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Nonlinear Behavior of Magnetohydrodynamic Modes Near Marginally Stable States I: General Formulation and Application to the Non-Resonant Kink Modes in a Reversed Field Pinch and to the Quasi-Interchange Modes in a Tokamak

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Abstract

Two types of nonlinear equations describing the time development of modes near marginally stable states in an inhomogeneous medium are obtained through a general formulation that employs a perturbation expansion around the marginally stable state under the assumption of a single helicity. One type of nonlinear equation has a Hamiltonian form that may be interpreted as the equation of motion for a particle in the potential field of a central force; the other type leads to the Landau equation, which is well known in fluid dynamics. The former equation is obtained when the linear operator is degenerate at the marginally stable state, which situation corresponds to the case when the linear dispersion relation has a double root for the frequency at the marginally stable state, whereas the latter is obtained when the linear operator is nondegenerate, i.e., the linear dispersion relation has a single root. In the framework of magnetohydrodynamics, the former corresponds to the nonresonant ideal modes, and the latter to the resistive modes. The nonlinear behavior of the nonresonant kink modes in a reversed field pinch and of the quasi-interchange modes in a tokamak are examined with the application of the general formulation. It is shown that new stable helical equilibria bifurcate near the initial axisymmetric equilibrium, so that the

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plasma nonlinearly oscillates around the new bifurcated equilibrium, which leads to nonlinear saturation of the nonresonant kink modes in a reversed field pinch and of the quasi-interchange mode in a tokamak. Compressibility reduces the nonlinear stabilizing effects and is important even when the modes are near marginally stable states.

1. Introduction

Understanding the nonlinear phenomena of unstable modes in inhomogeneous media such as fluids[1] and plasmas[2] is both a very interesting and important problem. Some investigations examine the turbulent states, whereas other studies consider the phenomena with coherent structures. In either case, these are very difficult problems when treated analytically, so that computer simulations are used as a powerful tool. Due to the complexity of the phenomena, however, it is important to treat these problems analytically even if assumptions are used that simplify the situation. These analytical treatments are a basis for understanding nonlinear phenomena. Here, we will focus our attention on nonlinear phenomena with coherent structures near marginally stable state. As an example, the nonlinear behavior of plasma waves near a marginally stable state is examined in a homogeneous one-dimensional medium.[3] The external kink mode[4] and the quasi-interchange mode[5] are investigated near marginally stable states with the use of a different scheme from Ref. 3.

In this paper, we generally treat the nonlinear behavior of the modes near marginally stable states in an inhomogeneous three-dimensional medium under the assumption of a single helicity. This means that the nonlinearity comes from the interactions of the mode under consideration with its harmonics, so that the three-dimensional problem reduces to a two-dimensional one. Then we consider the nonlinear behavior of weakly unstable ideal MHD modes, viz., the nonresonant kink mode in a reversed field pinch (RFP) and the quasi-interchange mode in a tokamak, applying the general treatment to them.

Using the assumption of single helicity and a perturbation expansion around the marginally stable state, together with the multiple-time-scale method, we can construct a general formulation to derive the nonlinear equation of the mode under consideration, which is applicable to modes near marginally stable states whether they are linearly unstable or not. From this formulation, we can obtain two types of nonlinear equations with respect to the complex amplitude of the mode near a marginally stable state, corresponding to the properties of the linear operator at the marginally stable state. One type of nonlinear equation has a Hamiltonian form that may be interpreted as the equation of motion for a particle in the potential field of a central force; the other type leads to the Landau equation, which is well known in fluid dynamics.[1] The former equation is obtained when the linear operator is degenerate at the marginally stable state, which corresponds to the case when the linear dispersion relation has a double root for the frequency at the marginally stable state, whereas the latter is obtained when the linear operator is nondegenerate, i.e., the linear dispersion relation has a single root for the frequency. The solutions of these two equations represent a wide class of nonlinear phenomena, some of which are bounded and others are unbounded. In particular, when the mode is linearly unstable but nonlinearly stable, new stable equilibria bifurcate near the initial equilibrium, which leads to nonlinear saturation of the mode. In the degenerate case, the mode exhibits either nonlinear oscillations around the new bifurcated equilibrium or more complicated behavior, depending on the initial conditions. In the nondegenerate case, the absolute value of the amplitude of the mode approaches a new bifurcated equilibrium asymptotically, whatever the initial conditions.

In the framework of magnetohydrodynamics (MHD), the degenerate case corresponds to nonresonant ideal modes or nonresonant modes whose linear dispersion relation does not include dissipative effects, while the nondegenerate case corresponds to resistive modes. Here, we apply the general formulation to nonresonant kink modes in a high temperature RFP plasma and to quasi-interchange modes in a high temperature tokamak plasma (degenerate case). The application to resistive modes will be described in a companion paper.[6] Nonresonant kink modes in an RFP are considered to be responsible for the self-reversal and also the sustainment of the reversed state.[7,8] Part of the present work on nonresonant kink modes in an RFP is an extension of previous work,[9,10] in which we had derived a nonlinear equation using the additional assumption of incompressibility.[10] Quasi-interchange modes[11,12] without mode rational surfaces have been proposed to explain the fast crash of sawteeth oscillations without precursor oscillations in recent large tokamaks.[13–16] These two modes correspond to the degenerate case, for which case we obtain the nonlinear equation with a Hamiltonian form. By examining wide classes of cylindrical equilibria that are unstable against each mode, we found that nonlinearity has a stabilizing effect. Hence, new stable helical equilibria bifurcate near the initial axisymmetric equilibrium. For initial conditions that are the same as those used in the linear and nonlinear calculations, a nonlinear oscillation occurs around the new bifurcated equilibrium, which leads to nonlinear saturation of the modes. For the quasi-interchange mode, these results are similar to those in Ref. 5. It is shown that compressibility is important even for modes near marginally stable states, because it reduces the nonlinear stabilizing effects.

In Sec. 2, the general formulation to obtain the nonlinear equations is given. Two types of nonlinear equations are obtained, according to the properties of the linear operator at the marginally stable state. The nonlinear properties of the nonresonant kink mode in an RFP and of the quasi-interchange mode in a tokamak are investigated in Sec. 3 by means of the general formulation, in which the appearance of the bifurcated stable equilibria is indicated. Section 4 contains conclusions and discussion.

2. General Formulation of the Perturbation Theory

We consider a nonlinear system defined in three-dimensional space (X_1, X_2, X_3) where the system is inhomogeneous in the direction of the coordinate X_1 and is periodic in the directions of the coordinates X_2 and X_3. The equilibrium is a function of X_1 only, and the perturbations have the phase dependence $e^{i(M_2X_2+M_3X_3-\omega t)}$ where M_2 and M_3 are the mode numbers and ω is the frequency. Although there are a number of normal modes in the system, we consider a particular normal mode having the mode numbers M_{2p} and M_{3p}. Under the assumption of single helicity, all the nonlinearly excited perturbations are harmonics of the mode under consideration. Then, the problem in three-dimensional space (X_1, X_2, X_3) reduces to one in two-dimensional space (r, ζ) where $r = X_1$ and $\zeta = M_{2p}X_2 + M_{3p}X_3$. Here, we consider traveling waves. The standing waves can be treated in a similar way.

We focus attention on one dependent variable u, because other dependent variables are linearly dependent upon u in the same order in a perturbation theory. We introduce the following expansion form of u with respect to the ordering parameter λ,

$$u = \sum_{n=1} \lambda^n u_n, \tag{2.1}$$

and assume that the first-order perturbation has the following form:

$$u_1 = A u_1(r) e^{i(\zeta - \omega t)} + \text{c.c.}, \tag{2.2}$$

where A is a complex coefficient and $u_1(r)$ is a real function of r. Substituting Eq. (2.1) into the original equations of the nonlinear system, we obtain the following general equations:

$$L\left(\frac{1}{i}\frac{\partial}{\partial \zeta}, i\frac{\partial}{\partial t}, p; r\right) u = M, \tag{2.3}$$

$$M = \sum_{n=2} \lambda^n M_n, \tag{2.4}$$

where L is a linearized differential operator with respect to r and M is a term containing the nonlinear terms. In the operator L, the expressions for the partial differentiations

$$\frac{1}{i}\frac{\partial}{\partial \zeta}, \quad i\frac{\partial}{\partial t}$$

are used instead of the Fourier mode number and the frequency, respectively. The quantity p in the operator L is a parameter characterizing whether or not the system is unstable with respect to the mode under consideration, which parameter may be associated with either the equilibrium or the system parameters. Note that Eqs. (2.3)–(2.4) are a closed system under adequate boundary conditions.

Firstly, we consider the linear solution of Eq. (2.3), substituting Eq. (2.2) into Eq. (2.3); i.e., we consider the solution of the following homogeneous boundary value problem (eigenvalue problem):

$$\lambda L\left(\frac{1}{i}\frac{\partial}{\partial\zeta}, i\frac{\partial}{\partial t}, p; r\right) u_1 \quad = \quad 0$$

or

$$\lambda A e^{i(\zeta-\omega t)} L(1,\omega,p;r) u_1(r) \quad = \quad 0. \tag{2.5}$$

When the parameter p is specified, the eigenvalue ω and the eigenfunction $u_1(r)$ are obtained from the boundary conditions. Then, the equation (2.5) is interpreted as the linear dispersion relation. At the marginally stable state indicated by $p = p_c$, the linear solution is given by

$$u_1 = A u_{1c}(r) e^{i(\zeta-\omega_c t)} + \text{c.c.}, \tag{2.6}$$

where $u_{1c}(r)$ and ω_c are the eigenfunction and the real frequency at the marginally stable state, respectively. Hereafter, it is assumed that the linear solution $u_{1c}(r)$ is unique. According to the properties of the linear operator, the marginally stable state is classified into two typical cases. One is the degenerate case in which the linear dispersion relation has a double root for the frequency at the marginally stable state; the other is the nondegenerate case in which the linear dispersion relation has a single root. In this theory, the degenerate case corresponds to

$$\frac{\partial}{\partial\omega} L(1,\omega_c,p_c;r) \quad = \quad 0 \tag{2.7}$$

and the nondegenerate case to

$$\frac{\partial}{\partial\omega} L(1,\omega_c,p_c;r) \quad \neq \quad 0. \tag{2.8}$$

The properties expressed by Eqs. (2.7)–(2.8) prescribe the nonlinear behavior of the mode near a marginally stable state as shown below.

In order to examine the nonlinear behavior of the mode near a marginally stable state, we consider the situation that slightly deviates from the marginally stable state either toward the unstable direction or toward the stable direction, which is realized by specifying the parameter p as follows:

$$p = p_c \pm \lambda^2. \tag{2.9}$$

This equation determines the sign and the value of the ordering parameter λ when the parameters p and p_c are given. According to the deviation of p from p_c, the linear solution at p slightly changes from the linear solution of Eq. (2.6) at a marginally stable state when the ordering parameter λ is small. When the parameter p deviates from p_c toward the unstable direction, a linear solution with a slow linear growth must be obtained. Therefore, we assume that the linear solution at p has the same form as Eq. (2.6) except for the slow time variation of the coefficient A. Taking the slow time variation of A into account, we introduce the following multiple-time-scale method:

$$\tau_1 = \lambda t, \qquad \tau_2 = \lambda^2 t, \ldots, \tag{2.10}$$

$$i\frac{\partial}{\partial \tau} = i\frac{\partial}{\partial \tau_0} + i\left(\lambda\frac{\partial}{\partial \tau_1} + \lambda^2\frac{\partial}{\partial \tau_2} + \cdots\right), \tag{2.11}$$

$$A = A\left(\tau_1, \tau_2, \ldots\right), \tag{2.12}$$

$$e^{i(\zeta - \omega_c t)} \longrightarrow e^{i(\zeta - \omega_c \tau_0)}. \tag{2.13}$$

As is clear from Eqs. (2.12)–(2.13), the coefficient A has been considered to be a function of the slow time scales $\tau_1, \tau_2, \ldots$, and the oscillation at the frequency ω_c has been interpreted as the phenomenon with the fastest time scale.

Expanding the operator

$$L\left(\frac{1}{i}\frac{\partial}{\partial \zeta}, i\frac{\partial}{\partial t}, p; r\right) u_1$$

around the marginally stable state together with the multiple-time-scale method given by Eqs. (2.10)–(2.13), we obtain the following expanded form:

$$
\begin{aligned}
&L\left(\frac{1}{i}\frac{\partial}{\partial\zeta}, i\frac{\partial}{\partial t}, p; r\right)\lambda A\left(\tau_1,\tau_2,\dots\right)u_{1c}(r)e^{i(\zeta-\omega_c\tau_0)} \\
&= L\left(\frac{1}{i}\frac{\partial}{\partial\zeta}, i\frac{\partial}{\partial\tau_0}+i\lambda\frac{\partial}{\partial\tau_1}+i\lambda^2\frac{\partial}{\partial\tau_2}+\cdots, p_c\pm\lambda^2; r\right)\lambda A\left(\tau_1,\tau_2,\dots\right)u_{1c}(r)e^{i(\zeta-\omega_c\tau_0)} \\
&= e^{i(\zeta-\omega_c\tau_0)}L\left(1, \omega_c+i\lambda\frac{\partial}{\partial\tau_1}+i\lambda^2\frac{\partial}{\partial\tau_2}+\cdots, p_c\pm\lambda^2; r\right)\lambda A\left(\tau_1,\tau_2,\dots\right)u_{1c}(r) \\
&= e^{i(\zeta-\omega_c\tau_0)}\left\{\lambda L\left(1,\omega_c,p_c;r\right)u_{1c}(r)A+\lambda^2 i\frac{\partial}{\partial\omega}L\left(1,\omega_c,p_c;r\right)u_{1c}(r)\frac{\partial A}{\partial\tau_1}\right. \\
&\quad+\lambda^3\left[i\frac{\partial}{\partial\omega}L\left(1,\omega_c,p_c;r\right)u_{1c}(r)\frac{\partial A}{\partial\tau_2}-\frac{1}{2}\frac{\partial^2}{\partial\omega^2}L\left(1,\omega_c,p_c;r\right)u_{1c}(r)\frac{\partial^2 A}{\partial\tau_1^2}\right. \\
&\qquad\left.\left.\pm\frac{\partial}{\partial p}L\left(1,\omega_c,p_c;r\right)u_{1c}(r)A\right]+O\left(\lambda^4\right)\right\}.
\end{aligned}
\tag{2.14}
$$

Substituting Eq. (2.14) into Eq. (2.3) and using Eqs. (2.1)–(2.4), we obtain an equation for each order of λ. According to Eqs. (2.7)–(2.8), we consider two cases separately.

(i) Degenerate case

In this case, Eq. (2.7) holds. From the equation at order λ, we obtain a linear solution at the marginally stable state, i.e., p_c, ω_c, and $u_{1c}(r)$. The equation at order λ^2 is

$$L\left(\frac{1}{i}\frac{\partial}{\partial\zeta}, \omega_c, p_c; r\right)u_2 = M_2, \tag{2.15}$$

where the inhomogeneous term M_2 includes no terms proportional to $e^{i(\zeta-\omega_c\tau_0)}$, but only terms proportional to $e^{i2(\zeta-\omega_c\tau_0)}$ and quasilinear terms. This inhomogeneous boundary value problem can be solved generally. Determination of the quasilinear solution, however, is complicated, as shown in Sec. 3. Although u_2 may have a solution proportional to u_1, such a solution could be transferred to the fundamental solution u_1 by redefinition of the coefficient A. Thus, we may choose u_2 not to have a component proportional to u_1, without loss of generality.

From the equation at order λ^3, for the solution u_{31} proportional to $e^{i(\zeta-\omega_c\tau_0)}$ we have the following equation:

$$\begin{aligned}
L\left(1,\omega_c,p_c;r\right)u_{31} &= \frac{1}{2}\frac{\partial^2}{\partial\omega^2}L\left(1,\omega_c,p_c;r\right)u_{1c}(r)\frac{\partial^2 A}{\partial\tau_1^2} \\
&- \left(\pm 1\right)\frac{\partial}{\partial p}L\left(1,\omega_c,p_c;r\right)u_{1c}(r)A + M_{31}(r)\left|A\right|^2 A \\
&\equiv I_{31}(A,r),
\end{aligned} \tag{2.16}$$

where $M_{31}(r)\left|A\right|^2 A$ comes from M_3. Now we introduce an adequate inner product, taking account of the boundary conditions:

$$\langle w,v\rangle, \tag{2.17}$$

which typically takes the form of an integral with respect to r. Moreover, we introduce the adjoint operator to L, i.e., L^+, and the solution of the adjoint operator L^+, i.e., $u_{1c}^+(r)$. Because we have assumed the uniqueness of $u_{1c}(r)$, a necessary and sufficient condition for a solution of Eq. (2.16) to exist—the solvability condition—is

$$\left\langle u_{1c}^+(r), I_{31}(A,r)\right\rangle = 0. \tag{2.18}$$

Consequently, we have the following nonlinear equation in the degenerate case:

$$C_0\frac{\partial^2 A}{\partial\tau_1^2} \pm C_1 A + C_3\left|A\right|^2 A = 0, \tag{2.19}$$

where

$$C_0 \equiv \left\langle u_{1c}^+(r), \frac{1}{2}\frac{\partial^2}{\partial\omega^2}L\left(1,\omega_c,p_c;r\right)u_{1c}(r)\right\rangle, \tag{2.20}$$

$$C_1 \equiv -\left\langle u_{1c}^+(r), \frac{\partial}{\partial p}L\left(1,\omega_c,p_c;r\right)u_{1c}(r)\right\rangle, \tag{2.21}$$

and

$$C_3 \equiv \left\langle u_{1c}^+(r), M_{31}(r)\right\rangle. \tag{2.22}$$

Rewriting Eq. (2.19) in terms of the original quantities, viz., the time $t = \tau_1/\lambda$ and the amplitude $\mathcal{A} = \lambda A$, we obtain

$$C_0 \frac{\partial^2 \mathcal{A}}{\partial t^2} + (p - p_c)\, C_1 \mathcal{A} + C_3 \left|\mathcal{A}\right|^2 \mathcal{A} = 0, \tag{2.23}$$

where $\pm\lambda^2$ is replaced by $p - p_c$ using Eq. (2.9).

Although the coefficients C_0, C_1, and C_3 may in general be complex, for wide classes of physical phenomena we are interested in the case when these coefficients are real. Hence, we examine the nonlinear behavior given by the solution of Eq. (2.23) for such a situation. Substituting $\mathcal{A} = |\mathcal{A}|\, e^{i\varphi}$ (φ is real) into Eq. (2.23), we have

$$C_0 \left[\frac{\partial^2 |\mathcal{A}|}{\partial t^2} - \left(\frac{\partial \varphi}{\partial t} \right)^2 |\mathcal{A}| \right] + (p - p_c)\, C_1 \, |\mathcal{A}| + C_3 \, |\mathcal{A}|^3 = 0, \tag{2.24}$$

$$|\mathcal{A}|^2 \, \frac{\partial \varphi}{\partial t} = \text{const.} \tag{2.25}$$

Then, Eq. (2.23) may be interpreted as the equation of motion of a particle in the potential field of a central force, where the potential field is given by

$$E_p\left(|\mathcal{A}| \right) = \frac{(p - p_c)\, C_1}{2} \, |\mathcal{A}|^2 + \frac{C_3}{4} \, |\mathcal{A}|^4 . \tag{2.26}$$

The linearized equation of Eq. (2.23) under the condition $|\mathcal{A}| \ll 1$ gives two independent linear solutions. The initial conditions of Eq. (2.23), i.e., $\mathcal{A}(t = 0)$ and $\frac{\partial}{\partial t}\mathcal{A}(t = 0)$ with sufficiently small magnitudes, are related to the choice of the magnitudes and the phases of the two independent linear solutions. Consequently, depending on the signs of the coefficients of C_0, C_1, and C_3 and on the initial conditions of Eq. (2.23), the mode under consideration exhibits a wide range of nonlinear phenomena. Some are bounded, and others are unbounded. We consider the typical case in which the mode under consideration is linearly unstable, i.e., $C_0 \, (p - p_c)\, C_1 < 0$ and both $\mathcal{A}$ and $\partial \mathcal{A}/\partial t$ in the initial conditions are real values with sufficiently small magnitudes. Such initial conditions correspond to the initial state having an unstable mode and a stable mode with a phase difference of π between each other. The amplitude $\mathcal{A}$ is always real, so that the motion of $\mathcal{A}$ is limited in the one-dimensional potential field given by Eq. (2.26) with real $\mathcal{A}$. When nonlinearity stabilizes the mode, i.e., $C_0 C_3 > 0$, the potential field given by Eq. (2.26) has wells, and so $\mathcal{A}$ is bounded. The stationary state of the potential field given by

$$\mathcal{A}_e = \pm\sqrt{-\frac{2\,(p-p_c)\,C_1}{C_3}} \tag{2.27}$$

corresponds to the bifurcated new stable equilibria. The quantity $\mathcal{A}_e$ is proportional to the linear growth rate and inversely proportional to nonlinearity. Accordingly, when the system becomes unstable, new stable equilibria indicated by Eq. (2.27) bifurcate, so that the system nonlinearly oscillates around the bifurcated equilibrium. In contrast, when the nonlinearity destabilizes the mode, i.e., $C_0 C_3 < 0$, the potential has a simple hill and so $\mathcal{A}$ is unbounded. In such a case, the amplitude $\mathcal{A}$ becomes large and higher order calculations are needed.

(ii) Nondegenerate case

In this case, Eq. (2.8) holds. From the equation at order λ, as well as the discussion in the degenerate case, we obtain a linear solution corresponding to a marginally stable state. From the equation at order λ^2, we have the following equation for the solution u_{21} proportional to $e^{i(\zeta-\omega_c\tau_0)}$:

$$L\,(1,\omega_c,p_c;r)\,u_{21} = -i\frac{\partial}{\partial\omega}L\,(1,\omega_c,p_c;r)\,u_{1c}(r)\frac{\partial A}{\partial\tau_1}. \tag{2.28}$$

Using a solvability condition similar to that in Eq. (2.18), we have

$$\frac{\partial A}{\partial\tau_1} = 0, \tag{2.29}$$

and we put $u_{21} = 0$ as in the argument in the degenerate case. Other components of the solution u_2 arising from M_2 are solvable. The equation for the solution u_{31} proportional to $e^{i(\zeta-\omega_c\tau_0)}$ at order λ^3 is given by

$$L\,(1,\omega_c,p_c;r)\,u_{31} \;=\; -i\frac{\partial}{\partial\omega}L\,(1,\omega_c,p_c;r)\,u_{1c}(r)\frac{\partial A}{\partial\tau_2}$$

$$-\quad (\pm 1)\frac{\partial}{\partial p}L(1,\omega_c,p_c;r)\,u_{1c}(r)A + M_{31}(r)\,|A|^2\,A, \tag{2.30}$$

where $M_{31}(r)\,|A|^2\,A$ comes from M_3. Using the solvability condition, we have the following nonlinear equation in the nondegenerate case:

$$d_0\frac{\partial A}{\partial \tau_2} \pm d_1 A + d_3\,|A|^2\,A = 0, \tag{2.31}$$

where

$$d_0 \equiv -i\left\langle u_{1c}^{+}(r), \frac{\partial}{\partial \omega}L(1,\omega_c,p_c;r)\,u_{1c}(r)\right\rangle, \tag{2.32}$$

$$d_1 \equiv -\left\langle u_{1c}^{+}(r), \frac{\partial}{\partial p}L(1,\omega_c,p_c;r)\,u_{1c}(r)\right\rangle, \tag{2.33}$$

and

$$d_3 \equiv \left\langle u_{1c}^{+}(r), M_{31}(r)\right\rangle. \tag{2.34}$$

Rewriting Eq. (2.31) in terms of the original quantities, viz., the time $t = \tau_2/\lambda^2$ and the amplitude $\mathcal{A} = \lambda A$, we have

$$d_0\frac{\partial \mathcal{A}}{\partial t} + (p-p_c)\,d_1\mathcal{A} + d_3\,|\mathcal{A}|^2\,\mathcal{A} = 0, \tag{2.35}$$

where $\pm\lambda^2$ is replaced by $p - p_c$. Putting $\mathcal{A} = |\mathcal{A}|\,e^{i\varphi}$ (φ is real), we obtain the following equations

$$\frac{\partial}{\partial t}|\mathcal{A}|^2 = 2\sigma\,|\mathcal{A}|^2 - l_r\,|\mathcal{A}|^4 \tag{2.36}$$

$$\frac{\partial}{\partial t}\varphi = \omega_r - \frac{1}{2}l_i\,|\mathcal{A}|^2, \tag{2.37}$$

where we use the following expressions

$$\sigma + i\omega_r = -\frac{(p-p_c)\,d_1}{d_0} \tag{2.38}$$

$$l_r + il_i = 2\frac{d_3}{d_0} \tag{2.39}$$

with σ, ω_r, l_r, and l_i being real. Equation (2.36) is well known as the Landau equation in fluid dynamics.[1] It has the following analytical solution:

$$|\mathcal{A}|^2 = \frac{\mathcal{A}_0^2}{\frac{l_r}{2\sigma}\mathcal{A}_0^2 + \left(1 - \frac{l_r}{2\sigma}\mathcal{A}_0^2\right)e^{-2\sigma t}}, \tag{2.40}$$

where $\mathcal{A}_0$ is the initial value of $|\mathcal{A}|$.

As understood from Eq. (2.40), the Landau equation has either a bounded solution or an unbounded one, depending on the signs of σ and l_r. We consider the typical case, i.e., the linearly unstable case, where $\sigma > 0$. If nonlinearity stabilizes the mode under consideration, i.e., $l_r > 0$, then $|\mathcal{A}|$ asymptotically approaches the following value:

$$|\mathcal{A}|_e = \sqrt{\frac{2\sigma}{l_r}} = \sqrt{-(p - p_c)\,\mathrm{Re}\left(\frac{d_1}{d_3}\right)}, \tag{2.41}$$

whatever the initial value of $|\mathcal{A}|$. Then, just as in the degenerate case, there is a bifurcation of the equilibrium in such a situation. When the system becomes unstable, a new stable equilibrium corresponding to $|\mathcal{A}| = |\mathcal{A}|_e$ bifurcates because of the nonlinear stabilization. In contrast, if the nonlinearity destabilizes the mode, i.e., $l_r < 0$, then the solution is unbounded, so that none of the higher order terms may be truncated and there is a fast transition to turbulence.

In the framework of an MHD plasma, the degenerate case corresponds to the nonresonant ideal modes or the nonresonant modes whose linear dispersion relations do not include dissipative terms, and the nondegenerate case to resistive modes. For resonant ideal MHD modes having mode rational surfaces, the eigenfunctions become singular as the modes approach marginally stable states, and hence higher-order radial derivatives become important. Consequently, resistive modes corresponding to resonant ideal modes are unstable beyond the marginally stable states of the ideal modes; then we treat the resistive modes for the MHD modes with mode rational surfaces.

3. Analysis of the Nonlinear Behavior of Non-Resonant Kink Modes in a High Temperature RFP and of Quasi-Interchange Modes in a High Temperature Tokamak

We examine the nonlinear behavior of nonresonant kink modes in a high temperature RFP and of quasi-interchange modes in a high temperature tokamak by applying the general formulation in Sec. 2. The nonresonant kink modes are important in association with self-reversal and the sustainment of the reversed state in an RFP.[7,8] The quasi-interchange modes are considered to be responsible for the fast crash of sawteeth oscillations in tokamaks.[11–19] These two modes are ideal MHD modes without mode rational surfaces, so that they correspond to the degenerate case of Sec. 2. The nonlinear equation describing the nonlinear behavior of these two modes is derived in Sec. 3.1. The coefficients of the nonlinear equation for each mode are calculated, and the nonlinear properties for each mode are shown in Sec. 3.2.

3.1. Derivation of the nonlinear equation

We begin with the usual MHD equations:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\nabla \cdot (\rho \vec{v}), \\
\rho \frac{\partial \vec{v}}{\partial t} &= -\rho \vec{v} \cdot \nabla \vec{v} + \vec{J} \times \vec{B} - \nabla P, \\
\frac{\partial P}{\partial t} &= -\vec{v} \cdot \nabla P - \Gamma P \nabla \cdot \vec{v} + \frac{\Gamma - 1}{S} \vec{J}^2, \\
\frac{\partial \vec{B}}{\partial t} &= \nabla \times \left(\vec{v} \times \vec{B} \right) - \nabla \times \left(\frac{1}{S} \vec{J} \right), \\
\vec{J} &= \nabla \times \vec{B}.
\end{aligned} \tag{3.1}$$

The set of equations (3.1) is normalized by the initial equilibrium toroidal magnetic field at the magnetic axis B_0, the initial equilibrium mass density at the magnetic axis ρ_0, the

Alfvén velocity $v_A = B_0/\sqrt{\mu_0 \rho_0}$, the minor radius a, and the Alfvén transit time $\tau_A = a/v_A$. Also, S is the Lundquist number defined by $S = \tau_D/\tau_A$ where $\tau_D = \mu_0 a^2/\eta$. The boundary conditions are $\hat{n} \cdot \vec{B} = 0$ and $\hat{n} \cdot \vec{v} = 0$, where $\hat{n}$ is the unit vector normal to the wall.

We approximate a toroidal plasma by a periodic cylindrical plasma with periodicity of $2\pi R$ and minor radius a. In what follows, we call R the major radius. Under the condition that only the mode with the poloidal mode number m and the toroidal mode number n is excited initially, only harmonics of the mode under consideration are excited, so that the assumption of single helicity holds. In such a situation there is helical symmetry, so that all physical quantities are dependent only upon t, r, and ζ, where ζ is a new angle defined by

$$\zeta = m\theta - kz, \tag{3.2}$$

with k the toroidal wavenumber

$$k = \frac{n}{R}. \tag{3.3}$$

The unit vectors $\hat{r} = \nabla r/|\nabla r|$, $\hat{\zeta} = \nabla\zeta/|\nabla\zeta|$, and $\hat{\eta} = \hat{r} \times \hat{\zeta}$ constitute a right-hand coordinate system. Consequently, the magnetic field $\vec{B}$ and the current density $\vec{J}$ can be expressed as follows:

$$\vec{B} = \frac{1}{h}\nabla\psi \times \hat{\eta} + B_\eta \hat{\eta} \tag{3.4}$$

$$\vec{J} = \frac{1}{h}\nabla(hB_\eta) \times \hat{\eta} + J_\eta \hat{\eta}, \tag{3.5}$$

where ψ is the helical flux function and

$$h = \sqrt{m^2 + (rk)^2}, \tag{3.6}$$

$$J_\eta = -h\Delta^*\psi + \frac{2mk}{h^2}B_\eta, \tag{3.7}$$

$$\begin{aligned}\Delta^*\psi &= \nabla \cdot \left(\frac{1}{h^2}\nabla\psi\right) \\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{h^2}\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\zeta^2}\psi.\end{aligned} \tag{3.8}$$

The velocity has the usual form

$$\vec{v} \;=\; v_r\hat{r} + v_\zeta\hat{\zeta} + v_\eta\hat{\eta}. \tag{3.9}$$

Substituting Eqs. (3.4)–(3.9) into Eq. (3.1), we have the following basic equations:

$$\frac{\partial\rho}{\partial t} \;=\; -\vec{v}\cdot\nabla\rho - \rho\nabla\cdot\vec{v}, \tag{3.10}$$

$$\begin{aligned}\rho\hat{\eta}\times\left(\frac{\partial\vec{v}}{\partial t}\times\hat{\eta}\right) \;=\;& -\rho\nabla\left(\frac{\vec{v}^2}{2}\right) + \rho\frac{v_\eta}{h}\nabla\left(hv_\eta\right) + \vec{v}\times\hat{\eta}\omega_\eta \\ & + \frac{J_\eta}{h}\nabla\psi - \frac{B_\eta}{h}\nabla\left(hB_\eta\right) - \nabla P,\end{aligned} \tag{3.11}$$

$$\rho\frac{\partial}{\partial t}\left(hv_\eta\right) \;=\; -\rho\vec{v}\cdot\nabla\left(hv_\eta\right) + \vec{B}\cdot\nabla\left(hB_\eta\right), \tag{3.12}$$

$$\frac{\partial P}{\partial t} \;=\; -\vec{v}\cdot\nabla P - \Gamma P\nabla\cdot\vec{v} + \frac{\Gamma-1}{S}\vec{J}^2, \tag{3.13}$$

$$\frac{\partial\psi}{\partial t} \;=\; -\vec{v}\cdot\nabla\psi - \frac{h}{S}J_\eta, \tag{3.14}$$

$$\frac{\partial}{\partial t}\left(\frac{B_\eta}{h} - \frac{2mk\psi}{h^4}\right) \;=\; \vec{B}\cdot\nabla\left(\frac{v_\eta}{h}\right) - \vec{v}\cdot\nabla\left(\frac{B_\eta}{h}\right) - \frac{B_\eta}{h}\nabla\cdot\vec{v} + \frac{1}{S}\Delta^*\left(hB_\eta\right) \tag{3.15}$$

with the boundary conditions

$$\frac{\partial}{\partial\zeta}\psi(t,a,\zeta) \;=\; 0, \tag{3.16}$$

$$v_r(t,a,\zeta) \;=\; 0. \tag{3.17}$$

We apply the general formulation in Sec. 2 to Eqs. (3.10)–(3.17). We expand the dependent variables as follows:

$$\begin{aligned}\psi \;&=\; \psi_0(r) + \lambda\psi_1 + \lambda^2\psi_2 + \lambda^3\psi_3 + \cdots \\ B_\eta \;&=\; B_{\eta0}(r) + \lambda B_{\eta1} + \lambda^2 B_{\eta2} + \lambda^3 B_{\eta3} + \cdots \\ J_\eta \;&=\; J_{\eta0}(r) + \lambda J_{\eta1} + \lambda^2 J_{\eta2} + \lambda^3 J_{\eta3} + \cdots \\ v_r \;&=\; \lambda v_{r1} + \lambda^2 v_{r2} + \lambda^3 v_{r3} + \cdots\end{aligned}$$

$$
\begin{aligned}
v_\zeta &= \lambda v_{\zeta 1} + \lambda^2 v_{\zeta 2} + \lambda^3 v_{\zeta 3} + \cdots \\
v_\eta &= \lambda v_{\eta 1} + \lambda^2 v_{\eta 2} + \lambda^3 v_{\eta 3} + \cdots \\
\rho &= \rho_0(r) + \lambda \rho_1 + \lambda^2 \rho_2 + \lambda^3 \rho_3 + \cdots \\
P &= P_0(r) + \lambda P_1 + \lambda^2 P_2 + \lambda^3 P_3 + \cdots
\end{aligned} \tag{3.18}
$$

and we assume the following form for ψ_1

$$
\psi_1 = \frac{A}{2}\psi_1(r) e^{i(\zeta - \omega t)} + \text{c.c.} \tag{3.19}
$$

The multiple-time-scale method is introduced:

$$
\begin{gathered}
\tau_1 = \lambda t, \qquad \tau_2 = \lambda^2 t, \ldots, \\
\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau_0} + \lambda \frac{\partial}{\partial \tau_1} + \lambda^2 \frac{\partial}{\partial \tau_2} + \cdots, \\
A = A(\tau_1, \tau_2, \ldots), \\
e^{i(\zeta - \omega t)} \longrightarrow e^{i(\zeta - \omega \tau_0)}.
\end{gathered} \tag{3.20}
$$

As the parameter p we choose the square of the toroidal wavenumber k,

$$
p = k^2, \tag{3.21}
$$

and introduce k_c as designating the marginally stable state, so that

$$
k^2 = k_c^2 \pm \lambda^2. \tag{3.22}
$$

This choice means that the major radius R changes as a continuous variable, without changing other variables. Since we consider ideal nonresonant modes in a high temperature plasma, we introduce the following ordering for the resistivity:

$$
\frac{1}{S} = \lambda^4 \frac{1}{S_4}. \tag{3.23}
$$

Substituting (3.18)–(3.20) and (3.22)–(3.23) into (3.10)–(3.15) and taking the boundary conditions (3.16)–(3.17) into account, we obtain equations for each order of λ.

order λ:

This order corresponds to the marginally stable state. Our purpose is to determine k_c, ω_c, and the eigenfunctions. The equation with respect to $\psi_1(r)$ obtained in this order is the same as the usual linearized equation except for the special condition $\mathrm{Im}(\omega) = 0$:

$$L\left(1, \omega, k^2; r\right) \psi_1(r) = 0,$$

$$|\psi_1(0)| < \infty, \qquad \psi_1(a) = 0, \qquad \mathrm{Im}(\omega) = 0. \tag{3.24}$$

Although the explicit form of L is not shown, it is well known that for ideal modes, the frequency ω appears in L in squared form, ω^2. Then, at the marginally stable state k_c, we have

$$\omega_c = 0, \tag{3.25}$$

and ω_c is the double root, so that we have

$$\frac{\partial}{\partial \omega} L\left(1, \omega_c, k_c^2; r\right) = 0. \tag{3.26}$$

Therefore, this situation corresponds to the degenerate case and in what follows we will use $\omega_c = 0$ explicitly.

From the induction law (3.14)–(3.15) and the equation of energy (3.13), we have

$$\vec{v} = 0. \tag{3.27}$$

Although ρ_1 is indefinite up to this order, Eq. (3.27) is consistent with the equation of continuity. Here we define the linearized operator L_l as follows.

$$L_l g(r) \equiv \Delta^*_{-l^2} g(r) + \left[\frac{1}{\psi_0'}\left(\frac{J_{\eta 0}}{h}\right)' + \frac{2kB_{\theta 0}\left(hB_{\eta 0}\right)'}{\left(h\psi_0'\right)^2}\right] g(r), \tag{3.28}$$

where

$$\Delta^*_{-l^2} g(r) \equiv \frac{1}{r}\frac{d}{dr}\left(\frac{r}{h^2}\frac{d}{dr}g(r)\right) - \frac{l^2}{r^2} g(r) \tag{3.29}$$

and $'$ indicates the derivative with respect to r. We obtain the following from the equations of motion (3.11)–(3.12):

$$\begin{aligned} L_1\psi_1(r) &= 0, \\ |\psi_1(0)| < \infty, \quad \psi_1(a) &= 0, \end{aligned} \tag{3.30}$$

and

$$B_{\eta 1} = \frac{A}{2} B_{\eta 1}(r) e^{i\zeta} + \text{c.c.}, \tag{3.31}$$

$$P_1 = \frac{A}{2} P_1(r) e^{i\zeta} + \text{c.c.}, \tag{3.32}$$

where

$$B_{\eta 1}(r) = \frac{(hB_{\eta 0})'}{h\psi_0'}\psi_1(r), \tag{3.33}$$

$$P_1(r) = \frac{P_0'}{\psi_0'}\psi_1(r). \tag{3.34}$$

Solving the eigenvalue problem (3.30), we have the eigenvalue k_c and the eigenfunction $\psi_1(r)$ at the marginally stable state. Therefore, in Eqs. (3.33)–(3.34) and in what follows, k should be read k_c whenever we do not specify otherwise. We see from a straightforward calculation that the operator given by Eq. (3.28) is identical to the operator $L\left(1, \omega_c, k_c^2; r\right)$:

$$L_1 = L\left(1, \omega_c, k_c^2; r\right), \tag{3.35}$$

since $\omega_c = 0$. The quantity $J_{\eta 1}$ is expressed by

$$J_{\eta 1} = \frac{A}{2} J_{\eta 1}(r) e^{i\zeta} + \text{c.c.}, \tag{3.36}$$

where

$$J_{\eta 1}(r) = -h\Delta_{-1}^{*}\psi_1(r) + \frac{2mk}{h^2} B_{\eta 1}(r). \tag{3.37}$$

order λ^2:

The velocity in order λ^2 is obtained from the induction law (3.14)–(3.15) and the equation of energy (3.13) to be as follows:

$$v_{r2} = \frac{1}{2}\frac{\partial A}{\partial \tau_1} v_{r2}(r) e^{i\zeta} + \text{c.c.}, \tag{3.38}$$

$$v_{\zeta 2} = -\frac{i}{2}\frac{\partial A}{\partial \tau_1} v_{\zeta 2}(r) e^{i\zeta} + \text{c.c.}, \tag{3.39}$$

$$v_{\eta 2} = -\frac{i}{2}\frac{\partial A}{\partial \tau_1} v_{\eta 2}(r) e^{i\zeta} + \text{c.c.}, \tag{3.40}$$

where

$$v_{r2}(r) = -\frac{1}{\psi_0'}\psi_1(r), \tag{3.41}$$

$$v_{\zeta 2}(r) = -\frac{1}{h}\frac{d}{dr}\left[r v_{r2}(r)\right], \tag{3.42}$$

$$v_{\eta 2}(r) = -\frac{2rkB_{\theta 0}}{h\psi_0'^2}\psi_1(r). \tag{3.43}$$

Because $\psi_1(a) = 0$, the boundary condition (3.17) is satisfied in Eq. (3.41). The above results indicate that the flow in this order is incompressible:

$$\vec{v}_2 = \frac{1}{h}\nabla\phi_2 \times \hat{\eta} + v_{\eta 2}\hat{\eta}, \tag{3.44}$$

where

$$\phi_2 = -\frac{i}{2}\frac{\partial A}{\partial \tau_1}\phi_2(r) + \text{c.c.}, \tag{3.45}$$

$$\phi_2(r) = -\frac{r}{\psi_0'}\psi_1(r). \tag{3.46}$$

The equation of continuity (3.10) gives the mass density at order λ:

$$\rho_1 = \frac{A}{2}\rho_1(r) e^{i\zeta} + \text{c.c.}, \tag{3.47}$$

where

$$\rho_1(r) = \frac{\rho_0'}{\psi_0'}\psi_1(r). \tag{3.48}$$

Just as at order λ, ρ_2 is indefinite at order λ^2. From the equations of motion (3.11)–(3.12), we see that ψ_2, $B_{\eta 2}$, and P_2 are expressed as follows:

$$\psi_2 = \psi_{20}(A,r) + \frac{1}{2}A^2\psi_{22}(r)e^{i2\zeta} + \text{c.c.}, \tag{3.49}$$

$$B_{\eta 2} = B_{\eta 20}(A,r) + \frac{1}{2}A^2 B_{\eta 22}(r)e^{i2\zeta} + \text{c.c.}, \tag{3.50}$$

$$P_2 = P_{20}(A,r) + \frac{1}{2}A^2 P_{22}(r)e^{i2\zeta} + \text{c.c.}, \tag{3.51}$$

where components proportional to $e^{i\zeta}$ are excluded in accordance with the general theory of Sec. 2, and the quasilinear components $\psi_{20}(A,r)$, $B_{\eta 20}(A,r)$, and $P_{20}(A,r)$ are all real functions and indefinite up to this order except for the following condition:

$$\begin{aligned} \frac{\partial P_{20}}{\partial r} &= \frac{J_{\eta 20}}{h}\psi_0' + \frac{J_{20}}{h}\frac{\partial \psi_{20}}{\partial r} - \frac{B_{\eta 20}}{h}(hB_{\eta 0})' - \frac{B_{\eta 0}}{h}\frac{\partial}{\partial r}(hB_{\eta 20}) \\ &+ \frac{1}{2}|A|^2\left(\frac{J_{\eta 1}(r)}{h}\psi_1'(r) - \frac{B_{\eta 1}(r)}{h}(hB_{\eta 1}(r))'\right). \end{aligned} \tag{3.52}$$

Using the definition (3.7) and Eqs. (3.49)–(3.50), we know

$$J_{\eta 2} = J_{\eta 20}(A,r) + \frac{1}{2}A^2 J_{\eta 22}(r)e^{i2\zeta} + \text{c.c.}, \tag{3.53}$$

where

$$J_{\eta 20}(A,r) = -h\Delta_0^*\psi_{20}(A,r) + \frac{2mk}{h^2}B_{\eta 20}(A,r), \tag{3.54}$$

$$J_{\eta 22}(r) = -h\Delta_{-4}^*\psi_{22}(r) + \frac{2mk}{h^2}B_{\eta 22}(r). \tag{3.55}$$

The quantity $\psi_{22}(r)$ is determined by the following boundary value problem:

$$L_2\psi_{22}(r) = M_{22}(r),$$

$$|\psi_{22}(0)| < \infty, \qquad \psi_{22}(a) = 0, \tag{3.56}$$

where

$$M_{22}(r) = -\frac{1}{2}\left\{\frac{kB_{\theta 0}}{(\psi_0' h)^2}\left[\frac{(hB_{\eta 0})'}{\psi_0'}\right]' + \frac{1}{2\psi_0'}\left[\frac{d}{dr}\left(\frac{J_{\eta 1}(r)}{h\psi_1(r)}\right) + \left(\frac{(hB_{\eta 0})'}{h^2\psi_0'}\right)^2 2rk^2\right]\right\}\psi_1^2(r). \tag{3.57}$$

This inhomogeneous boundary value problem is uniquely solvable, with $B_{\eta 22}(r)$ and $P_{22}(r)$ being obtained in terms of $\psi_{22}(r)$:

$$B_{\eta 22}(r) = \frac{(hB_{\eta 0})'}{h\psi_0'}\psi_{22}(r) + \frac{\psi_1^2(r)}{4h\psi_0'}\left[\frac{(hB_{\eta 0})'}{\psi_0'}\right]' \tag{3.58}$$

$$P_{22}(r) = \frac{J_{\eta 0}}{h}\psi_{22}(r) - \frac{B_{\eta 0}}{h}hB_{\eta 22}(r) + \frac{1}{4}\left\{\frac{J_{\eta 1}(r)}{h}\psi_1(r) - B_{\eta 1}^2(r)\right\}. \tag{3.59}$$

The results obtained up to this point exactly coincide with those of the incompressible case.[10]

order λ^3:

From the induction law (3.14), we have

$$v_{r3} = v_{r30}(A,r) + \frac{1}{2}\frac{\partial A}{\partial \tau_2}v_{r31}(r)e^{i\zeta} + \frac{1}{2}\frac{\partial A^2}{\partial \tau_1}v_{r32}(r)e^{i2\zeta} + \text{c.c.}, \tag{3.60}$$

where $v_{r30}(A,r)$ is a real function satisfying

$$\frac{\partial}{\partial \tau_1}\left\{\psi_{20}(A,r) + \frac{|A|^2}{4r}\left[\phi_2(r)\psi_1(r)\right]'\right\} + \psi_0' v_{r30}(A,r) = 0 \tag{3.61}$$

and

$$v_{r31}(r) = v_{r2}(r) \tag{3.62}$$

$$v_{r32}(r) = -\frac{1}{\psi_0'}\left\{\psi_{22}(r) - \frac{\psi_1^2(r)}{4r}\left[\frac{\phi_2(r)}{\psi_1(r)}\right]'\right\}. \tag{3.63}$$

The equation of energy and these results give

$$v_{\zeta 3} = -\frac{i}{2}\frac{\partial A}{\partial \tau_2}v_{\zeta 31}(r)e^{i\zeta} - \frac{i}{2}\frac{\partial A^2}{\partial \tau_1}v_{\zeta 32}(r)e^{i2\zeta} + \text{c.c.}, \tag{3.64}$$

where

$$v_{\zeta 31}(r) = v_{\zeta 2}(r), \tag{3.65}$$

$$\begin{aligned} v_{\zeta 32}(r) &= -\frac{1}{2h}\frac{d}{dr}\left[r v_{r32}(r)\right] \\ &\quad -\frac{r}{2\Gamma P_0 h}\left\{P_0' v_{r32}(r) + P_{22}(r) - \frac{P_1^2(r)}{4r}\left[\frac{\phi_2(r)}{P_1(r)}\right]'\right\}. \end{aligned} \tag{3.66}$$

Also, the periodicity of v_ζ with respect to ζ leads to

$$\frac{\partial}{\partial \tau_1}\left[P_{20}(A,r) + \frac{|A|^2}{4r}\left(\phi_2(r)P_1(r)\right)'\right] + \Gamma P_0 \frac{1}{r}\frac{\partial}{\partial r}\left[r v_{r30}(A,r)\right] + P_0' v_{r30}(A,r) = 0. \tag{3.67}$$

Using the induction law (3.15) and the preceding results, we obtain

$$v_{\eta 3} = -\frac{i}{2}\frac{\partial A}{\partial \tau_2} v_{\eta 31}(r) e^{i\zeta} - \frac{i}{2}\frac{\partial A^2}{\partial \tau_1} v_{\eta 32}(r) e^{i2\zeta} + \text{c.c.} \tag{3.68}$$

where

$$v_{\eta 31}(r) = v_{\eta 2}(r), \tag{3.69}$$

$$\begin{aligned} v_{\eta 32}(r) &= -\frac{rh}{2\psi_0'}\left\{\frac{1}{r}\frac{d}{dr}\left[\frac{rB_{\eta 0}}{h} v_{r32}(r)\right] + \frac{2B_{\eta 0}}{r} v_{\zeta 32}(r) + \frac{B_{\eta 22}(r)}{h}\right. \\ &\quad \left. -\frac{2mk\psi_{22}(r)}{h^4} - \frac{\psi_1^2(r)}{4r}\left[\frac{v_{\eta 2}(r)}{h\psi_1(r)}\right]' + \frac{\phi_2^2(r)}{4r}\left[\frac{B_{\eta 1}(r)}{h\phi_2(r)}\right]'\right\}. \end{aligned} \tag{3.70}$$

From the periodicity of v_η with respect to ζ, the following equation is obtained

$$\begin{aligned} &\frac{\partial}{\partial \tau_1}\left\{\frac{B_{\eta 20}(A,r)}{h} - \frac{2mk\psi_{20}(A,r)}{h^4} + \frac{|A|^2}{4r}\frac{d}{dr}\left[\frac{\psi_1(r)v_{\eta 2}(r) + \phi_2(r)B_{\eta 1}(r)}{h}\right]\right\} \\ &\qquad + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{rB_{\eta 0}}{h} v_{r30}(A,r)\right) = 0. \end{aligned} \tag{3.71}$$

Substituting Eq. (3.54) into Eq. (3.52) and combining Eqs. (3.52), (3.61), (3.67), and (3.71), we have the following equation with respect to $v_{r30}(A,r)$:

$$\begin{gathered} L_v y(A,r) = \frac{\partial |A|^2}{\partial \tau_1} M_v(r), \\ y(A,0) = 0, \qquad y(A,a) = 0, \end{gathered} \tag{3.72}$$

where

$$y(A,r) \equiv r v_{r30}(A,r), \tag{3.73}$$

$$L_v y \equiv r^2 \frac{\partial^2 y}{\partial r^2} + r\left[\frac{rQ'(r)}{Q(r)} - 1\right]\frac{\partial y}{\partial r} - \frac{r^3}{Q(r)}\left[\left(\frac{B_{\theta 0}}{r}\right)^2\right]' y, \tag{3.74}$$

$$Q(r) \equiv \Gamma P_0 + \left(\frac{\psi_0'}{h}\right)^2 + B_{\eta 0}^2, \tag{3.75}$$

$$\begin{aligned} M_v(r) \equiv \frac{r^3}{Q(r)}\Bigg\{ & \frac{\psi_0'}{r}\frac{d}{dr}\left(\frac{r}{h^2}\frac{dQ_1}{dr}\right) - \frac{J_{\eta 0}}{h}\frac{dQ_1}{dr} + \frac{1}{h^2}\frac{d}{dr}\left(\frac{2mkB_{\eta 0}}{h}Q_1 + h^3 B_{\eta 0} Q_2\right) \\ & - \frac{2mk\psi_0'}{h^3}\left(\frac{2mk}{h^3}Q_1 + hQ_2\right) + \frac{dQ_3}{dr} - Q_4\Bigg\}, \end{aligned} \tag{3.76}$$

$$Q_1 \equiv -\frac{1}{4r}\left[Q_2(r)\psi_1(r)\right]', \tag{3.77}$$

$$Q_2 \equiv -\frac{1}{4r}\left[\frac{\psi_1(r)v_{\eta 2}(r) + \phi_2(r)B_{\eta 1}(r)}{h}\right]', \tag{3.78}$$

$$Q_3 \equiv -\frac{1}{4r}\left[\phi_2(r)P_1(r)\right]', \tag{3.79}$$

$$Q_4 \equiv \frac{1}{2h}\left\{J_{\eta 1}(r)\psi_1'(r) - B_{\eta 1}(r)\left[hB_{\eta 1}(r)\right]'\right\}. \tag{3.80}$$

Putting

$$v_{r30}(A,r) = \frac{\partial |A|^2}{\partial \tau_1} v_{r30}(r), \tag{3.81}$$

we can solve Eq. (3.72) uniquely, so that from Eqs. (3.61), (3.67), and (3.71) ψ_{20}, $B_{\eta 20}$, and P_{20} are solved as follows for a sufficiently small initial value A

$$\psi_{20}(A,r) = |A|^2 \psi_{20}(r), \tag{3.82}$$

$$B_{\eta 20}(A,r) = |A|^2 B_{\eta 20}(r), \tag{3.83}$$

$$P_{20}(A,r) = |A|^2 P_{20}(r), \tag{3.84}$$

where

$$\psi_{20}(r) = -\frac{1}{4r}\left[\phi_2(r)\psi_1(r)\right]' - \psi_0' v_{r30}(r), \tag{3.85}$$

$$\begin{aligned} B_{\eta 20}(r) &= -\frac{2mk}{h^3}\frac{1}{4r}\left[\phi_2(r)\psi_1(r)\right]' - \frac{h}{4r}\left[\frac{\psi_1(r)v_{\eta 2}(r) + \phi_2(r)B_{\eta 1}(r)}{h}\right]' \\ &\quad -\frac{h}{r}\frac{d}{dr}\left[\frac{rB_{\eta 0}}{h}v_{r30}(r)\right] - \frac{2mk\psi_0'}{h^3}v_{r30}(r), \end{aligned} \tag{3.86}$$

$$P_{20}(r) = -\frac{1}{4r}\left[\phi_2(r)P_1(r)\right]' - \Gamma P_0\frac{1}{r}\frac{d}{dr}\left[rv_{r30}(r)\right] - P_0' v_{r30}(r). \tag{3.87}$$

It follows that $J_{\eta 20}$ is given by

$$J_{\eta 20}(A, r) = |A|^2 J_{\eta 20}(r), \tag{3.88}$$

where

$$J_{\eta 20}(r) = -h\Delta_0^* \psi_{20}(r) + \frac{2mk}{h^2}B_{\eta 20}(r). \tag{3.89}$$

The equation of continuity gives the following result, which is consistent with the preceding results:

$$\rho_2 = |A|^2 \rho_{20}(r) + \frac{1}{2}A^2\rho_{22}(r)e^{i2\zeta} + \text{c.c.}, \tag{3.90}$$

where

$$\rho_{20}(r) = -\frac{1}{4r}\left[\phi_2(r)\rho_1(r)\right]' - \rho_0\frac{1}{r}\frac{d}{dr}\left[rv_{r30}(r)\right] - \rho_0' v_{r30}(r), \tag{3.91}$$

$$\rho_{22}(r) = \frac{\rho_1^2(r)}{4r}\left[\frac{\phi_2(r)}{\rho_1(r)}\right]' - \frac{1}{r}\frac{d}{dr}\left[\rho_0 r v_{r32}(r)\right] - 2\frac{\rho_0 h}{r}v_{\zeta 32}(r). \tag{3.92}$$

Up to this point, we have obtained all the quantities of order λ^2.

Finally we consider the equations of motion (3.11) and (3.12). Since ψ_3 and $B_{\eta 3}$ have components proportional to $e^{i\zeta}$, we investigate only these components, i.e., $\psi_{31}(A, r)$ and $B_{\eta 31}(A, r)$.

From the equation of motion (3.12), we have

$$\frac{\psi_0'}{r}hB_{\eta 31} - \frac{(hB_{\eta 0})'}{r}\psi_{31} = \frac{1}{2}\frac{\partial^2 A}{\partial \tau_1^2}\rho_0 h v_{\eta 2}(r) - \frac{1}{2}A\delta\left\{\frac{\psi_0'}{r}hB_{\eta 1}(r) - \frac{(hB_{\eta 0})'}{r}\psi_1(r)\right\}$$

$$
\begin{aligned}
& - \frac{1}{2}|A|^2 A\Bigg\{\frac{\psi_{20}'(r)}{r}hB_{\eta 1}(r) - \frac{(hB_{\eta 20}(r))'}{r}\psi_1(r) \\
& + \frac{\psi_1'(r)}{r}hB_{\eta 22}(r) - \frac{(hB_{\eta 1}(r))'}{r}\psi_{22}(r) \\
& - \frac{1}{2}\left[\frac{\psi_{22}'(r)}{r}hB_{\eta 1}(r) - \frac{(hB_{\eta 22}(r))'}{r}\psi_1(r)\right]\Bigg\},
\end{aligned}
\tag{3.93}
$$

where $\delta\{\quad\}$ is an operator that projects out the order-λ^2 quantities from the operand, so that in the operand k should not be interpreted as k_c. The equation of motion (3.11) gives

$$
\begin{aligned}
& \frac{\psi_0'}{r}\Delta_{-1}^*\psi_{31} + \left(\frac{J_{\eta 0}}{h}\right)'\frac{\psi_{31}}{r} + \frac{2kB_{\theta 0}}{rh^2}hB_{\eta 31} = \frac{1}{2}\frac{\partial^2 A}{\partial \tau_1^2}\left\{\frac{1}{r}\frac{d}{dr}\left(\frac{\rho_0 r}{h^2}\frac{d\phi_2(r)}{dr}\right) - \frac{\rho_0}{r^2}\phi_2(r)\right\} \\
& - \frac{1}{2}A\delta\left\{\frac{\psi_0'}{r}\Delta_{-1}^*\psi_1(r) + \left(\frac{J_{\eta 0}}{h}\right)'\frac{\psi_1(r)}{r} + \frac{2kB_{\theta 0}}{rh^2}hB_{\eta 1}(r)\right\} \\
& - \frac{1}{2}|A|^2 A\Bigg\{ - \frac{\psi_{20}'(r)}{r}\frac{J_{\eta 1}(r)}{h} + \left(\frac{J_{\eta 20}(r)}{h}\right)'\frac{\psi_1(r)}{r} + \frac{2k^2}{h^4}hB_{\eta 20}(r)hB_{\eta 1}(r) \\
& - \frac{\psi_1'(r)}{r}\frac{J_{\eta 22}(r)}{h} + \left(\frac{J_{\eta 1}(r)}{h}\right)'\frac{\psi_{22}(r)}{r} + \frac{2k^2}{h^4}hB_{\eta 1}(r)hB_{\eta 22}(r) \\
& - \frac{1}{2}\left[-\frac{\psi_{22}'(r)}{r}\frac{J_{\eta 1}(r)}{h} + \left(\frac{J_{\eta 22}(r)}{h}\right)'\frac{\psi_1(r)}{r} + \frac{2k^2}{h^4}hB_{\eta 22}(r)hB_{\eta 1}(r)\right]\Bigg\}.
\end{aligned}
\tag{3.94}
$$

Substituting $B_{\eta 31}$ in Eq. (3.93) into Eq. (3.94), we have

$$
\begin{aligned}
L_1\psi_{31}(A,r) &= M_{31}(A,r) \\
&= M_{31}^{(0)}(r)\frac{\partial^2 A}{\partial \tau_1^2} + M_{31}^{(1)}(r)A + M_{31}^{(3)}(r)|A|^2 A,
\end{aligned}
\tag{3.95}
$$

$$
|\psi_{31}(A,0)| < \infty, \qquad \psi_{31}(A,a) = 0,
$$

where

$$
M_{31}^{(0)}(r) \equiv -\frac{1}{2\psi_1(r)}\left\{\frac{\phi_2(r)}{r}\frac{d}{dr}\left(\frac{\rho_0 r}{h^2}\frac{d\phi_2(r)}{dr}\right) - \frac{\rho_0}{r^2}\phi_2^2(r) - \rho_0 v_{\eta 2}^2(r)\right\}, \tag{3.96}
$$

$$
M_{31}^{(1)}(r) \equiv -\frac{1}{2}\delta\{L_1\psi_1(r)\} = -\frac{1}{2}\left(\pm\frac{1}{2k_c}\right)\frac{\partial L_1}{\partial k}\Big|_{k=k_c}\psi_1(r), \tag{3.97}
$$

$$
\begin{aligned}
M_{31}^{(3)}(r) \equiv\ & \frac{r}{2\psi_0'}\Bigg\{\frac{1}{r}\frac{d\psi_{20}(r)}{dr}\frac{J_{\eta 1}(r)}{h} - \frac{1}{r}\frac{d}{dr}\left(\frac{J_{\eta 20}(r)}{h}\right)\psi_1(r) - \frac{2k^2}{h^2}B_{\eta 20}(r)B_{\eta 1}(r) \\
& + \frac{1}{r}\frac{d\psi_1(r)}{dr}\frac{J_{\eta 22}(r)}{h} - \frac{1}{r}\frac{d}{dr}\left(\frac{J_{\eta 1}(r)}{h}\right)\psi_{22}(r) - \frac{2k^2}{h^2}B_{\eta 1}(r)B_{\eta 22}(r) \\
& - \frac{1}{2}\left[\frac{1}{r}\frac{d\psi_{22}(r)}{dr}\frac{J_{\eta 1}(r)}{h} - \frac{1}{r}\frac{d}{dr}\left(\frac{J_{\eta 22}(r)}{h}\right)\psi_1(r) - \frac{2k^2}{h^2}B_{\eta 22}(r)B_{\eta 1}(r)\right] \\
& + \frac{v_{\eta 2}(r)}{h\phi_2(r)}\Bigg\{\frac{1}{r}\frac{d\psi_{20}(r)}{dr}hB_{\eta 1}(r) - \frac{1}{r}\frac{d\left(hB_{\eta 20}(r)\right)}{dr}\psi_1(r) \\
& + \frac{1}{r}\frac{d\psi_1(r)}{dr}hB_{\eta 22}(r) - \frac{1}{r}\frac{d\left(hB_{\eta 1}(r)\right)}{dr}\psi_{22}(r) \\
& - \frac{1}{2}\left[\frac{1}{r}\frac{d\psi_{22}(r)}{dr}hB_{\eta 1}(r) - \frac{1}{r}\frac{d\left(hB_{\eta 22}(r)\right)}{dr}\psi_1(r)\right]\Bigg\}\Bigg\}.
\end{aligned} \tag{3.98}
$$

By straightforward calculation, we find

$$
M_{31}^{(0)}(r) = \frac{1}{4}\frac{\partial^2}{\partial\omega^2}L\left(1,\omega_c,p_c;r\right)\psi_1(r), \tag{3.99}
$$

$$
M_{31}^{(1)}(r) = -\left(\pm\frac{1}{2}\right)\frac{\partial}{\partial p}L\left(1,\omega_c,p_c;r\right)\psi_1(r), \tag{3.100}
$$

where $\omega_c = 0$, $p = k^2$, and $p_c = k_c^2$, so that we can reproduce the results of the general formulation in Sec. 2, taking account of the difference of the factor of 1/2 between Eq. (2.6) and Eq. (3.19).

Defining the inner product as

$$
\langle u, v\rangle \equiv \int_0^a ruv\,dr, \tag{3.101}
$$

we see that

$$
\langle\psi_1(r), L_1\psi_{31}(A,r)\rangle = \langle L_1\psi_1(r), \psi_{31}(A,r)\rangle, \tag{3.102}
$$

i.e., L_1 is self-adjoint. Therefore the solvability condition of Eq. (3.95) is

$$
\langle\psi_1(r), M_{31}(A,r)\rangle = 0. \tag{3.103}
$$

Consequently, we have the following nonlinear equation

$$C_0\frac{\partial^2 A}{\partial\tau_1^2} \pm C_1 A + C_3 |A|^2 A = 0, \tag{3.104}$$

where the coefficients C_0, C_1, and C_3 are all real and are given by

$$\begin{aligned}
C_0 &\equiv \left\langle \psi_1(r), M_{31}^{(0)}(r)\right\rangle \\
&= \int_0^a dr\, r\rho_0 \left\{v_{r2}^2(r) + v_{\zeta 2}^2(r) + v_{\eta 2}^2(r)\right\} > 0, && (3.105)\\
C_1 &\equiv \left\langle \psi_1(r), \pm M_{31}^{(1)}(r)\right\rangle \\
&= -\int_0^a dr \frac{r\psi_1(r)}{2k_c}\frac{\partial L_1}{\partial k}\bigg|_{k=k_c} \psi_1(r), && (3.106)\\
C_3 &\equiv \left\langle \psi_1(r), M_{31}^{(3)}(r)\right\rangle \\
&= 2\int_0^a dr\, r\psi_1(r) M_{31}^{(3)}(r) dr. && (3.107)
\end{aligned}$$

Note that we multiply Eq. (3.103) by the factor of 2 before calculation. Using the original quantities, viz., the time $t = \tau_1/\lambda$ and the amplitude $\mathcal{A} = \lambda A$, we obtain

$$C_0\frac{\partial^2 \mathcal{A}}{\partial t^2} + \left(k^2 - k_c^2\right) C_1 \mathcal{A} + C_3 |\mathcal{A}|^2 \mathcal{A} = 0, \tag{3.108}$$

where Eq. (3.22) was used. This is the nonlinear equation that prescribes the nonlinear behavior of nonresonant kink modes in an RFP and of quasi-interchange modes in a tokamak, near marginally stable states. Although the coefficients C_0 and C_1 do not change whether or not the plasma is compressible, compressibility is important because it affects the coefficient C_3 in the nonlinear term through the quasilinear components ψ_{20} and $B_{\eta 20}$. The resistivity does not affect Eq. (3.108) in the ordering we used.

We consider the special case in which the initial conditions of Eq. (3.108), i.e., $\mathcal{A}(t=0)$ and $\frac{\partial}{\partial t}\mathcal{A}(t=0)$, are real. Then the amplitude $\mathcal{A}$ is always real, so that we have the following expanded forms of the variables up to order λ^2:

$$\psi = \psi_0(r) + \mathcal{A}\psi_1(r)\cos\zeta + \mathcal{A}^2\left[\psi_{20}(r) + \psi_{22}(r)\cos 2\zeta\right] + \cdots,$$

$$
\begin{aligned}
B_\eta &= B_{\eta 0}(r) + \mathcal{A} B_{\eta 1}(r)\cos\zeta + \mathcal{A}^2\left[B_{\eta 20}(r) + B_{\eta 22}(r)\cos 2\zeta\right] + \cdots, \\
\rho &= \rho_0(r) + \mathcal{A}\rho_1(r)\cos\zeta + \mathcal{A}^2\left[\rho_{20}(r) + \rho_{22}(r)\cos 2\zeta\right] + \cdots, \\
P &= P_0(r) + \mathcal{A} P_1(r)\cos\zeta + \mathcal{A}^2\left[P_{20}(r) + P_{22}(r)\cos 2\zeta\right] + \cdots, \\
v_r &= \frac{\partial \mathcal{A}}{\partial t} v_{r2}(r)\cos\zeta + \cdots, \\
v_\zeta &= \frac{\partial \mathcal{A}}{\partial t} v_{\zeta 2}(r)\sin\zeta + \cdots, \\
v_\eta &= \frac{\partial \mathcal{A}}{\partial t} v_{\eta 2}(r)\sin\zeta + \cdots .
\end{aligned} \tag{3.109}
$$

The phase relationship of the basic Fourier modes is the same as for the usual linear and nonlinear calculations. The linearized version of Eq. (3.108) with $|\mathcal{A}| \ll 1$ gives two independent solutions. In the linearly unstable case, these two solutions are a growing mode and a damping mode. When $|\mathcal{A}(t=0)| \ll 1$ and $\left|\frac{\partial}{\partial t}\mathcal{A}(t=0)\right| \ll 1$, the above initial conditions correspond to the situation in which the initial amplitudes of both the growing mode and the damping mode are real.

When $\mathcal{A}$ is real, rewriting Eq. (3.108) we have the following conservative form:

$$
\frac{C_0}{2}\left(\frac{\partial \mathcal{A}}{\partial t}\right)^2 + \frac{(k^2 - k_c^2)\, C_1}{2}\mathcal{A}^2 + \frac{C_3}{4}\mathcal{A}^4 = E, \tag{3.110}
$$

where E is constant. We easily see that $C_0/2(\partial\mathcal{A}/\partial t)^2$ exactly expresses the order-λ^4 kinetic energy per unit volume, so that Eq. (3.110) corresponds to conservation of energy to order λ^4 and

$$
E_p = \frac{(k^2 - k_c^2)\, C_1}{2}\mathcal{A}^2 + \frac{C_3}{4}\mathcal{A}^4 \tag{3.111}
$$

corresponds to the potential energy to order λ^4. The total energy per unit volume, the total toroidal flux, and the total mass per unit volume are conserved as lower order conservation laws, up to order λ^2. The total toroidal current, however, is not conserved.

When the mode under consideration is both linearly and nonlinearly unstable, i.e., $(k^2 - k_c^2)\, C_1 < 0$ and $C_3 < 0$ (note that $C_0 > 0$), the potential given by Eq. (3.111) has a simple hill, so that the amplitude $\mathcal{A}$ can become large and higher nonlinear terms are needed. In contrast, when the mode is linearly unstable but nonlinearly stable, i.e., $(k^2 - k_c^2)\, C_1 < 0$ and $C_3 > 0$, the potential has wells, so that the nonlinear behavior is bounded. The stationary states of the potential given by

$$\mathcal{A}_e = \pm\sqrt{-\frac{(k^2 - k_c^2)\, C_1}{C_3}} \tag{3.112}$$

indicate bifurcated new stable equilibria, which correspond to helical equilibria in the vicinity of the initial axisymmetric equilibrium. There are three types of nonlinear behavior depending upon whether $E > 0$, $E = 0$, or $E < 0$, which all have analytic expressions. When $E < 0$ and $\mathcal{A}(t = 0) > 0$, the plasma oscillates nonlinearly around the bifurcated new equilibrium indicated by $\mathcal{A}_e > 0$ given in Eq. (3.112).

3.2. Nonlinear properties of nonresonant kink modes in a high temperature RFP plasma and of quasi-interchange modes in a high temperature tokamak plasma

3.2.1. The nonresonant kink mode in a high temperature RFP plasma

In RFP plasmas, the current density is very high and the pitch parameter $\mu(r)$ monotonically decreases in the radial direction. In such situations, nonresonant kink modes, which are the current-driven $m = 1$ modes without mode rational surfaces, become unstable. These modes are thought to be important for self-reversal and for sustainment of the reversed state.[7,8]

We consider the following force-free equilibrium used previously in Refs. 9 and 10:

$$\mu(r) = 0.3\left(1 - 1.8748r^2 + 0.8323r^4\right). \tag{3.113}$$

For this equilibrium, the toroidal wavenumber k_c corresponding to the marginally stable state of the nonresonant kink mode with $m = 1$ and the coefficients of the nonlinear equation (3.108) are shown in Table 1. From this table we see that nonlinearity stabilizes the perturbation and that compressibility reduces the stabilizing effects. For the weakly unstable modes with $k \gtrsim k_c$, i.e., $(k^2 - k_c^2)\,C_1 < 0$, the potential given by Eq. (3.111) has wells, so that new stable helical equilibria bifurcate near the initial axisymmetric equilibrium. These new bifurcated equilibria correspond to the stationary states of the potential given by Eq. (3.112), around which the plasma oscillates nonlinearly under the same initial condition as used in the usual linear and nonlinear calculations. The potential profile corresponding to $k = 1.4$ is given in Fig. 1. In this case, the ordering parameter $\lambda = \sqrt{k^2 - k_c^2}$ has the value $\lambda = 0.3571$ and for an aspect ratio of $A_s = 5$, the toroidal mode number is $n = 7$. From Fig. 1 we see that compressibility allows a larger amplitude than does incompressibility and that compressibility is important even near the marginally stable state.

In the incompressible case, the comparison between the results of the perturbation theory and those of the numerical simulation is made in detail in Ref. 10, in which the existence of the nonlinear oscillations is explored. The nonlinear oscillations in the numerical simulations are modified, which is thought to be due to a change of the potential field near the region $|\mathcal{A}| \ll 1$. This change of the potential field is caused by the variation of the initial equilibrium due to the nonlinear mode-mode coupling not included in the perturbation theory, the external boundary condition, i.e., the conservation of the total toroidal current and the dissipation due to resistivity.

It should be noted that although the perturbation theory is valid near the marginally stable state, the qualitative characteristics still persist even for modes far away marginally stable states, as is shown in Ref. 9 (incompressible case) and Refs. 7 and 8 (compressible case). As the modes deviate from marginally stable states, the changes of the potential field due to the variation of the initial equilibrium, etc., are larger, so that the nonlinear behavior appears as saturation with a slight nonlinear oscillation.

3.2.2. The quasi-interchange mode in a high temperature tokamak plasma

Sawteeth oscillations in recent large tokamaks show complicated characteristics that cannot be completely explained by the Kadomtsev reconnection model.[20] One such feature is the fast crash without precursor oscillations.[13–16] The quasi-interchange mode has been proposed to explain this fast crash.[11,12] This mode without a mode rational surface is essentially a pressure-driven mode with poloidal mode number $m = 1$ and toroidal mode number $n = 1$ in cylindrical geometry.

In order to examine the nonlinear behavior of this mode, we consider the following two types of equilibria:

$$\begin{aligned} \text{(I)} \qquad \mu(r) &= \mu_m + \Delta\mu \left[1 - \left(\frac{r}{r_m}\right)^2\right]^2 \\ P(r) &= P_0 \left(1 - r^2\right)^2 \end{aligned} \qquad (3.114)$$

$$\begin{aligned} \text{(II)} \qquad \mu(r) &= \mu_m \left[1 + \left(\frac{r}{r_0}\right)^{2\lambda_0}\right]^{1/\lambda_0} \\ P(r) &= P_0 \left(1 - r^2\right)^2, \end{aligned} \qquad (3.115)$$

where $\mu_m = 3.0597$, $\Delta\mu = 0.2703$, $r_m = 0.4$, $\lambda_0 = 5$, $r_0 = 0.55$, and $P_0 = 0.03$. The μ-profile of type (I) is hollow and that of type (II) has low shear. For these equilibria, the toroidal wavenumbers for the marginally stable states and the coefficients of the nonlinear equation (3.108) are shown in Table 2.

Table 2 shows that, just as for the nonresonant kink mode in an RFP, nonlinearity stabilizes the perturbation and compressibility reduces the stabilizing effect. For weakly unstable modes with $k \lesssim k_c$, i.e., $(k^2 - k_c^2)\, C_1 < 0$, the potential given by Eq. (3.111) has wells, so that new stable bifurcated helical equilibria indicated by Eq. (3.112) appear near the initial axisymmetric equilibrium, around which the quasi-interchange mode exhibits a nonlinear oscillation. Figure 2 indicates the potential profile for aspect ratio $A_s = 2.95$ with an equilibrium of type (I). The corresponding values of k and λ are 0.3390 and 0.06439, respectively. The minimum value of the safety factor is 1.037.

Although the effects of toroidicity are not included in this theory, toroidal curvature seems to prohibit flux tubes from interchanging and poloidal coupling due to toroidicity may decrease the energy of the $m = 1$ component in the nonlinear phase. Therefore, the qualitative characteristics may not change. Just as for the case of the nonresonant kink mode in an RFP, the saturation of the quasi-interchange mode due to bifurcation of the new equilibria appears with a slight nonlinear oscillation. These characteristics seem to be consistent with the results of Refs. 17 and 18.

4. Conclusions and Discussion

A general formalism to derive the nonlinear equation describing the nonlinear time development of modes near a marginally stable state in a three-dimensional inhomogeneous medium has been developed in this paper, with the use of a perturbation expansion around the marginally stable state, together with the multiple-time-scale method, under the assumption of single helicity. Two types of nonlinear equations with respect to the complex amplitude of the mode under consideration are obtained through this general formalism. One equation has a Hamiltonian form and can be viewed as the equation of motion for a particle in a potential field of a central force; the other equation is similar to the Landau equation that is well known in fluid dynamics. The former is obtained when the linear operator of the mode under consideration is degenerate at the marginally stable state, which situation corresponds to the fact that the linear dispersion relation has a double root for the frequency at the marginally stable state. The latter equation is obtained when it is nondegenerate; i.e., the linear dispersion relation has a single root.

The solutions of these two nonlinear equations describe a wide class of nonlinear phenomena, according to the situation under consideration. Some are bounded, and others are unbounded. In both the degenerate and the nondegenerate cases, when nonlinearity stabilizes the linearly unstable mode under consideration, new stable equilibria bifurcate near the initial equilibrium, corresponding to the steady states of the solutions. For the degenerate case, the mode exhibits either nonlinear oscillation about the new bifurcated equilibrium or more complicated behavior, depending on the initial conditions. In contrast, for the nondegenerate case, the absolute magnitude of the amplitude of the mode asymptotically approaches the new bifurcated equilibrium, whatever the initial conditions.

In the framework of MHD, the degenerate case corresponds to either nonresonant ideal modes or nonresonant modes without dissipative terms in the linear dispersion relation. The nondegenerate case, on the other hand, corresponds to resistive modes. The nonlinear behavior of nonresonant kink modes in a high temperature RFP and of quasi-interchange modes in a high temperature tokamak were then examined by the application of the general formalism. These two modes correspond to the degenerate case.

Nonresonant kink modes in an RFP are thought to be responsible for self-reversal and for sustainment of the reversed state. Quasi-interchange modes in a tokamak are thought to be responsible for the fast crash of sawteeth oscillations in recent large tokamaks. For the equilibria we examined, the effects of nonlinearity stabilize the perturbations for both modes. As a result, new stable helical equilibria bifurcate near the initial axisymmetric equilibrium, which leads to nonlinear oscillations of the plasma around the new bifurcated equilibrium under the same initial conditions as in the usual linear and nonlinear calculations.

Since the nonlinear stabilization occurs even for an incompressible plasma, it is thought to be due to magnetic tension. In the nonlinear calculation it should be noted that compressibility is important even for modes near a marginally stable state. It weakens the nonlinear stabilizing effect. Compressibility allows a mean radial flow, which changes the quasilinear components of the magnetic field, the pressure, and the density. Changes of the quasilinear components by an average radial flow decrease the nonlinear stabilizing effect.

However, the change of the initial equilibrium due to nonlinear mode-mode coupling not included in the perturbation theory, due to external boundary conditions, i.e., the conservation of the total toroidal current, and due to dissipative effects change the potential field, especially in the region where the amplitude is very small, and prohibit the plasma from returning to the initial state.[10] Then, the nonlinear oscillation is limited to be near the new bifurcated helical equilibrium, and the perturbation leads to saturation with a helical deformation.

What is presented here concerning the nonlinear behavior of modes near a marginally stable state is considered to be qualitatively applicable even to modes far away from marginally stable states, as shown in Refs. 9 (for an incompressible RFP), 7 and 8 (for a compressible RFP), 18 (for an incompressible tokamak), and 17 (for a compressible tokamak). As the modes deviate from the marginally stable states, the change of the potential field from the variation of the initial equilibrium due to the nonlinear mode-mode coupling and from higher order terms, etc., increases. Hence, the nonlinear oscillations are thought to become so modified that the nonlinear behavior leads to saturation with a slight nonlinear oscillation.

The effects of toroidicity, i.e., toroidal curvature and poloidal mode coupling, are important for quasi-interchange modes in tokamaks.[19] Both effects, however, are thought to stabilize $n = 1$ perturbations in the nonlinear phase. Hence, the qualitative results do not change. More extensive studies of quasi-interchange modes that include the effects of toroidicity are now underway.

The nondegenerate case, i.e., the resistive modes, will be reported in a future paper.

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Figure Captions

1. Potential E_p vs. amplitude $\mathcal{A}$ for a nonresonant kink mode in an RFP: solid curve (compressible case), broken curve (incompressible case).

2. Potential E_p vs. amplitude $\mathcal{A}$ for a quasi-interchange mode in a tokamak: solid curve (compressible case), broken curve (incompressible case).

Table 1

kc	c_0	c_1	c_3
1.354	0.7151	-0.02513	0.1573 (compressible)
			0.3044 (incompressible)

Table 2

	kc	c_0	c_1	c_3
(I)	0.3450	0.4824	1.000×10^{-2}	8.947×10^{-4} (compressible) 2.511×10^{-3} (incompressible)
(II)	0.3447	0.1750	2.815×10^{-3}	1.329×10^{-4} (compressible) 4.729×10^{-4} (incompressible)

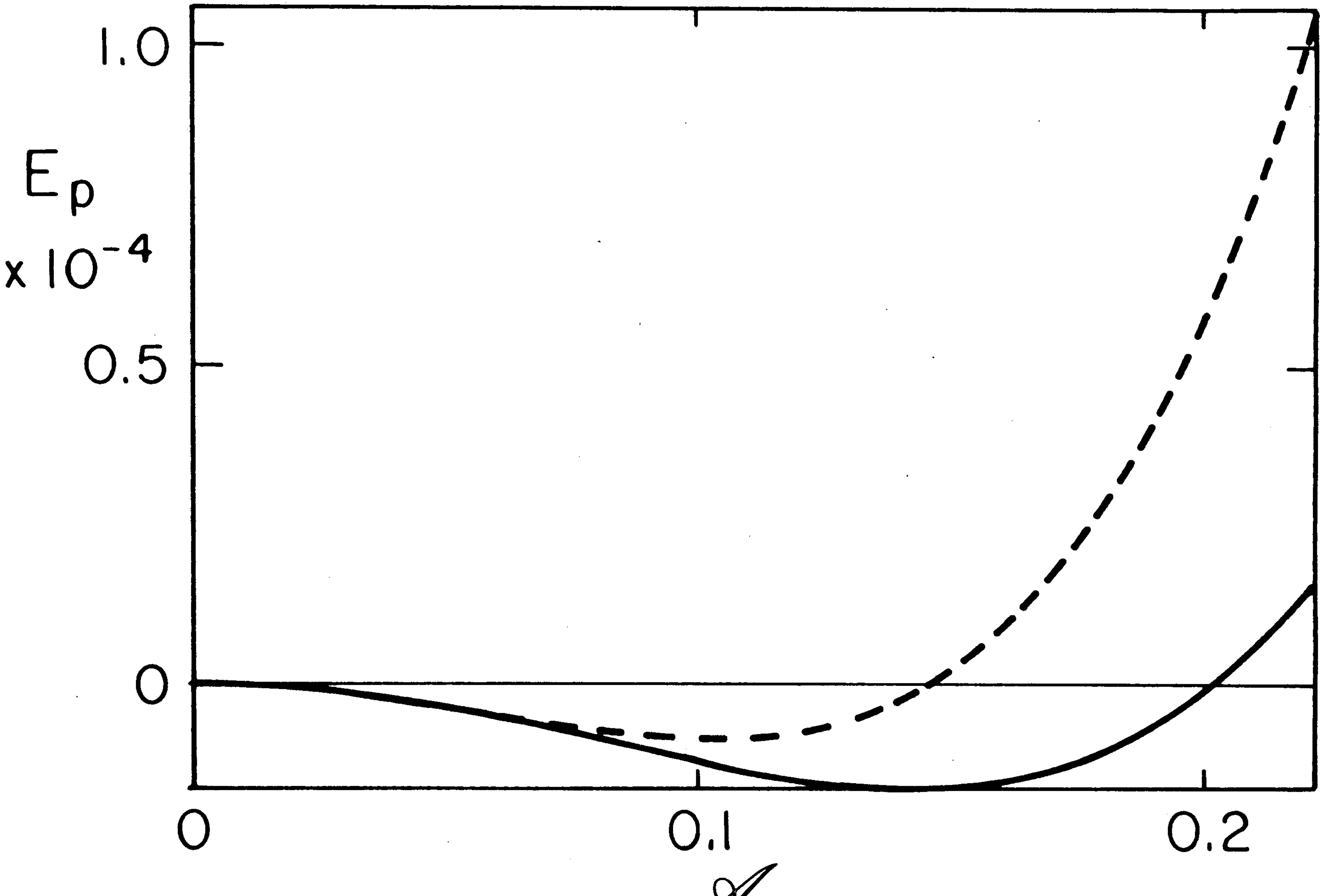

Figure 1

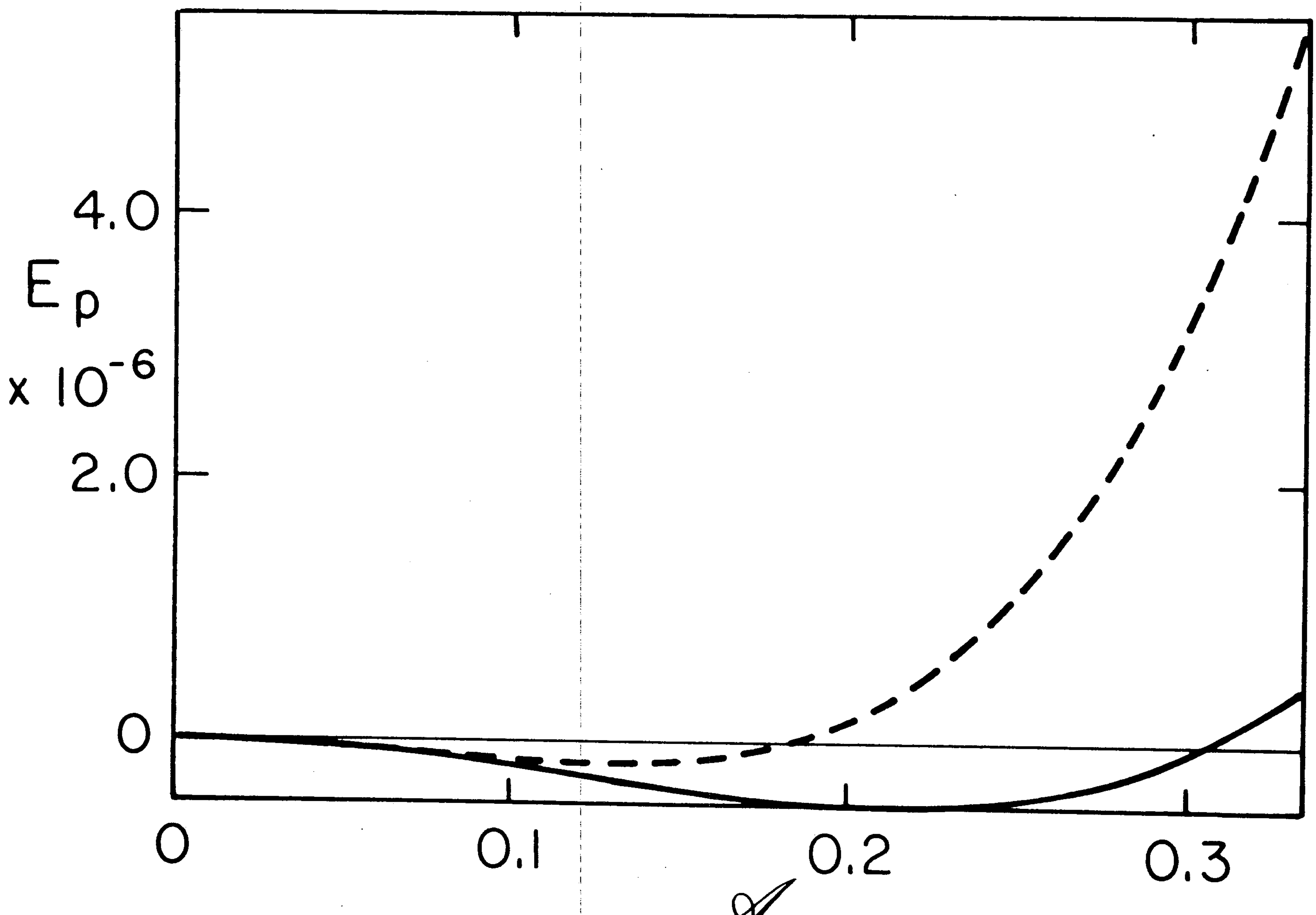

Figure 2