega - n\dot{\varphi} + (m+\ell)\dot{\theta})\left[\ell\phi_{m,\ell} + (\ell-1)\phi_{m+1,\ell-1}\right]^2 . \quad (16)$$

We can recover the dominant part of the previously obtained result in Ref. [6] when we assume $\Delta_m \gg \Delta_b$. In this case we expand $\phi_m(r)$ in Eq. (10) in a Taylor series, and as only the first derivative is important, we find

$$\phi_m(\bar{r} + \Delta_b\cos\theta) \approx \phi_m(\bar{r}) + \frac{d\phi_m}{dr}(\bar{r})\Delta_b\cos\theta. \quad (17)$$

(This result is also obtainable from Eqs. (10) and (11) by letting $|x+iy| \gg 1$ in z.) Therefore in the sum of (10) only the first sideband, $\ell = 1$, is present, whose amplitude is $\phi_{m,1} = \Delta_b d\phi_m/dr$. Using this expression in Eq. (16) we obtain

$$P_{\alpha 0} = -\frac{\pi^3}{2}\frac{c}{B}\frac{e_\alpha}{\omega}\int d\bar{r}d^3v\frac{v_\parallel^2}{Rq^2}\frac{\partial f_0}{\partial\bar{r}}\Delta_b{}^2 \times\left\{\left[(m+1)\delta(\omega - n\dot{\varphi} + (m+1)\dot{\theta}) + (m-1)\delta(\omega - n\dot{\varphi} + (m-1)\dot{\theta})\right]\left(\frac{d\phi_m}{dr}\right)^2 + \left[(m+2)\delta(\omega - n\dot{\varphi} + (m+2)\dot{\theta}) + m\delta(\omega - n\dot{\varphi} + m\dot{\theta})\right]\left(\frac{d\phi_{m+1}}{dr}\right)^2\right\}, \quad (18)$$

which is the dominant part of Eq. (9) of Ref. [6] when $m\phi_m/r \ll d\phi_m/dr$. It will be shown below that, if we apply this result to the narrow mode case, we would over-estimate the power transfer by roughly a factor of Δ_m/Δ_b.

We note that the resonance condition from Eq. (16) can be written in the form

$$\frac{v_\parallel}{v_A} = \frac{1}{2q_{mn}[n - (m+\ell)/q(\bar{r})]} \doteq \frac{1}{1-2\ell}\left[1 - \frac{2}{q_{mn}}\frac{(m+\ell)\delta q}{(1-2\ell)}\right], \quad (19)$$

where we have used $\omega = v_A/2Rq(r_m)$ for the mode frequency ($q(r_m) \approx (2m+1)/2n \equiv q_{mn}$), and $\delta q \equiv q(\bar{r}) - q(r_m)$. We also note from Eq. (19) that the speed of a resonant particle can be less than the Alfven speed, because (a) ℓ takes on all integer values; and (b) even for the principal resonances ($\ell = 0, 1$), the spread of δq can decrease the resonance speed.

We now wish to evaluate the phase-space integral in Eq. (16) explicitly in the two limiting cases $\Delta_b \gg \Delta_m$ and $\Delta_b \ll \Delta_m$. The major contribution to the linear power transfer comes from particles in the vicinity of the gap location, i.e., small δq. For resonances having $|v_{||}|$ below the Alfven speed, we can assume that the distribution function is slowly varying near the resonant velocity, and thus neglect δq in the resonance conditions. (When maximum particle speed $v_0 > v_A$, this occurs for all values of ℓ; it also occurs for $\ell \neq 0, 1$ in the case $v_0 \lesssim v_A$. The case of $v_0 \approx v_A$ and $\ell = 0, 1$ when such an approximation cannot be made will be discussed later.) With this approximation we find that the $\bar{r}$ integral only involves the Fourier amplitudes and can be calculated first, and the result for both limits can be written compactly as

$$P_\alpha = -\frac{\pi^3 c e_\alpha}{B\omega q_{mn}} \sum_\ell C_\ell^{(p)} \int d^3v \frac{v_A^2}{|1-2\ell|^3} \frac{\partial f_0}{\partial \bar{r}} |\Delta_b| \left| \frac{\Delta_b}{\Delta_m} \right|^{p-1} (m+\ell)\delta(v_{||} - v_A/(1-2\ell)), \quad (20)$$

where $p = 1, 2$ for $\Delta_b \gg \Delta_m$ and $\Delta_b \ll \Delta_m$ respectively. For $\Delta_b \gg \Delta_m$, we have taken $y \to 0$. This enables us to transform the $\bar{r}$ integral into an x integral which is independent of the magnitude of the velocity, except for a factor Δ_b as exhibited in Eq. (20). After considerable algebra $C_\ell^{(1)}$ is found to simplify to

$$C_\ell^{(1)} = \frac{16\ell^2 \hat{\Phi}_m^2}{\pi^2[4\ell^2 - 1]} + \frac{16(\ell-1)^2 \hat{\Phi}_{m+1}^2}{\pi^2[4(\ell-1)^2 - 1]} + \frac{2}{\pi}\frac{\Delta_b}{|\Delta_b|}(1-\delta_{\ell,0})(1-\delta_{\ell,1})(\alpha_m - \alpha_{m+1})\Phi_m \Phi_{m+1}. \quad (21)$$

For the opposite limit we used Eq. (18) for the power transfer. A straightforward calculation yields

$$C_\ell^{(2)} = \frac{\hat{\Phi}_m^2}{2\pi}(\delta_{\ell,1} + \delta_{\ell,-1}) + \frac{\Delta_m}{\Delta_{m+1}} \frac{\hat{\Phi}_{m+1}^2}{2\pi}(\delta_{\ell,2} + \delta_{\ell,0}). \quad (22)$$

We choose f_0 to be of the form of a slowing-down distribution

$$f_0 = \frac{3\beta_\alpha B^2}{16\pi^2 M_\alpha v_0^2} \frac{\Theta(v_0 - v)}{v^3 + v_I^3} g(\lambda, \Delta\lambda(v)), \quad (23)$$

where $\Theta(v)$ is the step function, and

$$g(\lambda, \Delta\lambda) = \frac{\exp[-(1-\lambda)/\Delta\lambda] + \exp[-(1+\lambda)/\Delta\lambda]}{\Delta\lambda[1-\exp(-2/\Delta\lambda)]}, \tag{24}$$

$$\Delta\lambda(v) = \Delta\lambda_0 + \frac{A}{3}\ln\left(\frac{1+v_I^3/v^3}{1+v_I^3/v_0^3}\right). \tag{25}$$

The physical meaning of the constants $\Delta\lambda_0$, A, and v_I is discussed in Ref. [10]. For an isotropic source such as alpha particles in an ignited plasma, we have $\Delta\lambda_0 \gg 1$, so $g = 1$; for a source from balanced-injected neutral beams we have $\Delta\lambda_0 \ll 1$, hence $\Delta\lambda \ll 1$ for $v \approx v_0$ and becomes larger as v decreases. At $v \ll v_I$ we have $\Delta\lambda \gg 1$ and $g \to 1$, indicating that the beam particles isotropize at speeds below v_I. The distribution function (23) is normalized so that $\beta_\alpha \equiv (8\pi M_\alpha/3B^2)\int d^3v v^2 f_0 + O(v_I^3/v_0^3)$ is the mean beta of the energetic particles.

Consider first the isotropic case where $g = 1$. For $v_I \ll v_A$ we can ignore v_I in the distribution function and obtain, for $\Delta_b \gg \Delta_m$,

$$P_\alpha = -\frac{3\pi^2}{16}\beta'_\alpha \frac{c^2 v_A}{\omega v_0}\sum_\ell C_\ell^{(1)} \frac{m+\ell}{(1-2\ell)^2}(1-\xi_\ell^2)\Theta(1-\xi_\ell), \tag{26}$$

where $\xi_\ell \equiv v_A/v_0|1-2\ell|$. For the opposite limit $\Delta_b \ll \Delta_m$ we have

$$P_{\alpha 0} = -\frac{\pi^2}{32}\beta'_\alpha \frac{c^2}{\omega}\frac{\Delta_{b0}}{\Delta_m}\sum_\ell C_\ell^{(2)} \frac{m+\ell}{|1-2\ell|}(1-\xi_\ell)(1+\xi_\ell+7\xi_\ell^2+3\xi_\ell^3)\Theta(1-\xi_\ell), \tag{27}$$

where $\Delta_{b0} \equiv q_{mn} v_0/\Omega$. We note from these expressions that the contribution from principal resonances $\ell = 0, 1$ (i.e., $|v_\parallel| = v_A$) vanishes continuously as v_0 approaches v_A, so higher sideband contributions may actually become dominant.

For the beam case, the v integral is easily carried out using the delta function, but the λ integral is more complicated. To be concrete we model the beam distribution as two separate components: a weakly scattered part with $v > v_I$ for which we have $\Delta\lambda \doteqdot$

$\Delta\lambda_0 + (A/3)[(v_I/v)^3 - (v_I/v_0)^3] \ll 1$, and a fully isotropized part with $v < v_I$, for which $\Delta\lambda \gg 1$. The weakly scattered part contributes mainly to the principal resonances, yielding

$$P_\alpha = -\frac{3\pi^2}{8}\beta'_\alpha \frac{c^2}{\omega}\left(\frac{v_A}{v_0}\right)^{p+1}\left(\frac{\Delta_{b0}}{\Delta_m}\right)^{p-1}\left[1-\exp\left(-\frac{1-v_A/v_0}{\Delta\lambda(v_A)}\right)\right](mC_0^{(p)}+(m+1)C_1^{(p)})\Theta(v_0-v_A). \tag{28}$$

When $|v_A/v_0 - 1| \ll 1$, the factor in the square bracket gives a larger contribution compared to the isotropic case, since

$$1-\exp\left(-\frac{1-v_A/v_0}{\Delta\lambda(v_A)}\right) \doteqdot \begin{cases} 1, & \text{if } \Delta\lambda(v_A) \ll |1-v_A/v_0| \ll 1; \\ (1-v_A/v_0)/\Delta\lambda(v_A), & \text{if } |1-v_A/v_0| \ll \Delta\lambda(v_A) \ll 1. \end{cases} \tag{29}$$

The contribution of other resonances is mainly from the isotropized component, and is somewhat difficult to calculate analytically. But qualitatively the scaling can be obtained using the same methods as for the case of an isotropic source, and the result is (apart from a numerical factor)

$$P_\alpha \propto -\beta'_\alpha \frac{c^2}{\omega}\left(\frac{v_A}{v_0}\right)^{p+1}\left(\frac{\Delta_{b0}}{\Delta_m}\right)^{p-1}\sum_\ell{}' \frac{(m+\ell)C_\ell^{(p)}}{|1-2\ell|^{p+2}}\left[\frac{v_I|1-2\ell|}{v_A}\right]^{2p-1}, \tag{30}$$

where the sum now excludes the principal resonances, i.e., $\ell \neq 0, 1$.

In the case when v_0 is very close to v_A, the principal resonances $\ell = 0, 1$ would have an abrupt cut-off, unless we take into account the δq spread in the resonance conditions. For the case $\Delta_b \gg \Delta_m$, our calculation shows that for the isotropic case, assuming $|v_0/v_A - 1| \ll 1$,

$$P_\alpha(\text{isotropic}) \propto \begin{cases} v_0/v_A - 1, & \text{if } v_0/v_A - 1 > \eta \text{ (same as } \delta q = 0); \\ \frac{1}{2}[\eta + v_0/v_A - 1], & \text{if } |v_0/v_A - 1| < \eta; \\ \approx 0, & \text{if } v_0/v_A - 1 < -\eta; \end{cases} \tag{31}$$

where $\eta \equiv 2ms(\Delta_{b0}/r_m)(1 + O(1/m))$. In the last line of Eq. (31), ≈ 0 means substantially less than $|v_0/v_A - 1|$. For a beam injected distribution we find

$$P_\alpha(\text{beam}) \propto \begin{cases} 1-\exp[-(v_0/v_A - 1)/\Delta\lambda_0], & \text{if } v_0/v_A - 1 > \eta \text{ (same as } \delta q = 0); \\ 1-\exp[-\frac{1}{2}(\eta + v_0/v_A - 1)/\Delta\lambda_0], & \text{if } |v_0/v_A - 1| < \eta; \\ \approx 0, & \text{if } v_0/v_A - 1 < -\eta. \end{cases} \tag{32}$$

The constants of proportionality can be obtained by comparing the top terms in Eqs. (31) and (32) with the principal resonance contributions from Eqs. (26) and (28). Qualitatively a similar behavior occurs in the $\Delta_m/\Delta_b \gg 1$ limit with η replaced by $2ms\Delta_m/r_m$.

We remark that there is an additional contribution from particles that resonate at a larger distance from the mode location than the banana width. This contribution is of higher order than our estimates, and can be estimated from the previously obtained formula [6] since the mode structure is now smooth along the particle orbit. We also note that the sum over ℓ should be terminated when $|v_{\|}|/v_0 = v_A/|2\ell - 1|v_0 \lesssim (r_m/R)^{1/2}$ because energetic particles become trapped.

In conclusion, we have presented a relatively simple method to predict the linear power transfer, taking into account finite banana orbit size from energetic particles that can resonate with the TAE modes. We have applied our formulas to a slowing down distribution generated either by parallel injected neutral beams, or by an isotropic source such as internally produced alpha particles. There is a significant difference in the response, with the beam distribution giving larger contributions near the injection energy than what would be obtained from an isotropic source.

ACKNOWLEDGMENTS

This work was supported by the US Department of Energy Contract No. DE-FG05-80ET-53088.

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