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Equilibrium of a Plasma in the Fluid- and Vlasov-Maxwell Systems

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Abstract

It is shown that a recently constructed exact solution of the Vlasov equation describing a plasma with density and temperature gradients can be expressed in terms of the constants of motion.The distribution function is then used to illustrate the differences between a Vlasov and a one fluid description. In fluid theory, only the pressure profile is determined (unless one postulates an equation of state), while the Vlasov description leads to a separate determination of density (g), and temperature (ψ^2) profiles; the equation of state, $g = \psi^{3-2/\beta}$, comes out naturally in the latter case.

I. INTRODUCTION

A realistic description of the current laboratory plasmas requires the consideration of density and temperature gradients. To deal with this problem, an exact phase-space distribution function (as an infinite series) satisfying the equilibrium Vlasov equation was recently constructed[1]. The expansion coefficients in the infinite series are determined as powers of the parameter $\lambda = u_{0\alpha}/v_{0\alpha}$, where $u_{0\alpha}$ and $v_{0\alpha}$ are respectively the drift and thermal speeds of αth species. For small λ, the series can be readily truncated, and one obtains (in this kinetic description) two separate differential equations relating the density (g) and temperature (ψ^2) profiles to the electromagnetic fields. The procedure results in a natural equation of state $g = g(\psi^2)$, and one is able to circumvent the problem associated with the fluid theories where an equation of state is to be externally imposed, otherwise one is left only with an equation determining the pressure profile alone. It is the primary purpose of this note to illustrate this important advantage of the Vlasov over the fluid description, i.e., the automatic and natural determination of individual density and temperature profiles in the Vlasov theory.

However, we begin with a digression in which we prove that the constructed infinite series distribution function can be expressed in terms of the constants of motion. Notice that the equivalence was shown only to $O(u_{0\alpha}/v_{0\alpha})$ in Ref. 1. Although retaining terms to $O(u_{0\alpha}/v_{0\alpha})$ is quite adequate for most of the magnetically confined fusion devices, the general proof, in addition to its aesthetic aspects, places the solution on a sound footing for other possible application.

The rest of this note contains a comparison between the Vlasov, and the steady-state one-fluid model, which replaces the Vlasov equation by the hydrodynamical force balance equation, $c\nabla p = \mathbf{J} \times \mathbf{B}$. In an axisymmetric geometry, the fluid-Maxwell system reduces to the well-known Grad-Shafranov differential equation[2,3] for the equilibrium magnetic surface function Φ. The standard method to solve this equation is to specify two parameters: the pressure and toroidal magnetic field in terms of Φ. Once a solution for Φ is found, other quantities are eventually determined. In this paper, we start from this method to find the fluid solutions. We consider the same geometry and examples as in Ref. 1: a circular cylindrical model varying only in the radial direction, and the simple pinch and tokamak equilibria.

We find that for the examples considered here, only one parameter is needed to obtain the solutions to the fluid-Maxwell system, which is consistent with the Vlasov-Maxwell model. We also show that the derived equilibrium profiles in both systems are exactly the same except that only the pressure profiles are obtained in the one-fluid model. This is well-known in the fluid theory in which an assumption of an equation of state is required for closure. This additionalequation of state is explicitly derived in the Vlasov-Maxwell system, and is $g = \psi^{3-2/\beta}$.

In Sec. II, we prove that the infinite series distribution function suggested in Ref. 1 is a function of the constants of the motion. Of particular note is that this function reduces to an ordinary displaced Maxwellian shape, if a constant parameter β, which measures the effect of the temperature gradient, is set equal to zero. In Sec. III, we start from the Grad-Shafranov equation, in a one-dimensional cylindrical form, to obtain the fluid solutions for a simple Z-pinch, and for a tokamak plasma. In Sec. IV, we state our conclusions by comparing the solutions in the fluid and kinetic theories.

II. SOLUTIONS IN TERMS OF INVARIANTS

We begin by considering the simple Z-pinch in which the plasma is embedded in a strong external field $B_0\hat{z}$ and has a current only in the z-direction, producing the self-consistent field $\mathbf{B} = B_\theta\hat{\theta}$. The infinite series distribution function describing the plasma with density and temperature gradients, given in Ref. 1, is rewritten here

$$f_\alpha = \frac{n_{0\alpha} g_\alpha}{\pi^{3/2} v_{0\alpha}^3 \psi_\alpha^3} \left[1 + \frac{2u_{0\alpha}}{v_{0\alpha}} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} C_{nm}^\alpha \left(\frac{v_z}{v_{0\alpha}} \right)^n \left(\frac{v}{v_{0\alpha}\psi_\alpha} \right)^{2m} \right] \exp\left(-\frac{v^2}{v_{0\alpha}^2 \psi_\alpha^2} \right), \quad (1)$$

where the density $n_\alpha = n_{0\alpha} g_\alpha$, the thermal velocity $v_\alpha = (2T_\alpha/m_\alpha)^{1/2} = v_{0\alpha}\psi_\alpha$, with g_α and ψ_α being the profile factors (at $r = 0$, $g_\alpha = \psi_\alpha = 1$), and $u_{0\alpha}$ is a measure of the drift velocity in the z-direction.

We first show the validity of the distribution function by proving that it is a function of the constants of the motion. Substituting Eq. (1) into the steady-state Vlasov equation, describing the equilibrium Z-pinch, yields the following general relationships for C_{nm}^α by equating power of v_z and v,

$$\frac{\psi_\alpha^3}{g_\alpha} \frac{d}{dr} \frac{g_\alpha}{\psi_\alpha^3} = -\frac{2u_{0\alpha} q_\alpha B_\theta}{m_\alpha v_{0\alpha}^2 c} C_{10}^\alpha = -h_\alpha B_\theta C_{10}^\alpha, \quad (2)$$

$$\frac{2}{\psi_\alpha} \frac{d\psi_\alpha}{dr} = -\frac{2u_{0\alpha} q_\alpha B_\theta}{m_\alpha v_{0\alpha}^2 c} C_{11}^\alpha = -h_\alpha B_\theta C_{11}^\alpha, \quad (3)$$

and

$$C_{nm}^\alpha = \frac{x^{n-1}}{n!m!} \sum_{s=0}^{m} (-1)^s (1 - sC_{11}^\alpha)^n D_{ms}, \quad (4)$$

where $h_\alpha = (q_\alpha u_{0\alpha})/(cT_{0\alpha})$, $x = 2u_{0\alpha}/v_{0\alpha}$, and $D_{ms} = m!/[(m-s)!s!]$ for $m \geq 0$ and $s \geq 0$. In Eq. (4), the coefficients have been generally expressed in terms of $C_{10}^\alpha = 1$ (chosen to be 1) and $C_{11}^\alpha = \beta_\alpha$. The summation from $m = 0$ to $m = \infty$ in Eq. (1) is, thus, given by

$$\sum_{m=0}^{\infty} C_{nm}^\alpha z_\alpha^m = \frac{x^{n-1}}{n!} \exp(z_\alpha) \sum_{s=1}^{\infty} \frac{(-z_\alpha)^s}{s!} (1 - s\beta_\alpha)^n, \quad (5)$$

where $z_\alpha = [v/(v_{0\alpha}\psi_\alpha)]^2$. Substituting Eq. (5) into Eq. (1), we can easily obtain

$$f_\alpha = \frac{n_{0\alpha}}{\pi^{2/3} v_{0\alpha}^3} \exp\left(\frac{u_{0\alpha}}{T_{0\alpha}} p_{z\alpha} \right) \exp\left[-\frac{H_\alpha}{T_{0\alpha}} \exp\left(-\frac{u_{0\alpha}}{T_{0\alpha}} \beta_\alpha p_{z\alpha} \right) \right], \quad (6)$$

where β_α measures the effect of the temperature gradient. Letting β_α equal to zero reduces Eq. (6) to the ordinary displaced Maxwellian distribution function. To derive Eq. (6), we have used Eqs. (2) and (3) to obtain $g_\alpha/\psi_\alpha^3 = \exp(h_\alpha A_z)$ and $\psi_\alpha^2 = \exp(\beta_\alpha h_\alpha A_z)$, where A_z satisfies $B_\theta = (\nabla \times \mathbf{B})_\theta = -dA_z/dr$, thus, $p_{z\alpha} = m_\alpha v_z + (q_\alpha/c)A_z(r)$. Obviously, it is a function of the constants of the motion, the total energy (H_α) and the canonical momentum ($p_{z\alpha}$).

III. SOLUTIONS TO FLUID-MAXWELL SYSTEM

A. A simple Z-pinch

The Grad-Shafranov equation, corresponding to fluid-Maxwell system in cylindrical geometry (variation only in the radial direction), takes the form

$$\frac{1}{r}\frac{d}{dr}r\frac{d\Phi}{dr} = -4\pi\frac{dp}{d\Phi} - I\frac{dI}{d\Phi}, \tag{7}$$

where p is the pressure and $I = B_z$ is the toroidal magnetic field. We note that the term on the left-hand side of Eq. (7) represents the part coming from $J_z\widehat{z} \times B_\theta\widehat{\theta}$, while the term $IdI/d\Phi$ on the right-hand side represents $J_\theta\widehat{\theta} \times B_z\widehat{z}$. Other quantities determined by Φ are the magnetic field $B_r = 0$, $B_\theta = d\Phi/dr$, and $B_z = I(\Phi)$, and the current components $J_r = 0$,

$$J_\theta = -\frac{c}{4\pi}\frac{dI}{dr}, \tag{8}$$

and

$$J_z = \frac{c}{4\pi}\frac{1}{r}\frac{d}{dr}rB_\theta. \tag{9}$$

Let us choose

$$p(\Phi) = n_0T_{0e}(1+\tau)\exp(h_1\Phi), \tag{10}$$

where $\tau = T_{0i}/T_{0e}$ and h_1 is a constant, to be identified later. Substituting Eq. (10) into Eq. (7) yields

$$\frac{1}{r}\frac{d}{dr}r\frac{d(h_1\Phi)}{dr} = -\frac{2}{\delta_{\text{eff}}^2}\exp(h_1\Phi), \tag{11}$$

where $\delta_{\text{eff}} = [2\pi n_0 T_{0e}(1+\tau)h_1^2]^{-1/2}$. Note that in the pinch equilibrium, $J_\theta = 0$ implying $IdI/d\Phi = 0$. The solution of Eq. (11) is

$$h_1\Phi = \ln\left[\left(1+\frac{r^2}{4\delta_{\text{eff}}^2}\right)^{-2}\right]. \tag{12}$$

To obtain equilibrium profiles explicitly, we need to calculate the current and pressure from the distribution function, Eq. (1). The results, retained to $O(u_{0\alpha}/v_{0\alpha})$, are

$$J_z = en_0 g\psi^2(5\beta/2-1)(1+\tau)u_{0e}, \tag{13}$$

and

$$p = n_0 T_{0e}(1+\tau)g\psi^2, \tag{14}$$

where $e > 0$. In deriving Eqs. (13) and (14), we have assumed that $g_i = g_e = g$ (quasineutrality) and $\psi_i = \psi_e = \psi$ (for long-lived equilibria), implying $T_{0i}/T_{0e} = -u_{0i}/u_{0e}$ and $\beta_i = \beta_e = -\beta$. The solutions to the Grad-Shafranov equation for the Z-pinch [with h_1 identified to be $-h_e(5\beta/2-1)$ from Eq. (13)] are thus given by

$$g\psi^2 = \left(1+\frac{r^2}{4\delta_{\text{eff}}^2}\right)^{-2}, \tag{15}$$

$$\frac{eu_{0e}}{cT_{0e}}B_\theta = (5\beta/2-1)^{-1}\frac{r}{\delta_{\text{eff}}^2\left(1+\frac{r^2}{4\delta_{\text{eff}}^2}\right)}, \tag{16}$$

and

$$J_z = en_0(1+\tau)(5\beta/2-1)u_{0e}\left(1+\frac{r^2}{4\delta_{\text{eff}}^2}\right)^{-2}, \tag{17}$$

where $\delta_{\text{eff}} = (c/\omega_{pe})(v_{0e}/u_{0e})(5\beta/2-1)^{-1}(1+\tau)^{-1/2}$.

The solutions are exact, except a constant (β) to be determined by either the boundary conditions, or possibly the experiment. For details, the readers may see Ref. 1.

B. A simple tokamak

A simple tokamak plasma is handled in the same manner. For simplicity, we shall assume that the electron current is in the z-direction, while the ion current is essentially in the θ direction. But, the self-consistent magnetic field produced by the J_θ current is to be ignored due to its smallness when compared with the strong external magnetic field $B_0\hat{z}$. Thus, J_θ through $J_\theta\hat{\theta} \times B_0\hat{z}$ manifests itself essentially in the force balance term (describing the ion motion), while only J_z is important for calculating the self-consistent field B_θ (representing the electron motion). It is a good approximation to separate the electron and ion parts from the Grad-Shafranov equation, Eq. (7),

$$\frac{1}{r}\frac{d}{dr}r\frac{d\Phi}{dr} = -4\pi\frac{dp_e}{d\Phi}, \tag{18}$$

and

$$4\pi\frac{dp_i}{d\Phi} + I\frac{dI}{d\Phi} = 0, \tag{19}$$

which is roughly equivalent to

$$\frac{dp_i}{dr} \approx \frac{1}{c}J_\theta B_0. \tag{20}$$

The solutions to Eq. (18), which is mathematically like the pinch system, are determined to be

$$g\psi^2 = \left(1 + \frac{r^2}{4\delta_t^2}\right)^{-2}, \tag{21}$$

$$\frac{eu_{0e}}{cT_{0e}}B_\theta = (5\beta/2 - 1)^{-1}\frac{r}{\delta t^2\left(1 + \frac{r^2}{4\delta_t^2}\right)}, \tag{22}$$

and

$$J_z = en_0(5\beta/2 - 1)u_{0e}\left(1 + \frac{r^2}{4\delta_t^2}\right)^{-2}, \tag{23}$$

where δ_t is equal to δ_{eff} without the factor τ, since we assume that there is no ion current in the z-direction. As to the ion part, we employ the distribution function, Eq. (1), by taking $u_{0i} = V_i$ and changing v_z to v_y, but assume that $g_i = g_e = g$ and $\psi_i = \psi_e = \psi$ to calaulate the current and pressure to $O(V_i/v_{0i})$,

$$J_\theta \;=\; -en_0(5\beta/2-1)V_i\, g\psi^2, \tag{24}$$

and

$$p_i \;=\; n_0 T_{0i} g \psi^2. \tag{25}$$

The combination of Eqs. (20), (24), and (25) yields

$$V_i \;=\; -\frac{cT_{0i}}{eB_0}(5\beta/2-1)^{-1}\frac{r}{\delta_t^2}\left(1+\frac{r^2}{4\delta_t^2}\right)^{-1}. \tag{26}$$

Equation (26) represents an expression for the poloidal drift speed as a result of the presence of B_0 and dp_i/dr, a force balance term for the ion motion, as we mentioned previously. Eqs. (21)-(23) and Eq. (26) form the complete solutions to the simple tokamak plasmas, which are very similar to the pinch profiles, except for the ion part.

IV. CONCLUSION

In our conclusion, we compare the solutions, Eqs. (15)-(17) for a simple Z-pinch and Eqs. (21)-(23) for a simple tokamak, with the derived equilibrium profiles in Ref. 1. We find that they are exactly the same, depending only on one parameter, with the exception that

$$g \;=\; \left(1+\frac{r^2}{4\delta_1^2}\right)^{-2(3\beta-2)/(5\beta-2)}, \tag{27}$$

and

$$\psi^2 \;=\; \left(1+\frac{r^2}{4\delta_1^2}\right)^{-4\beta/(5\beta-2)}, \tag{28}$$

were attained individually in the Vlasov-Maxwell system instead of the coupled solution $g\psi^2$ in the fluid-Maxwell system, where δ_1 is equal to δ_{eff} (pinch-like profiles) or to δ_t (tokamak-like profiles). An equation of state required to provide the complete fluid solutions is clearly shown in Eq. (46) of Ref. 1, i.e., $g = \psi^{3-2/\beta}$. Since separate knowledge of the density and temperature profiles is essential in understanding a vast class of phenomena in magnetically confined plasmas, the Vlasov approach is highly recommended.

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