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Power-Law Energy Spectrum and Orbital Stochasticity

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Abstract

The power-law energy distributions are observed in space and fusion plasmas. The power law decay of the two-time velocity correlation function and the corresponding frequency spectrum of the correlation function are shown to be related to the power law distribution of the time interval of acceleration, which produces a power-law energy distribution. In particular, the two time correlation function, the distribution of acceleration duration, namely the distribution of the trapping time of the quasi-trapped orbits in the vicinity of the magnetic null such as the geomagnetic tail configurations are shown to produce a power law energy distribution function. The statistical property is applicable under conditions given here to the energy spectra of cosmic rays, electrons in laser-plasma interaction and the radio-frequency heated confined plasmas.

I. Introduction

Power-law energy distributions of charged particles are observed and theoretically predicted in many fields of physics as diverse as cosmic particle energies of the solar flare[1] and the extra-heliospheric origin,[2](see Fig.1) gamma-ray busters,[3] magnetospheric energetic particles,[4] laser heated plasma electrons[5] and ion cyclotron resonance heating of tokamak plasmas.[6] In this work we show how the statistical properties resulting from Hamiltonian chaos of charged particles in electromagnetic fields can produce power-law energy spectra.

As an example, we consider the acceleration mechanism of the charged particles that are quasi-trapped in the earth's magnetic field on the night side where it is stretched into long-thin magnetic field loops and where there is a dawn-to-dusk electric field. The electric field E_y is driven by the solar wind velocity v_{SW} across the ambient north-south magnetic field, $E_y = V_{SW} B_z$, and varies in strength with the degree of magnetic activity arriving from the sun. We use this system where the orbits and velocity correlation functions have been well studied to develop a statistical model of a general acceleration mechanism (theory) that produces power-law energy distribution functions. The acceleration model developed here is related in its statistical assumption to a recent theory of Per Bak et al.[7] on the distribution of the strengths of earthquakes.

II. Temporal Correlation, Life-Time Probability and Particle Energy Spectrum

Let us assume that a charged particle is accelerated by an external force which is independent of, or a period function of time. When a particle is not scattered by particle-particle collisions or the orbit stochasticity, the particle may be accelerated monotonically by a static field or the resonance with the Rf field. When the duration of the monotonic acceleration T is given, the energy gain of the particle can be represented by

$$\Delta\varepsilon(T) = \pm \ell_{\pm}(T, \varepsilon) F, \tag{1}$$

where F is an effective acceleration force, $\ell_{\pm}$ (T, ε) are free flight paths between subsequent scatterings for acceleration (+ sign) and declaration (- sign) cases and ε is the particle energy before acceleration.

We assume that the distribution of the lift time T of such a particle is given by the probability density

$$p\left\{(\varepsilon/\varepsilon_o)^{-\eta}T\right\} = \theta(\Delta\varepsilon)P_+\left\{(\varepsilon/\varepsilon_o)^{-\eta}T\right\} + \theta(-\Delta\varepsilon)P_-\left\{(\varepsilon/\varepsilon_o)^{\eta}T\right\}$$

were P_+ and P_- are for acceleration and deceleration processes respectively. In this case, we introduce a transition probability from ε to $\varepsilon + \Delta\varepsilon$; W (ε, $\Delta\varepsilon$), and the function W (ε, $\Delta\varepsilon$) can be given by

$$W(\varepsilon, \Delta\varepsilon)\, d\Delta\varepsilon = P\left\{(\varepsilon/\varepsilon_o)^{-\eta}T\right\}(\varepsilon/\varepsilon_o)^{-\eta}\, dT \quad , \tag{2}$$

where $\underline{P}(y)$ satisfies

$$\int_0^\infty P(y)\, dy = 1$$

Equation(2) is rewritten to

$$W(\varepsilon, \Delta\varepsilon) = P\left\{(\varepsilon/\varepsilon_o)^{-\eta}T\right\}(\varepsilon/\varepsilon_o)^{-\eta} / \left[\frac{\partial\ell(T,\varepsilon)}{\partial T}F\right] \tag{3}$$

The Chapman-Kolmogorov equation (8, 9) for energy distribution function f (ε, t) may be given by

$$\frac{\partial f(\varepsilon,t)}{\partial t}=\int_{-\infty}^{\infty} d\,(\Delta\varepsilon)W(\varepsilon-\Delta\varepsilon;\Delta\varepsilon)\,f(\varepsilon-\Delta\varepsilon)$$

$$-\int_{-\infty}^{\infty} d\,(\Delta\varepsilon)W(\varepsilon,\Delta\varepsilon)f(\varepsilon)+C\left\{f(\varepsilon,t)\right\}, \qquad (4)$$

where C {f (ε, t)} is the collision operator for internal interactions.

Note here that the mean free path of a non-relativistic particle is given by

$$<\ell_{\pm}(T,\varepsilon)>=\int_{0}^{\infty} vTP_{\pm}\left\{(\varepsilon/\varepsilon_{o})^{-\eta}T\right\}(\varepsilon/\varepsilon_{o})^{-\eta}dT$$

$$=v(\varepsilon/\varepsilon_{o})^{\eta}\int_{0}^{\infty} yP_{\pm}(y)\,dy, \qquad (5)$$

where v is a velocity of the particle which is proportional to $\sqrt{\varepsilon}$.

For Coulomb collision of a high energy particle whose energy is much higher than the plasma thermal energy k_BT, the mean free path is proportional to ε^2. Therefore, Eq (5) indicates $\eta=3/2$.

The equation (4) is rewritten by using Eq. (2) to yield,

$$\frac{\partial f}{\partial t}=\int_{0}^{\infty} dT\left[(\varepsilon-\Delta\varepsilon)/\varepsilon_{o}\right]^{-\eta}P\left\{\left[(\varepsilon-\Delta\varepsilon)/\varepsilon_{o}\right]^{-\eta}T\right\}f(\varepsilon-\Delta\varepsilon)$$

$$-\int_{0}^{\infty} dT(\varepsilon/\varepsilon_{o})^{-\eta}P_{\sigma}\left\{(\varepsilon/\varepsilon_{o})^{-\eta}T\right\}f(\varepsilon)+C\left\{f(\varepsilon,t)\right\}. \qquad (6)$$

When the external acceleration is strong enough, the internal collision term can be neglected in Eq. (6). In this case, the temporal evolution of the energy distribution f (ε, t) is described by

$$f(\varepsilon,t)=\sum_{i=1} a_{i}(\varepsilon)\,e^{-\lambda_{i}t}, \qquad (7)$$

where λ_i is an eigen value of the integral equation

$$\int_{0}^{\infty} dT\left[(\varepsilon-\Delta\varepsilon)/\varepsilon_{o}\right]^{-\eta}P\left\{\left[(\varepsilon-\Delta\varepsilon)/\varepsilon_{o}\right]^{-\eta}T\right\}a_{i}(\varepsilon-\Delta\varepsilon)$$

$$-\int_{0}^{\infty} dT(\varepsilon/\varepsilon_{o})^{-\eta}P\left\{(\varepsilon/\varepsilon_{o})^{-\eta}T\right\}a_{i}(\varepsilon)=-\lambda_{i}a_{i}(\varepsilon) \qquad (8)$$

Here, we may require boundary conditions $a_i(\infty)=0$ and $a_i(0)=$constant. The eigen function for the zero eigen value ($\lambda_0=0$) corresponds to stationary energy distribution function.

For an example, let us assume that $P\{(\varepsilon/\varepsilon_0)^{-\eta}T\}$ has a power law probability for

$$T \leqq (\varepsilon/\varepsilon_0)^{-\eta} T \leqq T_{max}$$

$$P\left\{(\varepsilon/\varepsilon_0)^{-\eta} T\right\} = P_0\left[(\varepsilon/\varepsilon_0)^{-\eta} T/T_0\right]^{-\gamma}, \quad (\gamma>0). \qquad (9)$$

Note that $(\varepsilon/\varepsilon_0)^{\eta}T_c$ can be chosen to be the Coulomb collision mean free time which is proportional to $(\varepsilon/\varepsilon_0)^{3/2}$. In this case, $\eta=3/2$

This is the one observed in the vicinity of the KAM surface in the standard map and in the problem of the geomagnetic tail particle life time (see the next section). The stationary distribution is given by

$$f_S(\varepsilon) = a_0(\varepsilon) \propto (\varepsilon/\varepsilon_0)^{-\eta(\gamma-1)}. \qquad (10)$$

This result means that the power law distribution of the period of coherent motion yields the power law energy distribution of the accelerated particle.

The life time distribution function $P\{\varepsilon/\varepsilon_0)^{-\eta}T\}$ is related to the two time correlation function of the particle velocity in the direction of the acceleration field. The time correlation of $v(t)$; $C(\tau)$ is given

$$C(\tau) = \langle v(t)v(t+\tau)\rangle = \langle v^2\rangle \int_0^\infty \theta(T-\tau) P\{\varepsilon/\varepsilon_0)^{-\eta} T\}(\varepsilon-\varepsilon_0)^{-\eta}\, dT. \quad (11)$$

where $\theta(T-\tau)=1$ for $\tau<T$ and otherwise 0. For $P\{\varepsilon/\varepsilon_0)^{-\eta}T\}=P_0\{\varepsilon/\varepsilon_0)^{-\eta}T/T_0\}^{-\gamma}$, Eq. (11) leads to

$$C(\tau) = \langle v^2\rangle P_0 T_0 (\varepsilon/\varepsilon_0)^{(\gamma-1)\eta} \frac{1}{\gamma-1} (\tau/T_0)^{1-\gamma} \qquad (12)$$

This means that the power-law dependence of the correlation function is directly related to the power-law energy distribution.

By the Fourier transform of Eq. (12), the frequency power law spectrum of $C(\tau)$ is obtained as

$$S(f) = 2\,Re \int_0^\infty C(\tau)\, e^{2\pi i f\tau}\, d\tau$$

$$= 2Re\left\{\langle v^2\rangle P_0 T_0 \left(\varepsilon/\varepsilon_0\right)^{\gamma\eta} \frac{T_0}{\gamma-1}\left(\frac{1}{2\pi f T_0}\right)^{2-\gamma} \int_0^\infty y^{1-\gamma} e^{iy}\, dy\right\}$$

$$\propto (1/f)^{2-\gamma} \qquad (13)$$

Let us summarize the above result. We have shown that this power-law spectrum (13) and the power law decay of the two-time correlation function (12) are connected to the power-law energy distribution (10) of the accelerated particle in an external field.

Another example of the acceleration is the energy diffusion process by random acceleration and deceleration. Assuming $|\Delta\varepsilon| \ll \varepsilon$, the Chapman-Kolmogorov equation (4) can be reduced to the Fokker- Planck equation,

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial \varepsilon}\left\{\int_{-\infty}^{\infty} d(\Delta\varepsilon)\,\Delta\varepsilon\, W(\varepsilon, \Delta\varepsilon) + \frac{\partial}{\partial\varepsilon}\int_{-\infty}^{\infty} d(\Delta\varepsilon)\,\frac{\Delta\varepsilon^2}{2}\, W(\varepsilon, \Delta\varepsilon)\right\} \times f(\varepsilon) + \frac{\partial}{\partial\varepsilon}\left[R(\varepsilon) f(\varepsilon)\right],$$

where the last term is the Coulomb drag force and $R(\varepsilon) = R_0 (\varepsilon / \varepsilon_0)^{-1/2}$ for high energy particles. Assuming

$$\int_{-\infty}^{\infty} d(\Delta\varepsilon)\,\Delta\varepsilon\, W(\varepsilon, \Delta\varepsilon) = 0,$$

the stationary distribution satisfies

$$\frac{\partial}{\partial\varepsilon}\left\{A(\varepsilon) f(\varepsilon)\right\} + R(\varepsilon) f(\varepsilon) = const. \qquad (14)$$

Using Eqs. (1) and (3), and assuming $\ell(T, \varepsilon) = \ell_0 \sqrt{\varepsilon / \varepsilon_0}\,(T/T_0)$ because a velocity $v \propto \sqrt{\varepsilon}$ for a nonrelativistic particle, we obtain

$$A(\varepsilon) = A_0 (\varepsilon / \varepsilon_0)^{2\eta + 1} \qquad (15)$$

where

$$A_0 = (\ell_0^2 F^2 / 2T_0^2)\int_0^{\infty} dy\, y^2 P(y).$$

Since $f(\varepsilon) \rightarrow \infty$ for $\varepsilon \rightarrow \infty$, the integral constant of Eq (14) is zero. Therefore, the stationary distribution is given by

$$f(\varepsilon) = C_0 (\varepsilon / \varepsilon_0)^{-2\eta - 1} exp\left\{-\frac{R_0}{A_0}\int^{\varepsilon} (\varepsilon / \varepsilon_0)^{-2\eta - 3/2} d\varepsilon\right\}, \qquad (16)$$

This result indicates that the energy distribution will be a power law distribution as far as $\eta \geqq -1/4$ in the high energy limit.

III. Review of Orbit Analysis in Geomagnetic Tail

The ion and electron orbits and their two-time velocity correlation functions and the energy distributions have been investigated by Currant and Goetz[10] (1989) and W. Horton and T. Tajima[11] (1990). They discussed the particle orbits in the geomagnetic tail, where the magnetic field structure during the reconnection is as shown in Fig. 2. The storm-time electromagnetic fields can be represented by a two-dimensional magnetotail model[12] by the vector potential

$$A = Ay, \quad A = A_0 + A_1$$

$$A_o = -aB_{xo} \ln[\cosh z/a] + B_z x \cdots\cdots\cdots$$

$$A_1 = -\psi_o(t)\cos(kx). \qquad (17)$$

Here $-\partial A/\partial t = E_y$ the inductive dawn-dusk electric field. During the quiescent time, E_y is constant. However, ψ_o (t) grows like $\psi_o e^{\gamma t}$ during the storm time.

Particle orbits in the electron magnetic field given by Eq. (17) are described by the following equation of motion,

$$x = \frac{dx}{dt} = v \qquad (18)$$

$$v_s = \omega_z v_y \qquad (19)$$

$$v_y = \frac{q}{m}\frac{\partial \psi_o}{\partial t}\cos(k_x x) + \omega_x v_z \tanh(z/a) - \omega_z v_x \qquad (20)$$

$$v_z = -\omega_x v_y \tanh(z/a) \qquad (21)$$

where $\omega_z = qB_z/m$ and $\omega_x = qBx/m$. Without an external electric field, Eqs. (19), (20) and (21) contain two coupled oscillators : For a particle $|z/a| \ll 1$, $v_z / |\sqrt{a\omega_x v_y}| \ll 1$ and $v_y > 0$, one of the oscillators corresponds to $\ddot{z} = -\omega_x v_y z/a$ (which comes from Eq. (21)), and the characteristic frequency is $\Omega_d = \sqrt{v_y \omega_x / a}$. The other one is cyclotron motion in x-y plan which has a frequency ω_z. These trajectories with $|z/a| \ll 1$ are called meandering particles. Particles other than the meandering ones have one cyclotron frequency $\sqrt{\omega_x^2 \tanh^2(z/a) + \omega_z^2} \equiv \omega$ and the bouncing frequency around the minimum magnetic field plane(z=0), whose bouncing frequency is $\Omega_\mu = \omega_x \sqrt{\mu / a^2 \omega_x}$, where μ is magnetic moment for the cyclotron motion. We call those particles "mirroring particles".

In the geomagnetic tail the frequencies Ω_d, ω_z, ω_c and Ω_μ can be comparable around the z=0 plane. Since those oscillators couple with each other nonlinearly through Eqs. (18)-(21), the particles suffer transitions from the meandering orbits to the mirroring orbits and vice-versa. These transitions take place in a stochastic way. This means that the invariance of μ and the action of the meandering particles break down through the resonance of two oscillation motions.

The stochasticity or the chaotic tendency can be characterized by the ratio of the orbital frequencies defined by the parameter $\kappa = \omega_z / \Omega_d$. When κ is near unity (or an integer or rational number) the particle orbit will become chaotic. Since Ω_d is a function of v_y, these are a group of particles whose orbits are regular and others are chaotic. An example of phase space distribution of particles is shown in Fig. 3 for $k = 0.25$ We can see that there are two kinds of orbits. One of these is a kind of non-integrable orbit and the other one is the type of integrable orbit, the boundaries of which are indicated by the KAM surface in Fig. 3. At the boundary between the two regions, the orbits are periodic for a very long time until they become chaotic. Namely, particles in the vicinity of the KAM surface stay for a very long time in that region of phase space and gradually diffuse out from the region near the KAM surface. Since the particles in this region have a long time correlation, the two-time position or velocity correlation does not decay rapidly such as exponentially, but shows a slow power-law decay. In fact, the two-time velocity correlation function in the direction of dawn-dusk in our simulation has a long time tail as shown in Fig. 4. Figure 4 indicates that the two-time correlation function consists of two parts. One of those decays with slow oscillations. The frequency spectrum of this correlation function may be represented by[12]

$$s(f)=\int_0^\infty d\tau\, C(\tau)\, e^{2\pi i f\tau}=\sum_{j=1,2}\frac{C_j}{|f-f_j|^{\alpha_j}}, \qquad (22)$$

where

$$C(\tau)=\lim_{T_0\to\infty}\frac{1}{T_0}\int_{T-T_0}^{T} v_y(t)\, v_y(t-t)\, dt \qquad (23)$$

is the correlation function.

From Fig. 4, the exponent of the power-law decay of the correlation function is approximately 1. Therefore, γ of Eq (12) is 2. When $\eta=3/2$, Eq.(10) yields $f(\varepsilon)\propto(\varepsilon/\varepsilon_0)^{-3/2}$. This is slightly different from the energy distribution observed by Lui and Krimigis 4; $f\propto\varepsilon^{-2}$.

On the other hand, when particles are accelerated randomly, the energy distribution given by either Eq.(10) or (16) indicates $\eta=1$ or 1/3.

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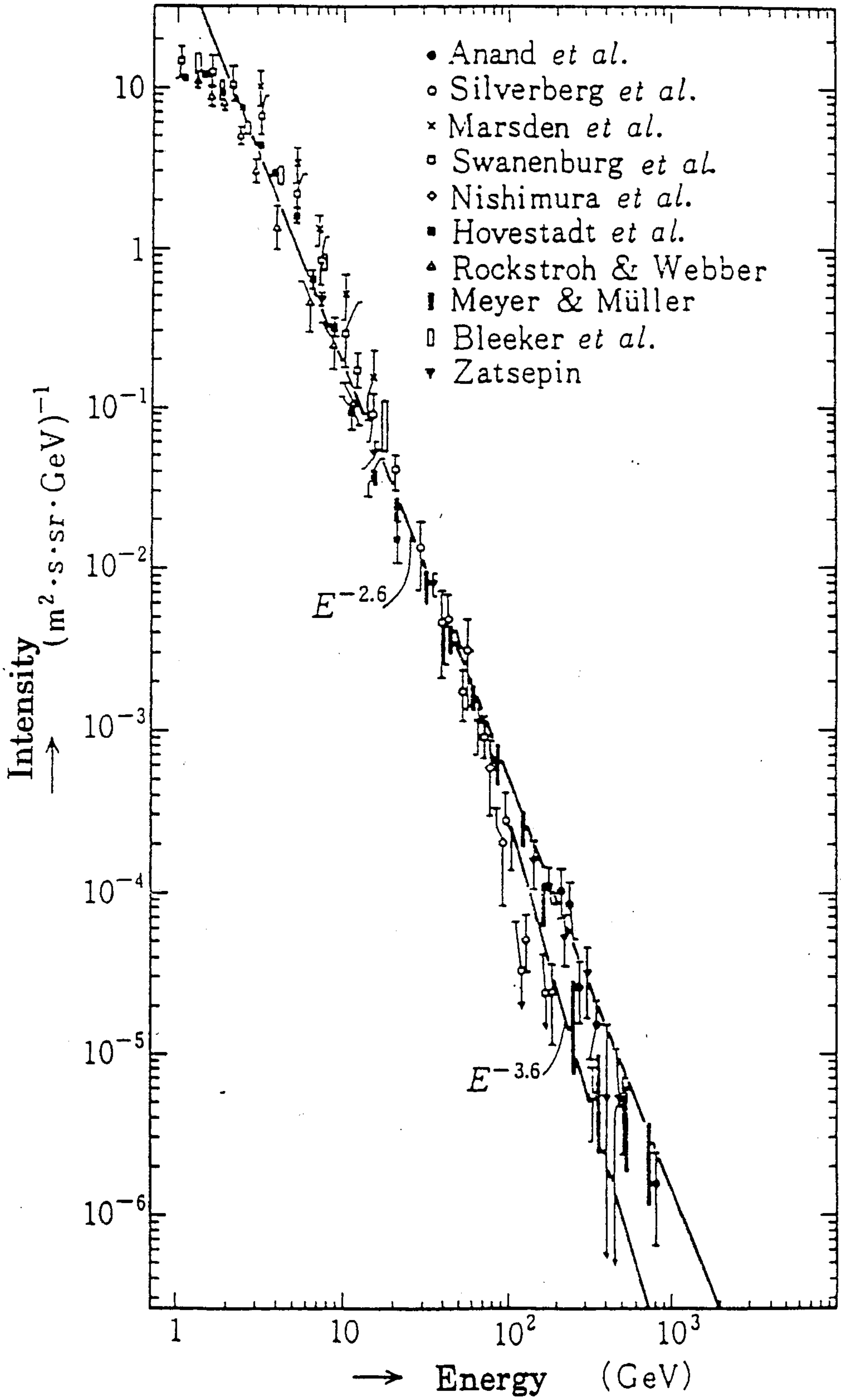


Fig.1 (a) Proton energy spectrum of cosmic ray.

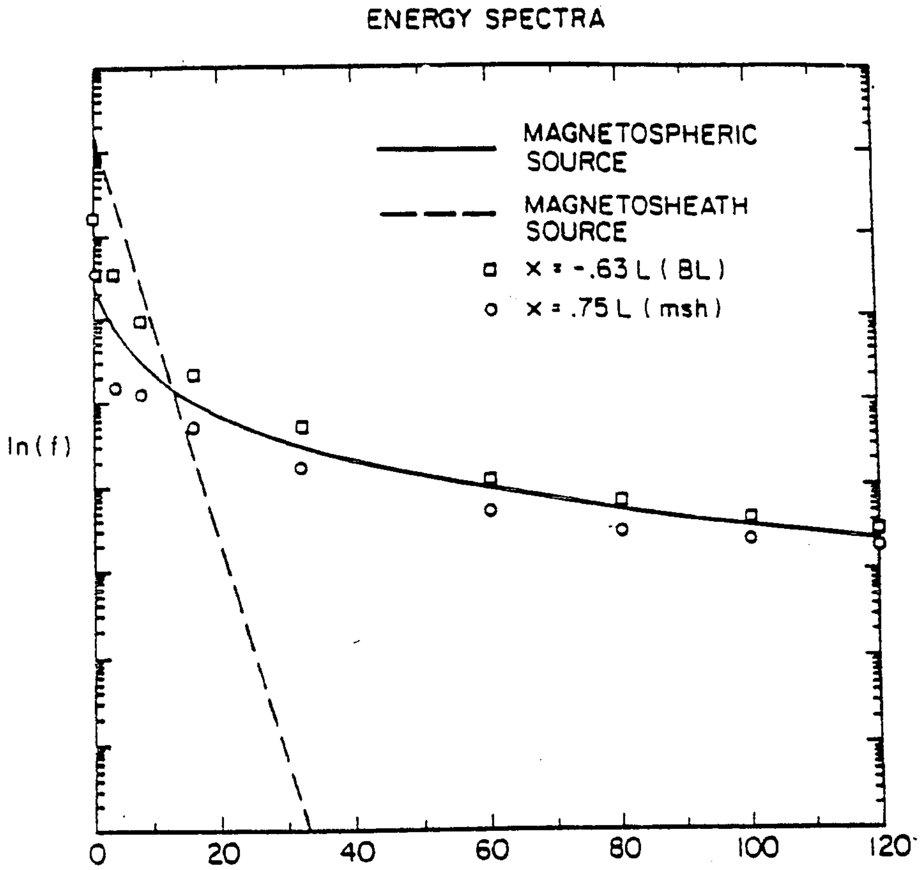
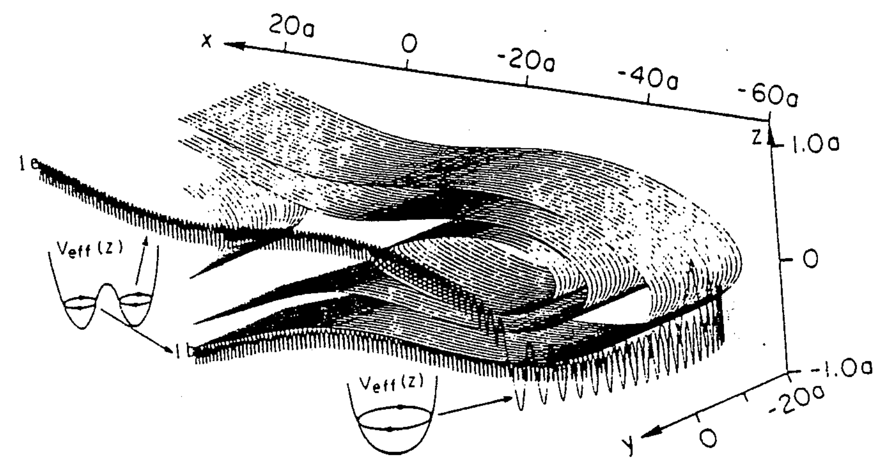


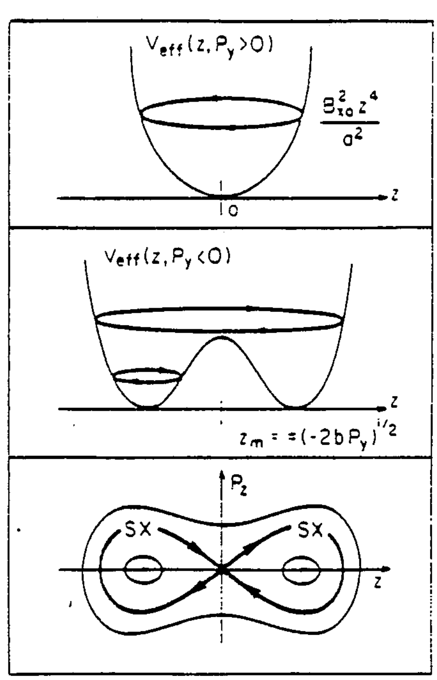
Fig.1 (b) Energy spectra in the magnetosheath and in the LLBL. The solid line is the source distribution in the magnetosphere . The broken line is the source distribution in the magnetosheath. The circles are in the LLBL, and the squares are in the magnetosheath. The distributions represented by both the circles and the squares are on open field line.

Fig.2 Magnetic structure in the geomagnetic tail. (a) and (b); effective potentials

A.



B.



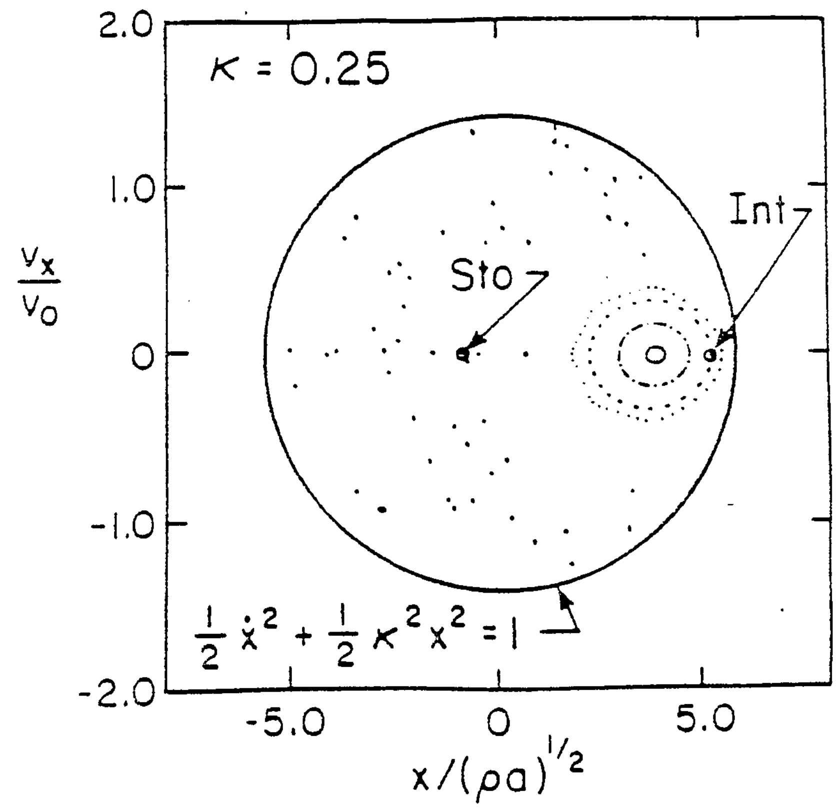


Fig. 3 Surface of section at the $z = 0$ plane for the 2D Hamiltonian in Eq. (2) for the stochasticity parameter $\kappa = 0.25$ and $H = 1$. The initial data for the integrable (Int) and the stochastic orbits (Sto) used in Figures 2 and 3 are marked.

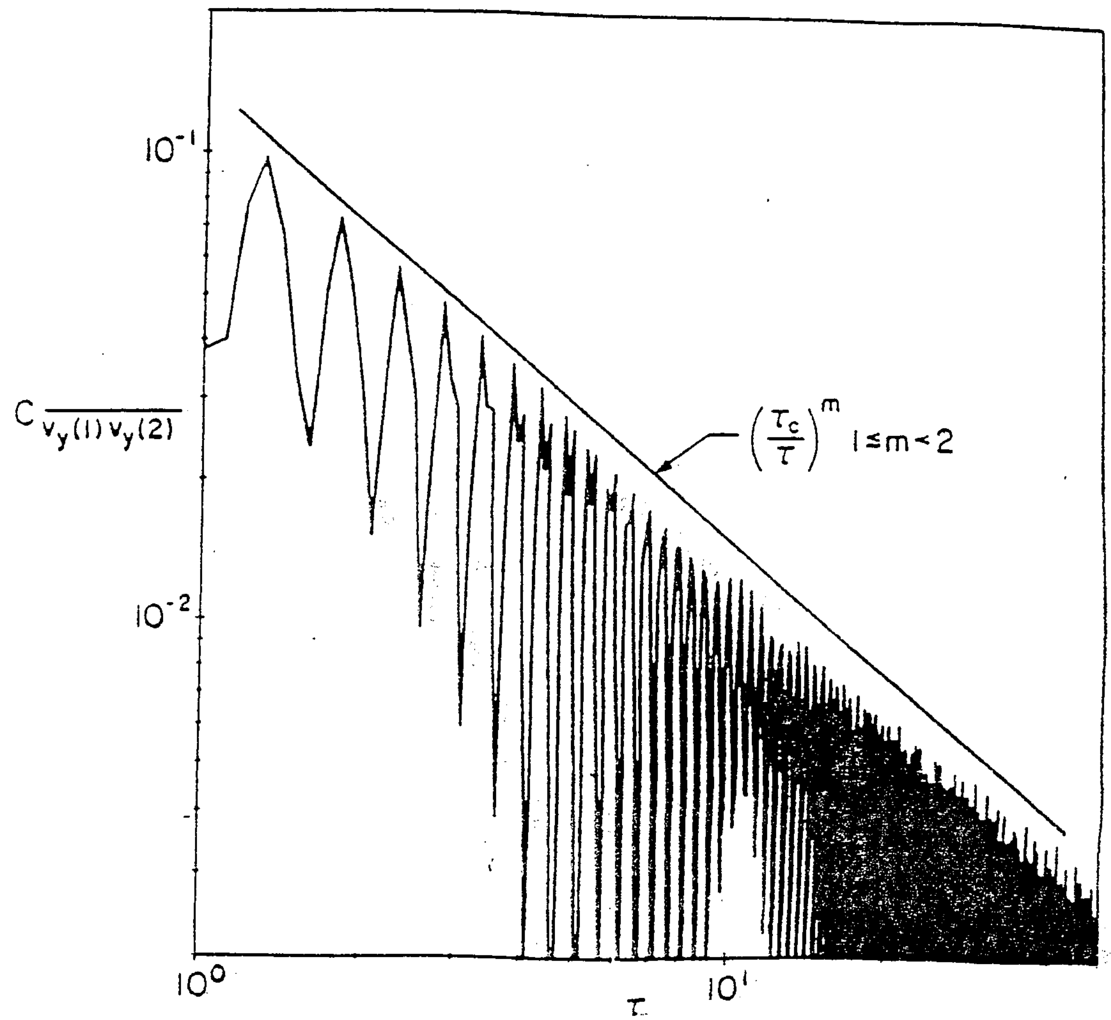


Fig.4

Power Law decay of two time velocity correlation.