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Abstract

The effect of equilibrium velocity shear on the resistive tearing instability has been systematically studied, using the boundary layer approach. Both the constant-ψ tearing mode, which has a growth rate that scales as $S^{-3/5}$, and the nonconstant-ψ tearing mode $\left(\Delta'(\alpha S)^{-1/3} \gg 1\right)$, which has a growth rate that scales as $S^{-1/3}$, are analyzed in the presence of flow. Here S is the usual ratio of the resistive diffusion and Alfvén times. It is found that the shear flow has a significant influence on both the external ideal region and the internal resistive region. In the external ideal region, the shear flow can dramatically change the value of the matching quantity Δ'. In the internal resistive region, the tearing mode is sensitive to the flow shear at the magnetic null plane: $G'(0)$. When $G'(0)$ is comparable with the magnetic field shear, $F'(0)$, the scalings of the "constant-ψ" tearing mode are changed and the $\Delta' > 0$ instability criterion is removed, provided $G'(0)G''(0) - F'(0)F''(0) \neq 0$. The scalings of the "nonconstant-ψ" tearing mode remain unchanged. When the flow shear is larger than the magnetic field shear at the magnetic null plane, both tearing modes are stabilized. Finally, the transition to ideal instability is discussed.

I. Introduction

The resistive tearing mode has been studied both analytically[1] and numerically.[2] In the absence of equilibrium flow, two kinds of tearing modes have been found: the "constant-ψ" tearing mode, whose growth rate scales as $S^{-3/5}$ and the "nonconstant-ψ" tearing mode, whose growth rate scales as $S^{-1/3}$, where S is the usual ratio of the resistive diffusion and Alfvén times. Since without resistivity the magnetic field is frozen into the flow, in the case of small resistivity it is to be expected that shear slow will have a profound influence on the tearing mode. This problem is of interest for laboratory and magnetospheric plasmas.[3,4,5] Because of the difficulty of this problem, approximations involving the many relevant parameters have been developed. Assuming the flow and Alfvén velocity have approximately the same spatial profile, Hofmann[3] derived a dispersion relation in which growth rate scales like $S^{-1/2}$. Paris and Sy[4] found that the scaling remains unchanged when the flow is significantly below the Alfvén speed. Dobrowolny et al.,[5] by using the "frozen-in" equation for the internal solutions, have shown the possible existence of a number of scalings with and without viscosity. Bondeson and Persson[6] used the "constant-ψ" approximation and Fourier transformed the internal equation in order to study the problem with and without viscosity. All of the above discussions pertain to the "constant-ψ" tearing mode. To our knowledge, no one has studied the effect of shear flow on the "nonconstant-ψ" tearing mode. Also, except for Hofmann,[3] the important effect of shear flow on the external ideal region has not been considered. Einaudi and Rubini[7] have studied the problem numerically. They do not find instabilities when the flow shear is large, in contrast to the results of Paris and Sy[4] and Bondeson and Persson.[6] In both Ref. 7 and Ref. 8, a transition to ideal instability is observed.

In this paper, we adopt the boundary layer approach to study the resistive tearing mode in the presence of shear flow. Both the "constant-ψ" and "nonconstant-ψ" tearing modes are treated. By introducing an assumption similar to that of Ref. 3, and carefully comparing the orders of the parameters involved, we arrive at general conclusions. In Table I we summarize the main conclusions that arise primarily from the affect of shear flow on the internal resistive region. Also, this table describes the transition to ideal instability. An additional main result of this paper is the recognition that the presence of flow affects the analysis in the external ideal region. Flow can drastically change the value of Δ'. We are able to explain the numerical results obtained in Refs. 7 and 8. In our present work, viscosity is neglected, but this possibly important effect is left to the next paper.

In the next section, the basic equations are written down, and the notations are indicated. Section III is devoted to the shear flow affect on the external solutions. In Sec. IV, we discuss the internal solution in the limits of slow growth and fast growth, which correspond to the "constant-ψ" and "nonconstant-ψ" tearing modes, respectively. Also, we consider the limits of small flow shear and flow shear that is comparable to magnetic shear at the magnetic null plane. Comparisons are made with previous work. In Sec. V, we discuss the transition to ideal instability. This is followed by a conclusion.

II. Basic Equations

We consider an incompressible plasma with uniform resistivity and density in the case of plane slab geometry. The starting point is the magnetohydrodynamic (MHD) model

$$\begin{aligned}
\rho\left(\frac{\partial \vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}\right) &= -\nabla p+\frac{1}{4\pi}\left(\nabla\times\vec{B}\right)\times\vec{B} \\
\frac{\partial \vec{B}}{\partial t}=-\nabla\times\vec{E} &= \frac{\eta}{4\pi}\nabla^2\vec{B}+\nabla\times\left(\vec{v}\times\vec{B}\right) \\
\nabla\cdot\vec{B} &= 0 \\
\nabla\cdot\vec{v} &= 0.
\end{aligned}$$

We assume the equilibrium quantities depend only on y. The equilibrium magnetic field and flow velocity are given respectively by

$$\vec{B}_0(y) = \hat{x}B_{ox}(y) + \hat{z}B_{oz}(y)$$

$$\vec{v}_0(y) = \hat{x}v_{ox}(y) + \hat{z}v_{oz}(y).$$

Since we are interested in the problem where the time scale is much smaller than the magnetic diffusion time scale, one can assume an ideal equilibrium and neglect the effect of $\frac{\eta}{4\pi}\nabla^2\vec{B}_0(y)$ in the equilibrium equations. With the assumed forms of $\vec{v}_0$ and $\vec{B}_0$, an ideal equilibrium exists provided there is a zeroth order pressure to support the magnetic field: $p_0(y) + \frac{B_0^2}{8\pi} =$ constant.

Now consider the linearization. We denote perturbed quantities by the subscript 1 and write them as follows:

$$f_1(\vec{r}, t) = f_1(y)\exp\left[i\left(k_x x + k_z z\right) + i\omega t\right].$$

The linearized equations, obtained by neglecting terms of second order, are

$$(\gamma + i\alpha G)\left(W'' - \alpha^2 W\right) - i\alpha G'' W = i\alpha F\left(\psi'' - \alpha^2\psi\right) - i\alpha F''\psi \tag{1}$$

$$(\gamma + i\alpha G)\psi - i\alpha F W = S^{-1}\left(\psi'' - \alpha^2\psi\right), \tag{2}$$

where we have used the following definitions:

$$\begin{array}{lll} \psi = \dfrac{B_{1y}}{B} & W = \dfrac{v_{1y}}{v_A} & v_A = \dfrac{B}{\sqrt{4\pi\rho}} \\ F = \dfrac{\vec{k}\cdot\vec{B}_0}{kB} & G = \dfrac{\vec{k}\cdot\vec{v}_0}{kv_A} & k = \left|\vec{k}\right| = \sqrt{k_x^2 + k_z^2} \\ \mu = \dfrac{y}{a} & \alpha = ka & \gamma = i\omega\tau_H \\ \tau_R = \dfrac{4\pi a^2}{\eta} & \tau_H = \dfrac{\sqrt{4\pi\rho}a}{B} & S = \dfrac{\tau_R}{\tau_H} \gg 1. \end{array} \tag{3}$$

Here prime denotes differentiation with respect to μ, B is a measure of the magnetic field, and a is the magnetic shear length. The above scalings are conventional,[1,4] except we have scaled the growth rate γ by τ_H, the Alfvén time, rather than the resistive diffusion time, and G and W, which represent respectively the equilibrium velocity and perturbed velocity, are scaled by the Alfvén velocity.

Defining $u = \gamma/\alpha + iG$ and $w = iW/u$, Eqs. (1) and (2) become

$$\left(u^2 w'\right)' - \alpha^2 u^2 w = -\left[F\left(\psi'' - \alpha^2\psi\right) - F''\psi\right] \tag{4}$$

$$u(\psi - Fw) = (\alpha S)^{-1}\left(\psi'' - \alpha^2\psi\right). \tag{5}$$

In the case of ideal MHD ($S \to \infty$), and Eqs. (4) and (5) reduce to

$$\left[\left(u^2 + F^2\right) w'\right]' - \alpha^2\left(u^2 + F^2\right) w = 0. \tag{6}$$

Extending Eq. (6) into the complex μ-plane, we note the presence of a singularity that occurs at a point where

$$u^2 + F^2 = \left(\frac{\gamma}{\alpha} + iG\right)^2 + F^2 = 0. \tag{7}$$

If we assume a magnetic null plane occurs at $\mu = 0$; i.e., $F(0) = 0$, then by selecting an appropriate reference frame, we can always let the equilibrium velocity be zero at $\mu = 0$; i.e., $G(0) = 0$. In the case of very small growth rate γ, the singularities of ideal MHD occur near $\mu = 0$. For the tearing mode under discussion, the small resistivity is only important in a thin layer around the ideal MHD singularity where $\mu = 0$. As in the usual tearing mode treatment, we adopt the boundary layer approach. In the discussion below, we assume $F'(0) \neq 0$, and without loss of generality $F'(0) > 0$. Also, we assume $\alpha \lesssim O(1)$, $G''(0)/F'(0) \lesssim O(1)$, and $F''(0)/F'(0) \lesssim O(1)$.

III. External Ideal Region

Away from the singularity discussed above, we can neglect resistivity ($S \to \infty$). This external ideal region is treated here with the assumption that the growth rate scales as follows:

$$\gamma \sim S^{-\sigma} \quad (0 < \sigma < 1).$$

Equations (4) and (5) reduce to

$$\begin{aligned} \psi - Fw &= 0 & (8) \\ \left[\left(F^2 - G^2\right) w'\right]' - \alpha^2 \left(F^2 - G^2\right) w &= 0. & (9) \end{aligned}$$

Now consider the behavior of Eqs. (8) and (9) as $\mu \to 0$. Taylor expanding the functions F and G, keeping the leading term in Eq. (8) and keeping terms to $O\left(\mu^3\right)$ in Eq. (9) yields

$$\psi \sim F'(0)\mu w \tag{10}$$

$$\begin{aligned} &\left\{\left[\left(F'(0)^2 - G'(0)^2\right)\mu^2 + \left(F'(0)F''(0) - G'(0)G''(0)\right)\mu^3\right] w'\right\}' \\ &-\alpha^2 \left[\left(F'(0)^2 - G'(0)^2\right)\mu^2 + \left(F'(0)F''(0) - G'(0)G''(0)\right)\mu^3\right] w \sim 0. \end{aligned} \tag{11}$$

The reason we retain the term $O\left(\mu^3\right)$ in Eq. (11) is to resolve the behavior for the case $F'^2(0) \sim G'^2(0)$.

Assuming $F'(0)^2 - G'(0)^2 \neq 0$, the solutions of Eqs. (10) and (11) behave as follows near $\mu = 0$

$$\begin{aligned} w &\sim \frac{C_0}{\mu}\left(1 + \frac{F'(0)F''(0) - G'(0)G''(0)}{F'(0)^2 - G'(0)^2}\mu \ln|\mu|\right) + C_{1\pm} + \cdots \\ \psi &\sim F'(0)C_0\left(1 + \frac{F'(0)F''(0) - G'(0)G''(0)}{F'(0)^2 - G'(0)^2}\mu \ln|\mu|\right) + F'(0)C_{1\pm}\mu + \cdots . \end{aligned}$$

Formally, this solution is the same as the case without flow $(G'(0) = G''(0) = 0)$, thus we can still define the matching quantity Δ'

$$\Delta' = \frac{1}{\psi}\frac{d\psi}{d\mu}\Bigg|_{0^-}^{0^+} = \frac{C_{1+} - C_{1-}}{C_0}.$$

Note that Eqs. (8) and (9) have the same structure as those without shear flow, although they differ by the presence of the term with G^2. Thus the shear flow can have as much influence on Δ' as the magnetic field. This is not a surprise, since in this region the magnetic field is frozen into the flow. Hofmann[3] has made some general comments on the shift of the wavenumber α_0, where $\Delta'(\alpha_0) = 0$, caused by shear flow. In Appendix A we consider two examples that demonstrate the importance of shear flow when calculating Δ'.

To conclude this section, we obtain the constraint on the internal solutions that is imposed by the external solutions. To this end, we assume that the internal scale length is ϵ, where $\epsilon \ll 1$. In the border between the internal and external regions, we obtain from Eq. (10)

$$\psi \sim F'(0)\epsilon w \sim \frac{iF'(0)\epsilon}{\gamma/\alpha + iG'(0)\epsilon} W. \tag{12}$$

Since the internal region is very thin, we can say that throughout, ψ, w, and W scale as in relation (12). This is something similar to what Dobrowolny et al.[5] called "use of the frozen-in law for internal solutions." In the case of no shear flow, this reduces to

$$\psi \sim \frac{i\alpha F'(0)\epsilon}{\gamma} W,$$

which is the assumption adopted in FKR.[1]

IV. Internal Resistive Region

The internal resistive region is so thin that the derivatives of ψ and W are very sensitive to the variation of ψ and W in this region. This suggests the introduction of a stretched variable ζ, defined as

$$\zeta = \frac{\mu}{\epsilon},$$

where, as noted above, ϵ is the scale length of the internal region. Using Eqs. (1) and (2), the rescaled internal equations become

$$\begin{aligned} &\left(\frac{\gamma}{\alpha F'(0)\epsilon} + i\frac{G'(0)}{F'(0)}\zeta + \frac{1}{2}i\frac{G''(0)}{F'(0)}\epsilon\zeta^2\right)\frac{\partial^2 W}{\partial\zeta^2} - i\epsilon\frac{G''(0)}{F'(0)}W = \\ &\qquad \left(i\zeta + \frac{1}{2}i\frac{F''(0)}{F'(0)}\epsilon\zeta^2\right)\frac{\partial^2\psi}{\partial\zeta^2} - i\epsilon\frac{F''(0)}{F'(0)}\psi + O\left(\epsilon^2\right) \end{aligned} \tag{13}$$

$$\left(\frac{\gamma}{\alpha F'(0)\epsilon}+i\frac{G'(0)}{F'(0)}\zeta+\frac{1}{2}i\frac{G''(0)}{F'(0)}\epsilon\zeta^2\right)\psi-\left(i\zeta+\frac{1}{2}i\frac{F''(0)}{F'(0)}\epsilon\zeta^2\right)W=$$
$$\left(\alpha F'(0)\epsilon^3 S\right)^{-1}\frac{\partial^2\psi}{\partial\zeta^2}+O\left(\epsilon^2\right). \qquad (14)$$

Because of the difficulty in solving the above equations directly, we are going to discuss them in different parameter limits. There are two parameters of interest. The first is $\left|\frac{\gamma}{\alpha F'(0)\epsilon}\right|$, which is the ratio of the local Alfvén time to the anticipated growth time, $1/\gamma$. Equivalently, this parameter is the ratio of the growth "phase velocity" to the Alfvén velocity in the resistive region. The second parameter is $|G'(0)/F'(0)|$, which is the ratio of the flow shear to the magnetic field shear at the magnetic null plane. We consider two cases: case A has $|\gamma/\alpha F'(0)\varepsilon| \ll 1$. Here the growth time is assumed to be long compared to the local Alfvén time scale. We refer to this as slow growth. Case B has $|\gamma/\alpha F'(0)\varepsilon| \sim 1$ which we term fast growth. The case

$$\left|\frac{\gamma}{\alpha F'(0)\epsilon}\right| \gg 1,$$

where the growth time scale is in the global Alfvén regime, i.e.,

$$\left|\frac{\gamma}{\alpha F'(0)a}\right| \sim 1$$

is not discussed in this paper.

A. Slow growth; $\left|\frac{\gamma}{\alpha F'(0)\epsilon}\right| \ll 1$.

As noted above, in this limit, the anticipated growth time scale is assumed to be much longer than the local Alfvén time scale. We expect that magnetic diffusion is going to be effective on this time scale. In the case of no flow, this limit corresponds to the classical "constant-ψ" tearing mode, which has a growth rate that scales as $S^{-3/5}$. Below we consider the problem in two different flow shear limits. In both cases, since the quantity that we want to match is $\psi \sim C_0 + C_\pm\mu = C_0 + C_\pm\epsilon\zeta$, order ϵ is the highest order that is matched.

1. Very small internal flow shear

In this limit,

$$\left|\frac{G'(0)}{F'(0)}\right| < \left|\frac{\tilde{\gamma}}{\alpha F'(0)\epsilon}\right|,$$

i.e., the flow shear is so small that the inertial terms still dominate the convection terms in the resistive region. Thus it is expected that the shear flow will lnot change the internal ordering in this limit.

We find it convenient to introduce a new variable φ defined by

$$\varphi = \left(W - \frac{G'(0)}{F'(0)}\psi\right) \Big/ \left(-i\frac{\gamma}{\alpha F'(0)\epsilon}\right). \tag{15}$$

Using (12), the constraint imposed by the outer solution, implies $\varphi \sim \psi$. Now, let

$$\gamma = \tilde{\gamma}\hat{\gamma}, \tag{16}$$

where $\tilde{\gamma}$ is the measure of γ, and $\hat{\gamma}$ is a factor of $O(1)$. Equations (13) and (14) become:

$$\begin{aligned}\left(\frac{\tilde{\gamma}}{\alpha F'(0)\epsilon}\right)^2 \hat{\gamma}\left(\hat{\gamma} + i\frac{\partial G'(0)\epsilon}{\tilde{\gamma}}\right)\frac{\partial^2\varphi}{\partial\zeta^2} &= -\left[\zeta\left(1 - \frac{G'(0)^2}{F'(0)^2}\right) + i\frac{\tilde{\gamma}\hat{\gamma}}{\alpha F'(0)\epsilon}\frac{G'(0)}{F'(0)}\right. \\ &+ \left.\epsilon\zeta^2\frac{F''(0)}{2F'(0)}\right]\frac{\partial^2\psi}{\partial\zeta^2} + \epsilon\frac{F''(0)}{F'(0)}\psi + O\left(\frac{\tilde{\gamma}}{\alpha F'(0)\epsilon}\epsilon\right) \end{aligned}\tag{17}$$

$$\left(\frac{\tilde{\gamma}}{\alpha F'(0)\epsilon}\right)^2 \hat{\gamma}(\psi - \zeta\varphi) = \frac{\tilde{\gamma}}{(\alpha F'(0))^2\epsilon^4 S}\frac{\partial^2\psi}{\partial\zeta^2} + O\left(\frac{\tilde{\gamma}}{\alpha F'(0)\epsilon}\epsilon\right). \tag{18}$$

Using $\left(\frac{\tilde{\gamma}}{\alpha F'(0)\epsilon}\right)^2$ as a small parameter, we expand φ and ψ as follows:

$$\begin{aligned}\varphi &= \sum_{n=0}^{\infty}\left(\frac{\tilde{\gamma}}{\alpha F'(0)\epsilon}\right)^{2n}\varphi_n \\ \psi &= \sum_{n=0}^{\infty}\left(\frac{\tilde{\gamma}}{\alpha F'(0)\epsilon}\right)^{2n}\psi_n.\end{aligned}\tag{19}$$

Consistency at leading order requires that

$$\frac{\tilde{\gamma}}{(\alpha F'(0))^2\epsilon^4 S} \sim O(1),$$

or

$$\left|\tilde{\gamma}\epsilon^2 S\right| \sim \left|\left(\frac{\tilde{r}}{\alpha F'(0)\epsilon}\right)^2\right| \ll 1,$$

which implies the resistive skin diffusion time is much shorter than the anticipated growth time. For convenience, we set

$$\frac{\tilde{\gamma}}{\left(\alpha F'(0)\right)^2 \epsilon^4 S} = 4. \tag{20}$$

Thus $\tilde{\gamma}$ is the growth rate for the flow free tearing mode. Inserting Eq. (19) into Eqs. (17) and (18) yields to leading order

$$\frac{\partial^2 \psi_0}{\partial \zeta^2} = 0; \tag{21}$$

hence, $\psi_0 = C_0 + C_1\zeta$. It is evident from Sec. III that matching to the external solutions can only be achieved if

$$\lim_{|\zeta|\to\infty} \psi_0 \quad = \quad \text{const.}$$

This implies $C_1 = 0$ and,

$$\psi_0 = C_0 \quad = \quad \text{const.} \tag{22}$$

This is the so-called "constant-ψ approximation."

In first order we obtain the equations

$$\hat{\gamma}\left(\hat{\gamma} + i\frac{\alpha G'(0)\epsilon}{\tilde{\gamma}}\zeta\right)\frac{\partial^2 \psi_0}{\partial \zeta^2} \quad = \quad -\zeta\frac{\partial^2 \psi_1}{\partial \zeta^2} + \frac{\epsilon}{\left(\frac{\tilde{\gamma}}{\alpha F'(0)\epsilon}\right)^2}\frac{F''(0)}{F'(0)}\psi_0 \tag{23}$$

$$\hat{\gamma}\left(\psi_0 - \zeta\varphi_0\right) \quad = \quad 4\frac{\partial^2 \psi_1}{\partial \zeta^2}. \tag{24}$$

Let $\hat{\gamma}\varphi_0 = -h/4$, $\psi_0 = 1$, and define

$$\lambda \quad = \quad \frac{\alpha G'(0)\epsilon}{\tilde{\gamma}} = \frac{G'(0)}{F'(0)}\frac{\alpha F'(0)\epsilon}{\tilde{\gamma}} \gg \frac{G'(0)}{F'(0)}$$

$$\lambda_F \quad = \quad \frac{4\epsilon F''(0)/F'(0)}{\left(\frac{\tilde{\gamma}}{\alpha F'(0)\epsilon}\right)^2} = \frac{1}{\tilde{\gamma}\epsilon S}\frac{F''(0)}{F'(0)}.$$

Using the above definition Eqs. (23) and (24) become

$$(\hat{\gamma} + i\lambda\zeta)\frac{\partial^2 h}{\partial\zeta^2} - \frac{1}{4}\zeta^2 h = \hat{\gamma}\zeta - \lambda_F \tag{25}$$

$$\frac{\partial^2\psi_1}{\partial\zeta^2} = \frac{1}{4}\left(\hat{\gamma} + \frac{1}{4}\zeta h\right). \tag{26}$$

These equations are equivalent to those obtained in Ref. 4, which yield the following upon enforcing matching to the external solutions

$$\begin{aligned}
\tilde{\gamma} &= \left(\frac{\Gamma(1/4)}{2\pi\Gamma(3/4)}\right)^{4/5}\left(\alpha F'(0)\Delta'^2\right)^{2/5} S^{-3/5},\\
\epsilon &= \left(\frac{\Gamma(1/4)}{2^{7/2}\Gamma(3/4)\pi}\right)^{1/5}\left(\alpha F'(0)\right)^{-2/5}\Delta'^{1/5}S^{-2/5},\\
\hat{\gamma} &= 1 - \frac{2}{5}i\lambda\lambda_F + \lambda^2\left(\frac{\pi}{16} + \frac{7}{50}\lambda_F^2\right) + O\left(\lambda^3\right),
\end{aligned} \tag{27}$$

where $\Gamma(x)$ is the gamma function. In the case where the internal flow shear is very small, the internal analysis remains the same as that without flow, as a result the scaling is unchanged. From the results of Eqs. (27), Paris and Sy[4] and Bondeson and Persson[6] conclude that small flow shear destabilizes the tearing mode. Even though the flow shear at the magnetic null plane is small, the flow in the external region could be large, in which case there is a significant influence on the value of Δ', as discussed in Appendix A.

Now let us check our assumptions. For the expansion in Eqs. (19) to be valid, we require

$$\left|\frac{\gamma}{\alpha F'(0)\epsilon}\right|^2 \sim (\alpha S)^{-2/5}\left|\Delta'\right|^{6/5} \sim \epsilon\left|\Delta'\right| \ll 1. \tag{28}$$

For the boundary layer approach to be valid, we must have $\epsilon \ll 1$; i.e.,

$$(\alpha S)^{-2/5}\left|\Delta'\right|^{1/5} \ll 1, \tag{29}$$

which implies the resistivity must be very small. When Δ' is very large, the above assumptions are not valid. When α is very small, we assume $\Delta' \sim 1/\alpha$ and Eq. (28) yields

$$\alpha \gg S^{-1/4}, \tag{30}$$

which is consistent with the limit obtained by FKR[1] for "constant-ψ" tearing mode in the case of no flow.

2. Comparable internal flow shear

In this limit we suppose $|G'(0)/F'(0)| \sim O(1)$, which implies that now the convection terms dominate the inertial terms in the resistive region. Thus there is a change in the tearing mode ordering. This limit has been studied by Hofmann[3] with the assumptions $G''(0) = 0$, $F''(0) = 0$, and $1 - G'(0)^2/F'(0)^2 > 0$. But here we remove these constraints. Equation (12) implies here $\psi \sim W$, and now in Eqs. (13) and (14), the natural expansion parameter is $\left(\frac{\gamma}{\alpha F'(0)\epsilon}\right)$, instead of $\left(\frac{\gamma}{\alpha F'(0)\epsilon}\right)^2$. Thus the equations analogous to Eqs. (19) are

$$\begin{aligned} \psi &= \sum_{n=0}^{\infty} \left(\frac{\gamma}{\alpha F'(0)\epsilon}\right)^n \psi_n \\ W &= \sum_{n=0}^{\infty} \left(\frac{\gamma}{\alpha F'(0)\epsilon}\right)^n W_n. \end{aligned} \tag{31}$$

This is the same expansion as that adopted by Hofmann.[3] Similarly, we assume $(\alpha F'(0)\epsilon^3 S)^{-1} \sim O(1)$; i.e.,

$$\left|\gamma\epsilon^2 S\right| \sim \left|\frac{\gamma}{\alpha F'(0)\epsilon}\right| \ll 1,$$

which implies the internal resistive skin time is much shorter than the anticipated growth time. For definiteness we choose $(\alpha F'(0)\epsilon^3 S)^{-1} = 1$, which implies that the internal scale length

$$\epsilon = (\alpha F'(0)S)^{-1/3}. \tag{32}$$

To leading order, the solutions that match to the external solutions, are

$$\frac{G'(0)}{F'(0)}\psi_0 = W_0 = \text{const.} \tag{33}$$

To first order Eqs. (13) and (14) yield

$$\frac{G'(0)}{F'(0)}\frac{\partial^2 W_1}{\partial \zeta^2} = \frac{\partial^2 \psi_1}{\partial \zeta^2} \tag{34}$$

$$\psi_0 + i\zeta\left(\frac{G'(0)}{F'(0)}\psi_1 - W_1\right) = \frac{\partial^2\psi_1}{\partial\zeta^2}. \tag{35}$$

Equation (34) implies $W_1 = \frac{F'(0)}{G'(0)}\psi_1$ (generality is not lost by dropping the two integration constants). We insert this into Eq. (35) and obtain

$$\psi_0 - i\zeta\frac{F'(0)}{G'(0)}\left(1 - \frac{G'(0)^2}{F'(0)^2}\right)\psi_1 = \frac{\partial^2\psi_1}{\partial\zeta^2}. \tag{36}$$

This is an inhomogeneous Airy equation, which has the following solution[9]:

$$\psi_1 = e^{-i4m\pi/3}\lambda^2\pi\psi_0 Hi\left(-\lambda^{-1}\zeta e^{i2m\pi/3}\right), \tag{37}$$

where m is an integer,

$$\lambda = \left[i\frac{F'(0)}{G'(0)}\left(1 - \frac{G'(0)^2}{F'(0)^2}\right)\right]^{-1/3},$$

and $Hi(\zeta)$ is the inhomogeneous Airy function. It is algebraic for large $|\zeta|$ when $\left|\arg\left(e^{i2m\pi/3}\lambda^{-1}\zeta\right)\right| < 2\pi/3$. Choosing $m = 0$ if

$$\frac{F'(0)}{G'(0)}\left(1 - \frac{G'(0)^2}{F'(0)^2}\right) < 0$$

and $m = -1$ if

$$\frac{F'(0)}{G'(0)}\left(1 - \frac{G'(0)^2}{F'(0)^2}\right) > 0,$$

yields a solution that is algebraic in a sector which includes the real axis. For large $|\zeta|$, the asymptotic behavior satisfies

$$\psi_1 \sim \frac{\lambda^3\psi_0}{\zeta}\left(1 + O\left(\frac{1}{\zeta^3}\right)\right), \tag{38}$$

which is valid in the sector $-7\pi/6 < \arg\zeta < \pi/6$, when

$$\frac{F'(0)}{G'(0)}\left(1 - \frac{G'(0)^2}{F'(0)^2}\right) < 0,$$

while if

$$\frac{F'(0)}{G'(0)}\left(1 - \frac{G'(0)^2}{F'(0)}\right) > 0,$$

it is valid in the sector $-\pi/6 < \arg\zeta < 7\pi/6$.

To second order, Eqs. (13) and (14) yield

$$\frac{G'(0)}{F'(0)}\frac{\partial^2 W_2}{\partial\zeta^2} = \frac{\partial^2\psi_2}{\partial\zeta^2} + \frac{i\frac{G'(0)}{F'(0)}\frac{\partial^2\psi_1}{\partial\zeta^2}}{\zeta - i\frac{\gamma}{\alpha G'(0)\epsilon}} - \frac{\epsilon}{\left(\frac{\gamma}{\alpha F'(0)\epsilon}\right)^2}\frac{\frac{F''(0)}{F'(0)}\psi_0 - \frac{G''(0)}{F'(0)}W_0}{\zeta - i\frac{\gamma}{\alpha G'(0)\epsilon}} \tag{39}$$

$$\psi_1 + i\frac{G'(0)}{F'(0)}\zeta\psi_2 - i\zeta W_2 + \frac{1}{2}i\frac{\epsilon}{\left(\frac{\gamma}{\alpha F'(0)\epsilon}\right)^2}\left(\frac{G''(0)}{F'(0)}\psi_0 - \frac{F''(0)}{F'(0)}W_0\right) = \frac{\partial^2\psi_2}{\partial\zeta^2}. \tag{40}$$

Note, we have kept the singularity at $\zeta = i\frac{\gamma}{\alpha G'(0)\epsilon}$ in Eq. (39). Equation (39) yields upon insertion of Eq. (35), and integration

$$\begin{aligned}\frac{G'(0)}{F'(0)}\int_{-\infty}^{\infty}\frac{\partial^2 W_2}{\partial\zeta^2}d\zeta &= \int_{-\infty}^{\infty}\frac{\partial^2\psi_2}{\partial\zeta^2}d\zeta + i\int_{-\infty}^{\infty}\frac{\psi_0 - \lambda^{-3}\psi_1\zeta}{\zeta - i\frac{\gamma}{\alpha G'(0)\epsilon}}d\zeta \\ &- \frac{\epsilon}{\left(\frac{\gamma}{\alpha F'(0)\epsilon}\right)^2}\int_{-\infty}^{\infty}\frac{d\zeta}{\zeta - i\frac{\gamma}{\alpha G'(0)\epsilon}}\frac{F'(0)F''(0) - G'(0)G''(0)}{F'(0)^2}\psi_0. \end{aligned} \tag{41}$$

Now consider each of the integrals of Eq. (41):

$$\int_{-\infty}^{\infty}\frac{\partial^2\psi_2}{\partial\zeta^2}d\zeta = \frac{\partial\psi_2}{\partial\zeta}\Big|_{-\infty}^{\infty} = \epsilon\Delta'\psi_0\Big/\left(\frac{\gamma}{\alpha F'(0)\epsilon}\right)^2, \tag{42}$$

where Δ' is the matching quantity defined in Sec. III. From Eq. (40) we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{\partial^2 W_2}{\partial\zeta^2}d\zeta &= \int_{-\infty}^{\infty}\frac{G'(0)}{F'(0)}\frac{\partial^2\psi_2}{\partial\zeta^2}d\zeta \\ &= \frac{G'(0)}{F'(0)}(\epsilon\Delta'\psi_0)\Big/\left(\frac{\gamma}{\alpha F'(0)\epsilon}\right)^2; \end{aligned} \tag{43}$$

$$\int_{-\infty}^{\infty}\frac{d\zeta}{\zeta - i\frac{\gamma}{\alpha G'(0)\epsilon}} = \pm i\pi, \tag{44}$$

where we take upper sign when $\frac{\mathrm{Re}(\gamma)}{G'(0)} > 0$, otherwise lower sign is taken. Using the results of Eq. (38)

$$\int_{-\infty}^{\infty}\frac{\psi_0 - \lambda^{-3}\psi_1\zeta}{\zeta - i\frac{\gamma}{\alpha G'(0)\epsilon}}d\zeta = \int_{\Gamma}\frac{\psi_0 - \lambda^{-3}\psi_1\zeta}{\zeta - i\frac{\gamma}{\alpha G'(0)\epsilon}}d\zeta,$$

where Γ is a contour in complex ζ plane. If

$$\frac{F'(0)}{G'(0)}\left(1-\frac{G'(0)^2}{F'(0)^2}\right) \; < \; 0,$$

Γ is closed in the lower half plane, while if

$$\frac{F'(0)}{G'(0)}\left(1-\frac{G'(0)^2}{F'(0)^2}\right) \; > \; 0,$$

Γ is closed in the upper half plane. So if

$$\frac{F'(0)}{G'(0)}\left(1-\frac{G'(0)^2}{F'(0)^2}\right)\frac{\mathrm{Re}(\gamma)}{G'(0)} \; < \; 0,$$

then

$$\int_{-\infty}^{\infty}\frac{\psi_0-\lambda^{-3}\psi_1\zeta}{z-i\frac{\gamma}{\alpha G'(0)\epsilon}}d\zeta \; = \; 0. \tag{45}$$

We obtain from Eq. (41) upon insertion of Eqs. (42), (43), (44), and (45)

$$\Delta' = \pm i\pi\frac{F''(0)F'(0)-G'(0)G''(0)}{F'(0)^2-G'(0)^2}.$$

Since a pure imaginary Δ' cannot be made equal to the external real Δ', matching cannot be achieved in this case.

If

$$\frac{F'(0)}{G'(0)}\left(1-\frac{G'(0)^2}{F'(0)^2}\right)\frac{\mathrm{Re}(\gamma)}{G'(0)} > 0, \tag{46}$$

$$\int_{-\infty}^{\infty}\frac{\psi_0-\lambda^{-3}\psi_1\zeta}{z-i\frac{\gamma}{\alpha G'(0)\epsilon}}d\zeta = \pm 2\pi i\psi_0 + O\left(\frac{\gamma}{\alpha G'(0)\epsilon}\right), \tag{47}$$

where we take the upper sign when $F'(0)/G'(0)\,(1-G'(0)^2/F'(0)^2) > 0$, otherwise the lower sign is taken. We obtain from Eq. (41) upon insertion of Eqs. (42), (43), (44), and (47)

$$\gamma = \pm(2\pi S)^{-1/2}\sqrt{\left|\alpha G'(0)\left(1-\frac{G'(0)^2}{F'(0)^2}\right)\right|\left(\Delta' \mp i\pi\frac{F''(0)F'(0)-G'(0)G''(0)}{F'(0)^2-G'(0)^2}\right)}, \tag{48}$$

where the sign of growth rate γ is determined by Eq. (46). Obviously, the above analysis is not valid when $1-G'(0)^2/F'(0)^2 = 0$.

From Eq. (46) we see that only when

$$\left(1 - \frac{G'(0)^2}{F'(0)^2}\right) \;>\; 0$$

does there exist a growing tearing mode. When

$$\left(1 - \frac{G'(0)^2}{F'(0)^2}\right) \;<\; 0,$$

the kinetic energy overpowers the magnetic energy in the internal resistive region, the flow freezes the magnetic field and suppress the tearing instability. This is not necessarily accompanied by an ideal mode. Numerically, Einaudi and Rubini[7] solved the initial value problem for the following equilibrium profiles: $F = \tanh\mu$, $G = G_0 \tanh b\mu$. They found the same scaling as Eq. (48) when $|G'(0)/F'(0)| \sim O(1)$. However, they also found that the tearing mode could be stabilized at some value $|G'(0)/F'(0)| < 1$, instead of $|G'(0)/F'(0)| = 1$. This can be explained by the influence of shear flow on the value of Δ'.

For the hyperbolic tangent profiles, $F''(0)F'(0) - G''(0)G'(0) = 0$, and the negative value of Δ' can stabilize the tearing mode. In the first example of Appendix A, we evaluate Δ' for a piecewise linear approximation of tanh (cf. Eqs. (A7) and (A2)). Assume that G_0, the quantity that measures the magnitude of the flow, is less than unity. From Eqs. (A6), (A7) and Fig. 1, we conclude that when the flow shear length b is less than the magnetic shear length, but

$$\frac{G'(0)^2}{F'(0)^2} = \frac{G_0^2}{b^2} < 1,$$

then at some value of b, $\Delta' = 0$. This qualitatively explains the stabilization seen in the numerical works of Ref. 7.

The result of Eq. (48) is different from that of Ref. 3 in that the second derivatives of the magnetic field and shear flow are included. This is far from trivial since it removes the $\Delta' > 0$ instability criterion if $F'(0)F''(0) - G'(0)G''(0) \neq 0$. In Refs. 4 and 6, they arrived at a similar conclusion by neglecting $G''(0)$, but in their growth rate expression, they omitted the very important factor $1 - G'(0)^2/F'(0)^2$. Thus their growth rate does not stabilize when

$G'(0)^2/F'(0)^2 > 1$. The authors of Ref. 7 noticed the discrepancy between their numerical results and the growth rate expression of Ref. 4. Our result explains this discrepancy.

To end this section, let us check our assumptions. From Eqs. (32) and (48) we obtain

$$\left|\frac{\gamma}{\alpha F'(0)\epsilon}\right| \sim \left|\sqrt{\epsilon\left(1-\frac{G'(0)^2}{F'(0)^2}\right)\Delta'}\right| \ll 1. \tag{49}$$

The validity of our boundary layer approach requires $\epsilon \ll 1$; i.e.,

$$(\alpha S)^{-1/3} \ll 1, \tag{50}$$

which requires that the resistivity must be small. Equation (49) is similar to Eq. (28), it cannot be satisfied when $|\Delta'|$ is very large except in the case $1-\frac{G'(0)^2}{F'(0)^2} \rightarrow 0$. We consider the case of very large $|\Delta'|$ in the next section.

B. Fast growth; $\left|\frac{\gamma}{\alpha F'(0)\epsilon}\right| \sim 1$

In this limit, the anticipated time scale is comparable with the local Alfvén time. In the case of no flow, this limit corresponds to the "nonconstant-ψ" tearing mode with a growth rate that scales as $S^{-1/3}$.[2,10]

Equation (12) implies in this limit that $\psi \sim W$, assuming $|G'(0)/F'(0)| \lesssim O(1)$.

Neglecting the terms of order $O(\varepsilon)$, Eqs. (13) and (14) become

$$\left(\frac{\gamma}{\alpha F'(0)\epsilon}+i\frac{G'(0)}{F'(0)}\zeta\right)\frac{\partial^2 W}{\partial\zeta^2} = i\zeta\frac{\partial^2\psi}{\partial\zeta^2} \tag{51}$$

$$\left(\frac{\gamma}{\alpha F'(0)\epsilon}+i\frac{G'(0)}{F'(0)}\zeta\right)\psi - i\zeta W = \frac{1}{\alpha F'(0)\epsilon^3 S}\frac{\partial^2\psi}{\partial\zeta^2}. \tag{52}$$

Defining

$$A=\frac{\gamma}{\alpha F'(0)\epsilon}+i\frac{G'(0)}{F'(0)}\zeta, \qquad \phi=\frac{iW}{A},$$

Eqs. (51), (52) become

$$\frac{\partial}{\partial\zeta}\left(A^2\frac{\partial\phi}{\partial\zeta}\right) = -\zeta\frac{\partial^2\psi}{\partial\zeta^2} \tag{53}$$

$$A(\psi-\zeta\phi) = \frac{1}{\alpha F'(0)\epsilon^3 S}\frac{\partial^2\psi}{\partial\zeta^2}. \tag{54}$$

Integration of Eq. (53) yields

$$\zeta\frac{\partial\psi}{\partial\zeta}-\psi=-A^2\frac{\partial\phi}{\partial\zeta}+C_0\equiv X, \tag{55}$$

where C_0 is a constant and we have defined a new dependent variable X.

Substituting Eq. (55) into Eq. (54), we obtain

$$\frac{1}{\alpha F'(0)\epsilon^3 S}\left[\frac{\partial^2 X}{\partial\zeta^2}-\left(\frac{2}{\zeta}+\frac{\partial A/\partial\zeta}{A}\right)\frac{\partial X}{\partial\zeta}\right]=AX+\frac{\zeta^2}{A}(X-C_0). \tag{56}$$

In the case of very small flow shear, i.e., $|G'(0)/F'(0)|\ll 1$ we expand X and γ as

$$\begin{aligned} X &= \sum_{n=0}^{\infty} X_n\left(i\frac{G'(0)}{F'(0)}\right)^n \\ \gamma &= \sum_{n=0}^{\infty}\gamma_n\left(i\frac{G'(0)}{F'(0)}\right)^n. \end{aligned} \tag{57}$$

The leading order of Eq. (56) is

$$\frac{1}{\alpha F'(0)\epsilon^3 S}\left(\frac{\partial^2 X_0}{\partial\zeta^2}-\frac{2}{\zeta}\frac{\partial X_0}{\partial\zeta}\right)=\frac{\gamma_0}{\alpha F'(0)\epsilon}X_0+\frac{\alpha F'(0)\epsilon}{\gamma_0}\zeta^2(X_0-C_0). \tag{58}$$

This equation has been solved in Ref. 10 in the case of $\Delta'<0$. In Appendix B we treat the case of $\left|\Delta'(\alpha S)^{-1/3}\right|\gg 1$. In the case where $\Delta'\to\infty$, Eq. (58) has the solution

$$\begin{aligned} &C_0=0, \qquad X_0=e^{-\zeta^2/2}, \\ &\frac{\gamma_0}{\alpha F'(0)\epsilon}=\frac{1}{\alpha F'(0)\epsilon^3 S}=1. \end{aligned} \tag{59}$$

To first order Eq. (56) yields

$$\frac{\partial^2 X_1}{\partial\zeta^2}-\frac{2}{\zeta}\frac{\partial X_1}{\partial\zeta}-\left(1+\zeta^2\right)X_1=\frac{\partial X_0}{\partial\zeta}+\zeta\left(1+\frac{\gamma_1}{\gamma_0}\right)X_0-\zeta^2\left(\zeta+\frac{\gamma_1}{\gamma_0}\right)X_0.$$

The appropriate solution for the above equation is

$$\begin{aligned} \gamma_1 &= 0 \\ X_1 &= \frac{1}{6}\zeta^3 e^{-\zeta^2/2}. \end{aligned} \tag{60}$$

To the second order, Eq. (56) yields

$$\begin{aligned}\frac{\partial^2 X_2}{\partial\zeta^2} - \frac{2}{\zeta}\frac{\partial X_2}{\partial\zeta} - \left(1+\zeta^2\right) X_2 &= \frac{\partial X_1}{\partial\zeta} - \left(\zeta + \frac{\gamma_1}{\gamma_0}\right)\frac{\partial X_0}{\partial\zeta} + \left(\frac{\gamma_1}{\gamma_0}X_1 + \frac{\gamma_2}{\gamma_0}X_0 + \zeta X_1\right) \\ &+ \zeta^2\left[-\left(\zeta+\frac{\gamma_1}{\gamma_0}\right)X_1 - \frac{\gamma_2}{\gamma_0}X_0 + \left(\zeta+\frac{\gamma_1}{\gamma_0}\right)^2 X_0\right] \\ &= e^{-\zeta^2/2}\left[-\frac{1}{6}\zeta^6 + \zeta^4 + \left(\frac{3}{2} - \frac{\gamma_2}{\gamma_0}\right)\zeta^2 + \frac{\gamma_2}{\gamma_0}\right].\end{aligned}$$

The appropriate solutions of the above equations are

$$\begin{aligned}X_2 &= \frac{1}{72}\zeta^6 - \frac{3}{32}\zeta^4 - \frac{5}{16}\zeta^2 \\ \frac{\gamma_2}{\gamma_0} &= \frac{5}{8}.\end{aligned} \tag{61}$$

Collecting the results of Eqs. (59), (60), and (61) yields

$$\begin{aligned}\gamma &= \gamma_0 + i\frac{G'(0)}{F'(0)}\gamma_1 + \left(i\frac{G'(0)}{F'(0)}\right)^2\gamma_2 + \cdots \\ &= \left(\alpha F'(0)\right)^{2/3} S^{-1/3}\left(1 - \frac{5}{8}\frac{G'(0)^2}{F'(0)^2} + \cdots\right).\end{aligned} \tag{62}$$

Thus we see that shear flow in this ordering tends to stabilize the $\Delta' = \infty$ tearing mode.

For the case where $\Delta' \neq \infty$, but $\left|(\alpha S)^{-1/3}\Delta'\right| \gg 1$, there is a correction of $O\left(\frac{1}{(\alpha s)^{-1/3}\Delta'}\right)$ to the case of no flow (see Appendix B). Including a small shear flow, the growth rate is

$$\frac{\gamma}{\gamma_0} = 1 - \frac{5}{8}\frac{G'(0)^2}{F'(0)^2} + O\left(\frac{1}{(\alpha s)^{-1/3}\Delta'}\frac{G'(0)}{F'(0)}\right). \tag{63}$$

For sufficient large $|\Delta'|$, we can say that the small shear flow is stabilizing.

When $G'(0)/F'(0) \sim O(1)$, i.e. the convection terms are comparable with the inertial terms in the internal region, the quantity A in Eq. (56) is still $\sim O(1)$. The scaling should remain unchanged except the case $1 - G'(0)^2/F'(0)^2 \to 0$. When $1 - G'(0)^2/F'(0)^2 \to 0$, the scalings changed to the "constant-ψ" tearing mode scaling as we discussed in the end of last section. Thus the tearing mode with $\left|(\alpha S)^{-1/3}\Delta'\right| \gg 1$ is stabilized when $1 - G'(0)^2/F'(0)^2 \to$

0^-, approaching zero from below. It is reasonable to ascribe this result to the idea that the flow freezes the magnetic field and suppresses the tearing mode with $\left|(\alpha S)^{-1/3}\Delta'\right| \gg 1$ when $1 - G'(0)^2/F'(0)^2 < 0$. This conjecture agrees with the numerical results in Ref. 7.

Our assumption $\left|\frac{\gamma}{\alpha F'(0)\epsilon}\right| \sim 1$ is always satisfied if $|\epsilon\Delta'| \gg 1$ and $1 - G'(0)^2/F'(0)^2 \not\to 0$. This is seen by examination of Eqs. (59), (63), and (B15). The requirement of $\epsilon \ll 1$ leads to $(\alpha S)^{-1/3} \gg 1$, which requires very small resistivity.

V. Transition to Ideal Instability

Since shear flow itself can drive Kelvin-Helmholtz instability, a potentially powerful instability, the results of the preceding sections could be overshadowed. However, the necessary condition for this to happen is that the flow velocity not be bounded by the magnetic field everywhere, in any reference frame.[11] For all of the tearing modes treated here there exist velocity profiles that are Kelvin-Helmholtz stable.

An interesting case is where the Kelvin-Helmholtz instability is near marginality, since here its growth rate can be comparable to that of tearing. Also, the tearing analysis in the external ideal region corresponds to that of marginal ideal instability, so here is a natural place to begin tracking the transition from tearing to Kelvin-Helmholtz instability. In Refs. 7 and 8 this transition was tracked numerically as the appropriate values of the flow parameters were varied. The profiles considered were $F = \tanh\mu$ and either $G = G_0\,\mathrm{sech}(\mu/b)$ or $G = G_0\,\mathrm{sech}^2(\mu/b)$. In this section we track this transition analytically by expanding about the ideal instability at marginality.

For tractability we approximate the hyperbolic profiles by piecewise linear profiles, as discussed in example 2 of Appendix A, although we have changed frame so that $G(0) = 0$. These profiles are linear in three regions: $|\mu| < b$, $\mu > b$, and $\mu < -b$. Consider first the ideal problem with $b \leq 1$ and $|\mu| < b$. For convenience we define

$$H = \sqrt{u^2 + F^2}w; \tag{64}$$

hence, Eq. (6) can be rewritten as

$$\frac{d^2H}{d\mu^2} - \left[\alpha^2 - \frac{\omega^2/\alpha^2}{(\mu^2 - \omega^2/\alpha^2)^2}\right] H = 0, \tag{65}$$

where $\omega = -i\gamma$.

Marginal solution are given by solving the equation

$$\frac{d^2H_N}{d\mu^2} - \alpha^2 H_N = 0 \tag{66}$$

in the region $|\mu| < b$, and matching the solutions at $\mu = \pm b$ and $\mu = \pm 1$ to the appropriately decaying solution as $\mu \to \pm\infty$. Equation (66) has two solutions:

$$H_N^\infty = \sinh \alpha\mu \tag{67}$$

with the matching condition $\bar{\alpha} - \tanh \bar{\alpha} + \bar{\alpha}\beta \tanh \bar{\alpha} = 0$; and

$$H_N^0 = \cosh \alpha\mu \tag{68}$$

with the matching condition $1 - \bar{\alpha} \tanh \bar{\alpha} - \bar{\alpha}\beta = 0$. In the above expressions, $\bar{\alpha}$ and β are defined as in Eq. (A10). For the details of matching, we refer the reader to Appendix A. Hereafter, we denote the quantities corresponding to neutral solution by N, and fix the wavevector α.

From Eqs. (67), (68), (A3), and (A10), we see that H_N^∞ corresponds to the external tearing solution for the case where $\Delta' = \infty$; similarly, H_N^0 corresponds to $\Delta' = 0$.

Now upon multiplying Eq. (65) by H_N and Eq. (66) by H, and subtracting, we obtain

$$\frac{d}{d\mu}\left[H_N \frac{dH}{d\mu} - H\frac{dH_N}{d\mu}\right] = -\frac{\omega^2/\alpha^2}{[\mu^2 - \omega^2/\alpha^2]^2} H H_N. \tag{69}$$

Defining $y = w'/w$, yields with Eq. (64)

$$\frac{1}{H}\frac{dH}{d\mu} = y(\mu) + \frac{\mu}{\mu^2 - \omega^2/\alpha^2}. \tag{70}$$

Shortly, we will need to use Eq. (70).

Consider now, instability that is near to the neutral mode; i.e., $H \to H_N$ and $\omega = \delta\omega$. Correspondingly, we assume the flow parameters

$$G_0 = G_{0N} + \delta G, \qquad b = b_N + \delta b.$$

In the discussion below we neglect terms of second order.

Integration of Eq. (69) yields

$$HH_N\left(\frac{1}{H}\frac{dH}{d\mu} - \frac{1}{H_N}\frac{dH_N}{d\mu}\right)\Big|_{-b_m^+}^{b_m^-} = -\int_{-b_m^+}^{b_m^-} \frac{\omega^2/\alpha^2}{\left(\mu^2-\omega^2/\alpha^2\right)^2} HH_N d\mu, \tag{71}$$

where the limit b_m is the smaller of b and b_N. The upper and lower sign is used to avoid the discontinuity at $\mu = bm$ as seen in Eq. (75). Using Eq. (70) and the symmetry of the problem, we obtain

$$\left(\frac{1}{H}\frac{dH}{d\mu} - \frac{1}{H_N}\frac{dH_N}{d\mu}\right)\Big|_{-b_m^+}^{b_m^-} \approx 2\left[y\left(b_m^-\right) - y_N\left(b_m^-\right)\right]. \tag{72}$$

Since the solution $y(\mu)$ depends implicitly on ω, and the flow parameters G_0 and b, we have

$$\begin{aligned} y\left(\omega, b, G_0; \mu\right) - y_N\left(0, b_N, G_{0N}; \mu\right) \quad &\approx \quad \frac{\partial y(\mu)}{\partial \omega}\Bigg|_{\substack{\omega=0\\ b=b_N\\ G_0=G_{0N}}} \delta\omega + \frac{\partial y(\mu)}{\partial b}\Bigg|_{\substack{\omega=0\\ b=b_N\\ G_0=G_{0N}}} \delta b \\ &+ \frac{\partial y(\mu)}{\partial G_0}\Bigg|_{\substack{\omega=0\\ b=b_N\\ G_0=G_{0N}}} \delta G_0. \end{aligned} \tag{73}$$

Using Eq. (6), $y(\mu)$ satisfies the Riccati equation

$$\begin{aligned} \frac{dy}{d\mu} &= \alpha^2 - y^2 - \frac{\left(u^2+F^2\right)'}{u^2+F^2} y, \\ y(\pm 1) &= \mp\alpha. \end{aligned} \tag{74}$$

$(u^2+F^2)\,y$ is continuous, as seen by the integration of Eq. (74). $y(\mu)$ must have a jump at $\mu = b$

$$y\left(b^-\right) = \frac{b^2 - \left(\frac{\omega}{\alpha} - G_0\right)^2}{b^2 - \omega^2/\alpha^2} y\left(b^+\right). \tag{75}$$

For $|\mu| < b$, we can write the solution of Eq. (74) as follows

$$y(\omega, b, G_0; \mu) \;=\; y(w, b, G_0; \mu) - y(w, b, G_0; b^-) + \frac{b^2 - \left(\frac{\omega}{\alpha} - G_0\right)^2}{b^2 - \omega^2/\alpha^2} y\left(b^+\right), \qquad (76)$$

Assuming $\mu^2 - G_0^2 \neq 0$ for $b < |\mu| < 1$, $\frac{\partial y(b_m^-)}{\partial \omega}\Big|_{\substack{\omega=0 \\ b=b_N \\ G_0=G_{0N}}}$ is a real quantity up to first order.

Let

$$g \equiv \frac{\partial y\left(b_m^-\right)}{\partial \omega}\Bigg|_{\substack{\omega=0 \\ b=b_N \\ G_0=G_{0N}}} . \qquad (77)$$

Using Eqs. (74) and (76), we have

$$\begin{aligned}
\frac{\partial y(b_m^-)}{\partial b}\Bigg|_{\substack{\omega=0 \\ b=b_N \\ G_0=G_{0N}}} &= -\frac{G_{0N}^2}{b_N^2}\alpha\left[\alpha - \beta_N y_N\left(b_N^+\right)\right] \\
\frac{\partial y(b_m^-)}{\partial G_0}\Bigg|_{\substack{\omega=0 \\ b=b_N \\ G_0=G_{0N}}} &= -\frac{2G_{0N}}{b_N^2} y_N\left(b_N^+\right) + \left(1 - \frac{G_{0N}^2}{b_N^2}\right)\frac{\partial y_N(\mu)}{\partial G_{0N}}\Bigg|_{\mu=b_N^+}, \qquad (78)
\end{aligned}$$

where β_N is defined as in Eq. (A10), and $y_N\left(b_N^+\right)$ satisfies Eq. (A11). Note, $y(\mu)$ is independent of the parameter b for $b < \mu < 1$.

Combining Eqs. (72), (73), (77), (78), the right-hand side of Eq. (71) becomes

$$\begin{aligned}
HH_N\left(\frac{1}{H}\frac{dH}{d\mu} - \frac{1}{H_N}\frac{dH_N}{d\mu}\right)\Bigg|_{-b_m^+}^{b_m^-} \approx\; & 2H_N\left(b_N\right)^2\Bigg[g\delta\omega - \frac{G_{0N}^2}{b_N^2}\alpha\left[\alpha - \beta_N y_N\left(b_N^+\right)\right]\delta b \\
& - \left(2\frac{G_{0N}}{b_N^2} y_N\left(b_N^+\right) - \left(1 - \frac{G_{0N}^2}{b_N^2}\right)\frac{\partial y\left(b_N^+\right)}{\partial G_0}\right)\delta G_0\Bigg] . (79)
\end{aligned}$$

For the right-hand side of Eq. (71), we evaluate the integral by considering the contour shown in Fig. 2a. Assuming the imaginary part of ω is less than zero, we obtain

$$\begin{aligned}
-\int_{-b_m}^{b_m} \frac{\omega^2/\alpha^2}{\left(\mu^2 - \omega^2/\alpha^2\right)^2} HH_N d\mu \;&=\; -\frac{2\pi i\omega^2}{\alpha^2}\frac{d}{d\mu}\left[\frac{H_N^2}{\left(\mu - \omega/\alpha\right)^2}\right]\Bigg|_{\mu=-\omega/\alpha} \\
&= \begin{cases} \frac{1}{2} i\pi\alpha\omega & \text{for } H_N = H_N^\infty = \sinh\alpha\mu, \\ -\frac{1}{2} i\pi\frac{\alpha}{\omega} & \text{for } H_N = H_N^0 = \cosh\alpha\mu. \end{cases} \qquad (80)
\end{aligned}$$

We insert Eq. (79) and (80) into Eq. (71). For the case $H_N = H_N^0$, there is no valid solution, since Eq. (80) diverges and Eq. (79) vanishes in the limit $\delta b = \delta\omega = \delta G_0 = 0$. This means the neutral mode corresponding to $\Delta' = 0$ is an isolated mode (at fixed α).

For the case $H_N = H_N^\infty$ the mode is not isolated and we obtain the following:

$$\delta\omega = \frac{2\left[-\frac{G_{0N}^2}{b_N^2}\alpha\left(\alpha - \beta_\infty y_N\left(b_N^+\right)\right)\delta b - \left(\frac{2G_{0N}}{b_N^2}y_N^\infty\left(b_N^+\right) - \left(1 - \frac{G_{0N}^2}{b_N^2}\right)\frac{\partial y_N^\infty\left(b_N^+\right)}{\partial G_0}\right)\delta G_0\right]\sinh^2(\alpha b)}{\frac{1}{2}i\alpha\pi - 2\left(\sinh^2\left(\alpha b_N\right)\right)g}. \tag{81}$$

For the profile analogous to that studied in Ref. 10, b is set to 1, $y_N(\mu) = -\alpha$ for $|\mu| > 1$, and $\beta = 1 - G_0^2$. Then Eq. (81) becomes

$$\delta\omega = \frac{4\alpha G_{0N}\sinh^2\alpha}{\frac{1}{2}i\alpha\pi - 2g\sinh^2\alpha}\delta G_0.$$

There exists instability when $\delta G_0 > 0$, meanwhile $\delta\beta < 0$. Now we evaluate G_{0N}, the flow parameter corresponding to the neutral mode for $\alpha = 0.45$. This is done in order to compare with Ref. 10. Using Eq. (A15) gives

$$1 - G_{0N}^2 = \beta_\infty = -\frac{\alpha - \tanh\alpha}{\alpha},$$

whence, $G_{0N} \approx 1.03$. In Ref. 10, they observed strong ideal instability at $G_0 \approx 1.2$. This roughly agrees with our above analysis.

For the profile discussed in Ref. 7, G_0 is fixed at unity. Thus Eq. (78) becomes

$$\delta\omega = \frac{-2\frac{G_{0N}^2}{b_N^2}\alpha\left(\alpha - \beta_\infty y_N\left(b_N^+\right)\right)\sinh^2\left(\alpha b_N\right)\delta b}{\frac{1}{2}i\alpha\pi - 2g\sinh^2(\alpha b)}.$$

We see from Eq. (A12) and Fig. 1a that $\left|y_N^\infty\left(b_N^+\right)\right| < \alpha$, and $|\beta_\infty| < 1$. Thus the instability appears when $\delta b < 0$, meanwhile $\delta\beta < 0$, as a result of Eq. (A13).

In both cases, when the flow parameter is varied so that β is decreased from β_∞, there exists instability. Using Fig. 1b, we conclude that an ideal instability appears when Δ' becomes negative through ∞.

When a small resistivity is included in the above problem, there is no influence on the neutral mode corresponding to $\Delta' = 0$; while for the neutral mode corresponding to $\Delta' = \infty$, the growth rate is increased from zero to $(\alpha F'(0))^{2/3}S^{-1/3}$. When the flow parameters are perturbed further, the Δ' value becomes negative and there exists a mixture of tearing and ideal instabilities. This connection of the tearing and ideal modes is similar to that discussed in Ref. 10.

VI. Summary

In the present paper, we have systematically studied the tearing mode in the presence of shear flow. It is found that the shear flow has a significant influence on both the "constant-ψ" and the "nonconstant-ψ" tearing modes. In the external ideal region the magnetic field is frozen into the flow, hence the shear flow can dramatically change the value of the matching quantity Δ'. Some flow profiles can change the scaling from "constant-ψ" to "nonconstant ψ" tearing. In the internal resistive region, the tearing mode is very sensitive to the flow shear at the magnetic null plane; i.e., $G'(0)$. $G'(0)$ changes the order of the convection terms. When $G'(0)$ is very small the inertial terms still dominate the convection terms. Thus the scaling remains unchanged for both tearing modes, while $G'(0)$ stabilizes the "nonconstant-ψ" tearing mode with sufficiently large Δ', while the "constant-ψ" tearing mode is destabilized. In the case where $G'(0)$ is comparable with the magnetic field shear, $F'(0)$, the convection terms overtake the inertial terms in the "constant-ψ" tearing mode, thus its growth rate is changed from $S^{-3/5}$ to $S^{-1/2}$. The scale length of the singular layer is changed from $S^{-2/5}$ to $S^{-1/3}$, and the $\Delta' > 0$ instability criterion is removed provided $G'(0)G''(0) - F'(0)F''(0) \neq 0$. For the "nonconstant-$\psi$" tearing mode, the inertial terms are comparable with convection terms, the scaling remains unchanged. When the flow shear is larger than the magnetic field shear at the magnetic null plane, the flow freezes the magnetic field and stabilizes the tearing mode. Additionally, we have shown the parameter regions for the validity of the "constant-ψ" and "nonconstant-ψ" tearing modes. Finally, since the shear flow can drive ideal instability, we discussed the transition from the tearing mode to the ideal mode in two examples. It is found that this happens when the value of the matching quantity Δ' goes to negative through $\Delta' = \infty$, which is similar to the $m = 1$ tearing mode in Ref. 10.

Acknowledgments

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Appendix A: Δ' Value in the Presence of Equilibrium Shear Flow

Here we evaluate the Δ' value in the presence of the equilibrium shear flow. We assume the equilibrium magnetic field has the form

$$F = |\mu|, \quad |\mu| < 1; \qquad F = 1, \quad \mu > 1; \qquad F = -1, \quad \mu < -1. \tag{A1}$$

This piecewise linear profile can be viewed as a rough approximation of the profile $F = \tanh \mu$.

In the *first example*, we assume the equilibrium shear flow to be

$$G = \frac{G_0}{b}\mu, \quad |\mu| < b; \qquad G = G_0, \quad \mu > b; \qquad G = -G_0, \quad \mu < -b. \tag{A2}$$

This piecewise linear profile can be viewed as an approximation of the profile $G = G_0 \tanh \mu / b$.

For convenience we define

$$y \equiv \frac{w'}{w},$$

and

$$f(x_1, x_2) = \frac{1 - x_1 \tanh x_1 - x_1 x_2}{x_1 - \tanh x_1 + x_1 x_2 \tanh x_1}. \tag{A3}$$

The reason for these definitions will become clear below. Consider first the case where $b < 1$. In the region $|\mu| < b$, Eqs. (8) and (9) become

$$\psi = \mu w$$

$$w'' - \frac{2w'}{\mu} - \alpha^2 w = 0,$$

which have the solution

$$\psi = A_\pm \frac{\sinh \alpha\mu}{\mu} + B_\pm \frac{\cosh \alpha\mu}{\mu}.$$

Here the A_+ and B_+ are as yet undetermined constants for the solution in the region $0 < \mu \leq b$, and A_- and B_- are constants for the solution in the region $-b \leq \mu < 0$. We have allowed for the discontinuity at $\mu = 0$ that arises because of the resistive layer. Thus Δ' is given by

$$\Delta' = \frac{\psi'}{\psi}\Big|_{0_-}^{0+} = \alpha\left(\frac{A_+}{B_+} - \frac{A_-}{B_-}\right).$$

The constants $A_\pm$ and $B_\pm$ are determined by the boundary conditions at $\mu = \pm\infty$. To find these constants we must trace the solution for $|\mu| > 1$ through the regions $b < \mu < 1$ and $-b < \mu < -1$. In these regions Eqs. (8) and (9) are

$$\psi = \mu w$$

$$w'' + \frac{2\mu w'}{\mu^2 - G_0^2} - \alpha^2 w = 0. \tag{A4}$$

Equation (A4) has no simple solution, but it is transformed into a Riccatti equation by $y \equiv w'/w$. We obtain

$$y' = \alpha^2 - y^2 - \frac{2\mu y}{\mu^2 - G_0^2}. \tag{A5}$$

In the outer region, $|\mu| > 1$ the solutions are trivially given by

$$w \sim e^{-\alpha|\mu|}.$$

From this we obtain two conditions $y(1) = -\alpha$ and $y(-1) = \alpha$. We can replace the two unknown quantities in Δ'; i.e., $A_\pm/B_\pm$ by $y(\pm b) \equiv y_\pm$. These quantities are in turn determined by solving the Riccatti equation (A5) subject to the boundary conditions $y(1) = -\alpha$ and $y(-1) = \alpha$. Matching at $\mu = \pm b$ yields

$$y_\pm = \frac{\frac{A_\pm}{B_\pm}\alpha b \pm \alpha b \tanh \alpha b - \frac{A_\pm}{B_\pm} \tanh \alpha b \mp 1}{b \pm \frac{A_\pm}{B_\pm} b \tanh \alpha b},$$

which implies

$$\alpha\frac{A_\pm}{B_\pm} = \frac{\pm\alpha^2 b \tanh \alpha b \mp \alpha - \alpha b y_\pm}{\pm y_\pm b \tanh \alpha b - \alpha b + \tanh \alpha b}.$$

Using the symmetry of the Riccatti equation: $y \to -y$; $\mu \to -\mu$, and the symmetry in the boundary conditions $y(\pm 1) = \mp\alpha$, we conclude that $y_+ = -y_-$. Finally we obtain the following expression for Δ':

$$\Delta' = 2\alpha f\left(\bar{\alpha}, \beta\right), \tag{A6}$$

where the function f was defined above in Eq. (A3), and $\beta = -\frac{1}{\alpha}y(b)$. The complete determination of Δ' has been reduced to finding β, which as noted requires solving Eq. (A5). However, the qualitative nature of the solution can be estimated. Assuming $G_0^2 < 1$ and $G_0^2/b^2 < 1$, it can be shown that $-\infty < y(b) < -\alpha$. This implies

$$1 < \beta < \infty. \tag{A7}$$

Moreover, as $G_0^2/b^2 \to 1$, $\beta \to \infty$.

Similarly in the case where $b > 1$ we obtain

$$\Delta' = 2\alpha f(\alpha, \beta),$$

where

$$\beta = -\frac{1}{\alpha}y(1),$$

and y satisfies the Riccatti equation

$$\begin{aligned} \frac{dy}{d\mu} &= \alpha^2 - y^2 - \frac{2\mu}{\mu^2 - b^2/G_0^2}y, \\ y(b) &= -\alpha. \end{aligned} \tag{A8}$$

Assuming $G_0^2 < 1$, we obtain from the above $-\alpha < y(1) < 0$. This implies

$$0 < \beta < 1. \tag{A9}$$

Note, as $b^2/G_0^2 \to \infty$, $\beta \to 1$.

In the *second example*, we assume the equilibrium shear flow to be

$$G = 0, \quad |\mu| < b; \qquad G = -G_0, \quad |\mu| > b.$$

This profile is a linear approximation for either of the profiles

$$
\begin{aligned}
G &= G_0\left(\operatorname{sech}\frac{\mu}{b}-1\right),\\
G &= G_0\left(\operatorname{sech}^2\frac{\mu}{b}-1\right).
\end{aligned}
$$

The linear profile has a discontinuity at $|\mu| = b$. Since $(F^2 - G^2)\, w'(\mu)$ is continuous, as seen by integration of Eq. (9), $w'(\mu)$ must have a jump at $\mu = b$, and therefore so does $y(\mu)$.

Following the procedure used in the first example, but accounting for this jump, we obtain when $b < 1$

$$
\Delta' = 2\alpha f(\bar{\alpha}, \beta),
$$

where $\bar{\alpha} = \alpha b$,

$$
\begin{aligned}
\beta &= -\frac{1}{\alpha} y\left(b^-\right)\\
&= -\frac{1}{\alpha}\left(1-\frac{G_0^2}{b^2}\right) y\left(b^+\right),
\end{aligned}
\tag{A10}
$$

and $y(\mu)$ satisfies the Riccati equation for $b^+ \le \mu \le 1$

$$
\begin{aligned}
\frac{dy}{d\mu} &= \alpha^2 - y^2 - \frac{2\mu}{\mu^2 - G_0^2} y,\\
y(1) &= -\alpha.
\end{aligned}
\tag{A11}
$$

Assuming $G_0^2 \to 1$ from above, we obtain from Eqs. (A11), $-\alpha < y\left(b^+\right) < 0$, which implies

$$
\beta < 0. \tag{A12}
$$

Using Eqs. (A10) and (A11), we obtain

$$
\begin{aligned}
\frac{\partial\beta}{\partial b} &= -2\frac{G_0^2}{\alpha b^3} y\left(b^+\right) - \frac{1}{\alpha}\left(1-\frac{G_0^2}{b^2}\right)\frac{dy}{d\mu}\bigg|_{\mu=b^+}\\
&= \frac{1}{\alpha}\left(1-\frac{G_0^2}{b^2}\right)\left[\frac{2}{b} y\left(b^+\right) - \left(\alpha^2 - y\left(b^+\right)^2\right)\right] > 0.
\end{aligned}
\tag{A13}
$$

As $b \to 0$, $\beta \to -\infty$. When $b > 1$,

$$\Delta' = 2\alpha f(\alpha, \beta),$$

where

$$\beta = \frac{\left(2 - G_0^2\right) e^{-2\alpha} - G_0^2 e^{-2\alpha b}}{\left(2 - G_0^2\right) e^{-2\alpha} + G_0^2 e^{-2\alpha b}} < 1.$$

In all the above cases, Δ' has the form

$$\Delta' = 2\alpha f\left(\bar{\alpha}, \beta\right),$$

where $\bar{\alpha} = \alpha b$ if $b < 1$, otherwise $\bar{\alpha} = \alpha$. In the above expression, $\bar{\alpha}$ and β measure the influence of the shear flow. In the case of no flow, $\bar{\alpha} = \alpha$ and $\beta = 1$.

At criticality $\Delta' = 0$, which implies $f\left(\bar{\alpha}, \beta_0\right) = 0$. This defines a curve

$$\beta_0\left(\bar{\alpha}\right) \equiv \frac{1 - \bar{\alpha}\tanh\bar{\alpha}}{\bar{\alpha}}. \tag{A14}$$

Similarly, at $\Delta' = \infty$, $f\left(\bar{\alpha}, \beta_\infty\right) = \infty$ implies

$$\beta_\infty\left(\bar{\alpha}\right) \equiv -\frac{\bar{\alpha} - \tanh\bar{\alpha}}{\bar{\alpha}}. \tag{A15}$$

Both $\beta_0\left(\bar{\alpha}\right)$ and $\beta_\infty\left(\bar{\alpha}\right)$ are monotonic decreasing functions of $\bar{\alpha}$, which are shown in Fig. 1a. Also the variation of Δ' with β at fixed $\bar{\alpha}$ is shown in Fig. 1b. We see that the shear flow can drastically change the value of Δ'.

Appendix B: Nonconstant-ψ Tearing Mode with $|(\alpha F'(0))S^{-1/3}\Delta'| \gg 1$

The case of the no flow tearing mode with $\Delta' \sim O(1)$ and $\Delta' < 0$ has been analyzed in Refs. 1 and 10. Here we discuss the case where $|\Delta'| \gg 1$.

Let

$$\frac{(\alpha F'(0))^2 \epsilon^4 S}{r_0} = 1, \qquad \hat{\lambda} = \frac{r_0}{(\alpha F'(0))^{2/3} S^{-1/3}}, \tag{B1}$$

and rewrite Eq. (58) as[8]

$$\frac{\partial^2 X_0}{\partial \zeta^2} - \frac{2}{\zeta}\frac{\partial X_0}{\partial \zeta} = \hat{\lambda}^{3/2} X_0 + \zeta^2 (X_0 - C_0), \tag{B2}$$

where

$$X_0 = \zeta \frac{\partial \psi}{\partial \zeta} - \psi = -\left(\frac{r}{\alpha F'(0)\epsilon}\right)^2 \frac{\partial \phi}{\partial \zeta} + C_0.$$

The solution of the above that matches to the external solution should be asymptotic to C_0. We obtain from Eq. (B2)

$$X_0 \underset{|\zeta|\to\infty}{\longrightarrow} C_0 - \frac{\hat{\lambda}^{3/2}}{\zeta^2} C_0. \tag{B3}$$

We redefine

$$\Delta' = \frac{\int_{-\infty}^{\infty} \frac{1}{\zeta}\frac{\partial X_0}{\partial \zeta} d\zeta}{\epsilon C_0}. \tag{B4}$$

Let $\hat{X} = X_0 - C_0$, $\zeta^2 = t$, and rewrite Eq. (B2) as

$$t\frac{\partial^2 \hat{X}}{\partial t^2} - \frac{1}{2}\frac{\partial \hat{X}}{\partial t} - \frac{1}{4}\left(\hat{\lambda}^{3/2} + t\right)\hat{X} \ = \ \frac{1}{4}\hat{\lambda}^{3/2} C_0.$$

We find it convenient to convert this equation to a homogeneous equation by differentiation[12]

$$t\frac{\partial^3 \hat{X}}{\partial t^3} + \frac{1}{2}\frac{\partial \hat{X}}{\partial t} - \frac{1}{4}\left(\hat{\lambda}^{3/2} + t\right)\frac{\partial \hat{X}}{\partial t} - \frac{1}{4}\hat{X} \ = \ 0. \tag{B5}$$

Assume $\hat{X} = K \int_C e^{zt} v(z) dz$, where K is a constant. C is the path decided later. Substituting it into Eq. (B5) we obtain

$$\int_C \left[-z\left(z^2 - \frac{1}{4}\right)\frac{dv}{dz} + \left(-\frac{5}{2}z^2 - \frac{\hat{\lambda}^{3/2}}{4} z\right) v\right] e^{zt} dz + z e^{zt}\left(z^2 - \frac{1}{4}\right) v(z)\Big|_C = 0.$$

Let

$$\left[-z\left(z^2-\frac{1}{4}\right)\frac{dv}{dz}+\left(-\frac{5}{2}z^2-\frac{\hat{\lambda}^{3/2}}{4}z\right)v\right]=0,$$

which yields

$$v=\left(z-\frac{1}{2}\right)^{-(5+\hat{\lambda}^{3/2})/4}\left(z+\frac{1}{2}\right)^{-(5-\hat{\lambda}^{3/2})/4}. \tag{B6}$$

Now we need to choose a path so that Eq. (B3) is satisfied and

$$ze^{zt}\left(z^2-\frac{1}{4}\right)v(z)\Big|_C=ze^{zt}\left(z-\frac{1}{2}\right)^{-(1+\hat{\lambda}^{3/2})/4}\left(z+\frac{1}{2}\right)^{-(1-\hat{\lambda}^{3/2})/4}\Big|_C=0.$$

When $\hat{\lambda}^{3/2}>1$, the path can be chosen from $z=-1/2$ to $z=0$. By a substitution

$$z=-\frac{1-y}{2(1+y)},$$

we get the solution obtained in Ref. 8. For the case $\hat{\lambda}^{3/2}<1$, $z=-1/2$ becomes a singular point. In order to extend our solution to include $\hat{\lambda}^{3/2}<1$, we modify our path as in Fig. 2b. Thus

$$\begin{aligned}\hat{X} &= K\int_C e^{zt}\left(z-\frac{1}{2}\right)^{-(5+\hat{\lambda}^{3/2})/4}\left(z+\frac{1}{2}\right)^{-(5-\hat{\lambda}^{3/2})/4}dz\\ &= K\int_{C_\delta} e^{zt}\left(z-\frac{1}{2}\right)^{-(5+\hat{\lambda}^{3/2})/4}\left(z+\frac{1}{2}\right)^{-(5-\hat{\lambda}^{3/2})/4}dz\\ &+ K\left(-1+e^{-i\pi(5-\hat{\lambda}^{3/2})/2)}\right)\int_{-1/2+\delta}^{0} e^{zt}\left(z-\frac{1}{2}\right)^{-(5+\hat{\lambda}^{3/2})/4}\left(z+\frac{1}{2}\right)^{-(5-\hat{\lambda}^{3/2})/4}.\end{aligned} \tag{B7}$$

Now consider the case where $\left|1-\hat{\lambda}^{3/2}\right|\ll 1$. Define $\left(1-\hat{\lambda}^{3/2}\right)/4=\sigma$, and rewrite Eq. (B7) as

$$\begin{aligned}\hat{X} &\approx K\int_{C_\delta} e^{zt}\left(z-\frac{1}{2}\right)^{-3/2+\sigma}\left(z+\frac{1}{2}\right)^{-1-\sigma}dz\\ &+ i2\pi\sigma K\int_{-1/2+\delta}^{0} e^{zt}\left(z-\frac{1}{2}\right)^{-3/2+\sigma}\left(z+\frac{1}{2}\right)^{-1-\sigma}dz.\end{aligned} \tag{B8}$$

When $t\to\infty$,

$$\hat{X}\sim -iK\int_{-1/2+\delta}^{0} e^{zt}\left(\frac{1}{2}\right)^{-5/2}(2\pi i\sigma)dz=2^{7/2}\pi\sigma\frac{K}{t}. \tag{B9}$$

Comparison with (B3) yields

$$K = -\frac{\hat{\lambda}^{3/2} C_0}{2^{7/2}\pi\sigma}. \tag{B10}$$

We choose the radius δ of C_δ so that $1 \gg \delta \gtrsim \sigma$. Let $z + 1/2 = \delta e^{i\theta}$ in C_δ, then

$$\begin{aligned} \int_{C_\delta} e^{zt}\left(z - \frac{1}{2}\right)^{-3/2+\sigma}\left(z + \frac{1}{2}\right)^{-1-\sigma} dz &= \int_{C_\delta} e^{zt}\left(-i + \delta e^{i\theta}\right)^{-3/2+\sigma} \delta^{-\sigma} e^{-i\sigma\theta} d(i\theta) \\ &= 2\pi e^{-\frac{1}{2}t} + O(\sigma). \end{aligned} \tag{B11}$$

For the second term of Eq. (8), we estimate the order of magnitude as below

$$\begin{aligned} \left|\int_{-1/2+\delta}^{0} e^{zt}\left(z - \frac{1}{2}\right)^{-3/2+\sigma}\left(z + \frac{1}{2}\right)^{-1-\sigma} dz(2\pi i\sigma)\right| &\lesssim 2\pi\sigma \int_{-1/2+\delta}^{0}\left(z + \frac{1}{2}\right)^{-1-\sigma} dz \\ &\sim \sigma \ln\delta \sim O(\sigma). \end{aligned} \tag{B12}$$

Substituting Eqs. (B11) and (B12) into Eq. (B8), we obtain

$$\hat{X} = -\frac{\hat{\lambda}^{3/2} C_0}{2^{5/2}\sigma}\left(e^{-\frac{1}{2}t} + O(\sigma)\right). \tag{B13}$$

For definiteness, we choose

$$-\frac{\hat{\lambda}^{3/2} C_0}{2^{5/2}\sigma} = 1. \tag{B14}$$

Substituting Eq. (B13) into Eq. (B4), we obtain

$$\Delta' = \frac{\hat{\lambda}^{3/2}}{4\sigma\epsilon}\sqrt{\pi} \gg 1,$$

or

$$\gamma \approx \left(1 - \frac{2\sqrt{\pi}}{3\left(\alpha F'(0) S\right)^{-1/3}\Delta'}\right)\left(\alpha F'(0)\right)^{2/3} S^{-1/3}. \tag{B15}$$

The above analysis requires $\left|\left(\alpha F'(0) S\right)^{-1/3}\Delta'\right| \gg 1$.

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Table I: Summary of the affect of equilibrium shear flow on the tearing mode

	"constant-ψ" tearing mode	"nonconstant-ψ" tearing mode
$\left\|\frac{G'(0)}{F'(0)}\right\| \ll 1$	(a) The growth rate and scale length of the resistive region are respectively $\gamma \sim \alpha^{2/5}\Delta'^{4/5}S^{-3/5}$, $\epsilon \sim (\alpha S)^{-2/5}\Delta'^{1/5} \ll 1$ (b) The "constant-ψ" approximation is valid if $\epsilon\|\Delta'\| \ll 1$ (c) Small flow shear $G'(0)$ destabilizes the "constant-ψ" tearing mode	(a) The growth rate and scale length of the resistive region are respectively $\gamma \sim \alpha^{2/3}S^{-1/3}$, $\epsilon \sim (\alpha S)^{-1/3} \ll 1$ (b) In this limit, we have $\epsilon\|\Delta'\| \gg 1$, $1 - G'(0)^2/F'(0)^2 \not\to 0$ (c) Small flow shear $G'(0)$ stabilizes the "nonconstant-ψ" tearing mode with sufficiently large Δ'
$\left\|\frac{G'(0)}{F'(0)}\right\| \lesssim 1$	(a) The growth rate and scale length of the resistive region are respectively $\gamma \sim (\alpha\|\Delta'\|)^{1/2} S^{-1/2}$, $\epsilon \sim (\alpha S)^{-1/3} \ll 1$ (b) If $G'(0)G''(0) - F'(0)F''(0) \neq 0$, $\Delta' > 0$ instability criterion is removed (c) The "constant-ψ" approximation is valid if $\left\|\sqrt{(1 - G'(0)^2/F'(0)^2)}\,\Delta'\epsilon\right\| \ll 1$	(d) There exists a transition to ideal instability when Δ' becomes negative through $\Delta' = \infty$ (which is made possible by the flow on the external region)
$\left\|\frac{G'(0)}{F'(0)}\right\| > 1$	stabilized	stabilized

Figure Captions

1. Influence of equilibrium shear flow on the matching quantity Δ'.

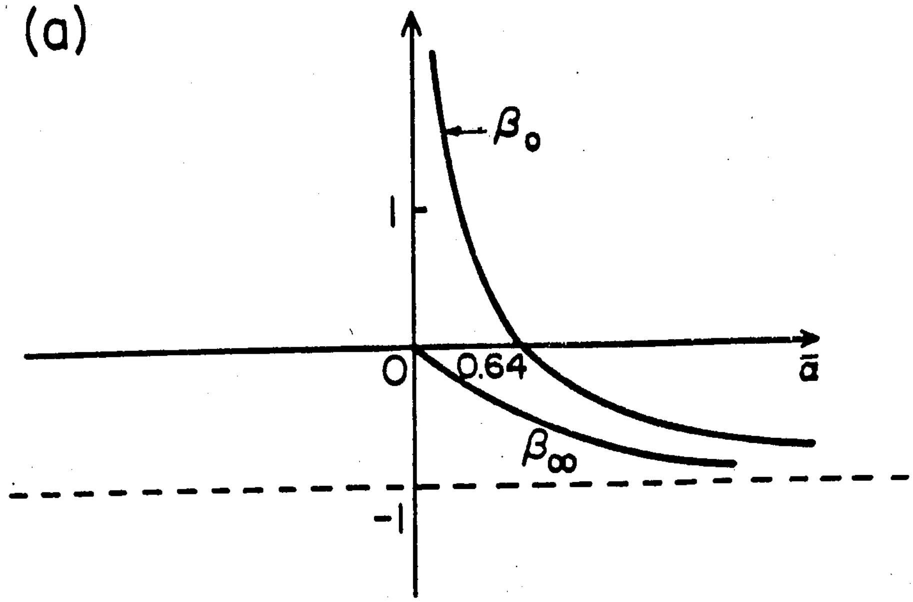
   (a) Sketch of functions $\beta_0(\bar{\alpha})$ and $\beta_\infty(\bar{\alpha})$. β_0 and β_∞ are the parameters at which $\Delta'(\beta_0, \bar{\alpha}) = 0$, and $\Delta'(\beta_\infty, \bar{\alpha}) = \infty$, respectively.

   (b) Sketch of variation of Δ' with parameter β.

2. Integral contours

   (a) The integral contour used in Sec. V.

   (b) The integral contour used in Appendix B.


(a)
β_0
1
0
0.64
$\bar{a}$
β_∞
-1

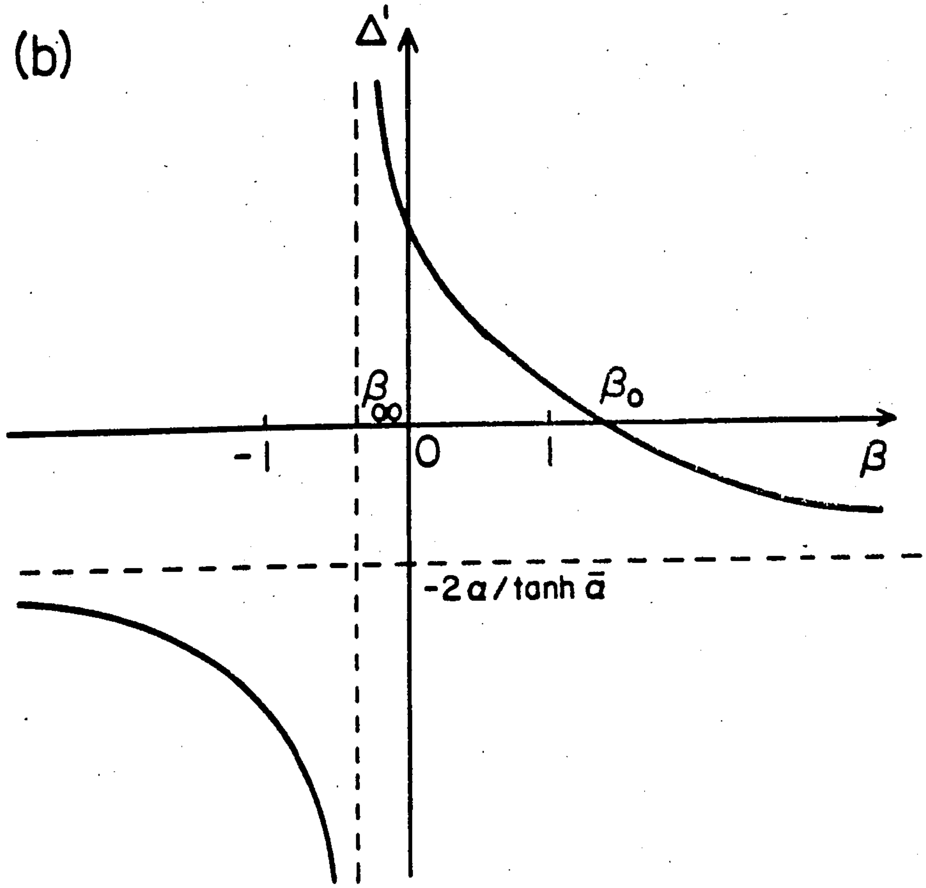

(b)
Δ'
β_∞
β_0
-1
0
1
β
$-2a/\tanh\bar{a}$

Figure 1

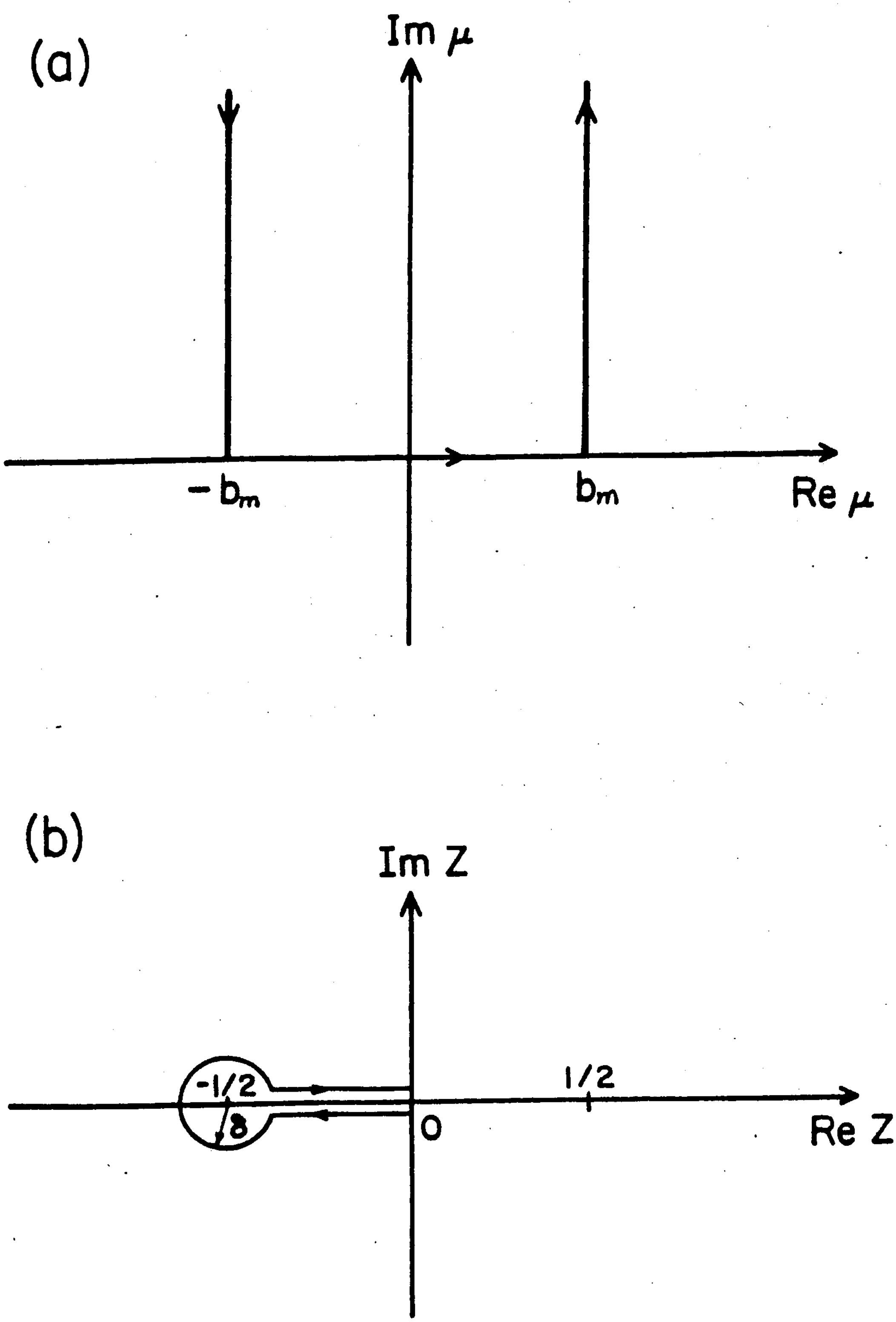


Figure 2