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Anomalous Electron Thermal Conduction from Magnetic Turbulence

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Abstract

The electron thermal balance equation from Braginskii equations with the finite Larmor radius heat flux is analyzed for the space-time averaged power balance in the presence of electromagnetic fluctuations. Formulas for the anomalous thermal flux associated with the $\mathbf{E} \times \mathbf{B}$ motion and the magnetic $\delta \mathbf{B}_{\perp}$ fluctuation are derived and evaluated for the c/ω_{pe} scale electromagnetic turbulence typical of the ∇T_e driven short wavelength drift modes. The result is compared with several Ohkawa-type formulas for the anomalous electron thermal energy transport.

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In this brief communication, we analyze the anomalous electron thermal flux from Braginskii equations[1] and evaluate the formula for the skin depth (c/ω_{pe}) electromagnetic turbulence using the mixing length estimate of the saturated amplitude of the fluctuations.[2]

The electron thermal energy balance equation is given by[1]

$$\frac{3}{2}\frac{\partial}{\partial t}\left(n_e T_e\right) + \nabla \cdot \left[\frac{5}{2} n_e T_e \left(\mathbf{v}_E + \mathbf{v}_{de}\right) + \mathbf{q}_{de} + \mathbf{q}_c\right] = \left(\mathbf{v}_E + \mathbf{v}_{de}\right) \cdot \nabla p_e \ , \tag{1}$$

where

$$\mathbf{v}_E = \frac{c\mathbf{E}\times\mathbf{B}}{B^2} \quad , \quad \mathbf{v}_{de} = -\frac{c\widehat{\mathbf{b}}\times\nabla p_e}{e n_e B} \quad , \quad \mathbf{q}_{de} = \frac{5}{2}\left(p_e \widehat{\mathbf{b}}\times\nabla T_e\right)/m_e\omega_{ce} \ ,$$

$$\mathbf{q}_c = -\kappa_\parallel^e \nabla_\parallel T_e - \kappa_\perp^e \nabla_\perp T_e = \widehat{\mathbf{b}} q_\parallel + \mathbf{q}_\perp \ ,$$

$$\kappa_\parallel^e = 3.16\,\frac{n_e T_e}{m_e \nu_e} \qquad \text{and} \qquad \kappa_\perp^e = 4.66\,\frac{n_e T_e \nu_e}{m_e \Omega_e^2} \ .$$

Averaging Eq. (1) over the rapid fluctuations gives

$$\frac{3}{2}\frac{\partial}{\partial t}\left\langle n_e T_e\right\rangle + \frac{\partial}{\partial x}\left\langle \frac{5}{2} n_e T_e \left(v_{Ex} + v_{dex}\right) + q_{de,x} + \left(b_x q_\parallel + q_x\right)\right\rangle = \left\langle \mathbf{v}_E \cdot \nabla p_e\right\rangle \ . \tag{2}$$

Using

$$\left\langle \frac{5}{2} n_e T_e\, v_{de,x} + q_{de,x}\right\rangle = \frac{5}{2}\frac{c}{eB}\left\langle \frac{\partial}{\partial y}\left(p_e T_e\right)\right\rangle = 0$$

and

$$\left\langle \mathbf{v}_E \cdot \nabla p_e\right\rangle = \frac{\partial}{\partial x}\left\langle n_e T_e v_{Ex}\right\rangle + 2\,\frac{\partial \ell n B}{\partial x}\left\langle n_e T_e v_{Ex}\right\rangle \ ,$$

we obtain the space-time averaged power balance equation

$$\frac{3}{2}\frac{\partial}{\partial t}\left\langle n_e T_e\right\rangle + \frac{\partial}{\partial x}\left\langle \frac{3}{2} n_e T_e v_{Ex} + \left(b_x q_\parallel + q_x\right)\right\rangle = 2\,\frac{\partial \ell n B}{\partial x}\left\langle n_e T_e v_{Ex}\right\rangle \ . \tag{3}$$

The source term in Eq. (3) is the toroidal pumping term and proportional to the toroidicity parameter and the electron thermal flux associated with the $\mathbf{E}\times\mathbf{B}$ motion.

The physics of the toroidal pumping term is understood as follows: The weaker toroidal magnetic field on the outside of the torus makes the $\mathbf{E} \times \mathbf{B}$ drift wave motion compress and expand the plasma fluid according to

$$\nabla \cdot \mathbf{v}_E = \mathbf{B} \times \nabla\phi \cdot \nabla \left(\frac{c}{B^2}\right) = -\frac{2}{BR_0}\left(\frac{\cos\theta}{r}\frac{\partial\phi}{\partial\theta} + \sin\theta\,\frac{\partial\phi}{\partial r}\right) ,$$

where we use $\mathbf{E} = -\nabla\phi$ and consider the usual circular cross-section tokamak with $B_T = B_0/(1 + r\cos\theta/R)$. The work done $(p\nabla \cdot \mathbf{v}_E)$ during this expansion and compression over several drift wave periods gives rise to a net energy transfer between the fluctuation kinetic energy and the ambient thermal energy density. The transfer rate, which we call toroidal pumping by the drift wave oscillations, is given by

$$\langle p\nabla \cdot \mathbf{v}_E\rangle = \frac{2}{R_0}\cos\theta\,\langle pv_{Ex}\rangle ,$$

where we use that $\langle pv_{Ey}\rangle = 0$ on the outside of the torus. The net $\mathbf{E} \times \mathbf{B}$ energy flux is $\frac{3}{2}\langle p\mathbf{v}_E\rangle = \frac{3}{2}\langle pv_{Ex}\rangle\,\widehat{e}_x$. In an outward anomalous energy flux $\langle p\,v_{Ex}\rangle > 0$, the net toroidal compressional work transfers energy from the ambient thermal energy to the drift wave fluctuations on the outside $(\cos\theta > 0)$ of the torus. This direction of energy transfer is consistent with the expansion of the plasma flux tubes in the outward motion on the outside of the torus.

From guiding-center theory of the orbits the toroidal energy transfer term may be understood from the effective toroidal force

$$F_{\text{eff}} = -\frac{P_\perp}{B}\nabla B - \frac{P_\|}{R^2}\mathbf{R}$$

acting on the guiding centers. The rate of work done by the force is

$$\langle \mathbf{v}_E \cdot \mathbf{F}_{\text{eff}}\rangle = -\frac{\left\langle \left(P_\perp + P_\|\right)\,v_{Ex}\right\rangle\cos\theta}{R_0} = -\frac{\langle p\,v_{Ex}\rangle}{R_0}\cos\theta ,$$

where we assumed $P_\perp = P_\| = P$. In the guiding-center picture the energy taken from the thermal plasma and delivered to the fluctuation is understood to be due to guiding centers falling down the effective force field $\mathbf{F}_{\text{eff}}$.

The anomalous electron thermal flux is given by

$$q_{ex}^{A} = \frac{3}{2}\langle n_e T_e v_{Ex}\rangle + \left\langle b_x q_{\|} + q_x \right\rangle = \frac{3}{2}\Gamma_{ex} T_e + K_{ex} , \tag{4}$$

as found in Eq. (3). In Eq. (4) we define

$$\mathbf{\Gamma}_e = \langle n_e \mathbf{v}_E \rangle \tag{5}$$

and

$$\mathbf{K}_e = \frac{3}{2}\langle n_e \rangle \langle T_e \mathbf{v}_E \rangle + \left\langle \mathbf{q}_{\|} + \mathbf{q}_{\perp} \right\rangle . \tag{6}$$

The anomalous electron thermal flux given by Eq. (4) includes the three different types of physics which are associated with the $\mathbf{E} \times \mathbf{B}$ motion of the electron guiding centers, the radial component of magnetic fluctuation and collisions.

Now we calculate the anomalous electron thermal transport given by Eq. (6) for the c/ω_{pe} scale electromagnetic turbulence typical of the ∇T_e driven short wavlengths drift modes.[2,3]

For the electron thermal transport associated with $\mathbf{E} \times \mathbf{B}$ motion it is given by the first term of Eq. (6);

$$K_{ex}^{\mathrm{ES}} = \frac{3}{2}\langle n_e \rangle \langle T_e\, v_{Ex} \rangle = -\chi_e^{\mathrm{ES}} \frac{dT_{e0}}{dx}$$

and

$$\chi_e^{\mathrm{ES}} \cong \left(\frac{m_e}{M_i}\right)^{1/2} \left(\frac{2q}{s}\right) \sqrt{\eta_e}\, \frac{\rho_s}{r_n} \frac{cT_i}{eB} , \tag{7}$$

where we use the saturated amplitude for temperature fluctuation $\left\langle \widetilde{T}_e^2 \right\rangle^{1/2} \sim \frac{1}{|k_x|}\left|\frac{dT_{e0}}{dx}\right|$ and for electrostatic potential fluctuation $\left|\frac{e}{T_e}\Phi\right| \sim \frac{\omega}{\omega_{*T_e}} \frac{\widetilde{T}_e}{T_e}$ as derived in Ref. 2. The value of $\langle k_x^2 \rangle$ in Ref. 2 is taken from the ballooning mode calculation for the linear fluctuations and gives the $(2q/s)$ dependence of Eq. (7). Equation (7) shows that the small scale ($\sim \rho_e$) electrostatic component of the turbulence produces a small, of order $(m_e/M_i)^{1/2}$, ρ_s scale drift wave anomalous transport. The transport in Eq. (7) scales similar to the neoclassical plateau formula $\chi_e^{\rm NC} \sim (\rho_e q)^2\, v_e/qR \sim \frac{\rho_e}{r_n}\, q\varepsilon_n\, \frac{cT_e}{eB}$ but is larger by the factor $\chi_e^{\rm ES}/\chi_e^{\rm NC} \sim \sqrt{\eta_e}/s\varepsilon_n$ as in the case of the toroidal η_i-mode.

The heat conduction from the radial component of the magnetic fluctuations is given by the second term of Eq. (6);

$$K_{ex}^{M} = \left\langle b_x^{*}\, q_{\|} \right\rangle = -\kappa_{\|}^{e} \left\langle \left(i\, k_{\|}\, \widetilde{T}_e + b_x\, \frac{dT_{e0}}{dx} \right) b_x^{*} \right\rangle . \tag{8}$$

The relation between the temperature fluctuation $\widetilde{T}_e$ and the magnetic fluctuation b_x is given by the following equations which are studied in Ref. 4 for the ∇T_e driven short wavelengths drift mode;

$$\left(\frac{\beta_i}{2} - \nabla_{\perp}^2 \right) \frac{\partial A}{\partial t} = -\tau(1+\tau)\, \frac{\partial \phi}{\partial z} + \tau\, \frac{\partial \widetilde{T}_e}{\partial z} - \frac{\tau(1+\eta_e)}{2}\, \beta_i\, \frac{\partial A}{\partial y} , \tag{9}$$

$$\frac{\partial \widetilde{T}_e}{\partial t} = -\tau \left[\eta_e - 2\varepsilon_n(1+\tau)(\Gamma-1) \right] \frac{\partial \phi}{\partial y} - 2\tau\varepsilon_n(2\Gamma-1)\, \frac{\partial}{\partial y}\, \widetilde{T}_e$$

$$-(\Gamma-1)\, \frac{\partial}{\partial z}\, \nabla_{\perp}^2 A + \chi_{\perp e} \nabla_{\perp}^2 \widetilde{T}_e + \chi_{\| e} \nabla_z^2 \widetilde{T}_e - \chi_{\| e}\, \frac{\beta_i}{2}\, \eta_e\, \frac{\partial^2}{\partial y \partial z}\, A , \tag{10}$$

where we use the dimensionless variables of $(x,y) \to \rho_{ei}(x,y)$, $z \to r_n z$ and $t \to r_n t/v_{ei}$ with $v_{ei} = (T_i/m_e)^{1/2}$, $\rho_{ei} = v_{ei}/\omega_{ce}$ and rescale the hydrodynamic fluctuations by

$$\left(\frac{e\Phi}{T_e},\, \frac{e v_{ei}}{cT_i}\, \frac{2}{\beta_i}\, A_{\|},\, \frac{\widetilde{T}_e}{T_{e0}} \right) = \frac{\rho_{ei}}{r_n}\, (\phi, A, \widetilde{T}_e) .$$

In terms of these dimensionless variables, collisional thermal diffusivities are

$$\chi_{\| e} = \kappa_{\|}^{e}\, \frac{2}{3n_e}\, \frac{1}{v_{ei}\, r_n} \qquad \text{and} \qquad \chi_{\perp e} = \kappa_{\perp}^{e}\, \frac{2}{3n_e}\, \frac{r_n}{v_{ei}\, \rho_{ei}^2} .$$

Substituting Eq. (9) into Eq. (10), we express the temperature fluctuation in terms of the magnetic fluctuation

$$\widetilde{T}_e \;=\; \left\{ \left[\omega \left(k_{\perp}^2 + \frac{\beta_i}{2} \right) - \tau\, \frac{\beta_i}{2}\, (1+\eta_e)\, k_y \right] (\eta_e - 2\varepsilon_n(\Gamma-1)(1+\tau))\, \frac{k_y}{1+\tau} \right.$$

$$-\left(\Gamma-1\right)k_z k_\perp^2 + i\chi_{\|e}\frac{\beta_i}{2}\eta_e k_z k_y\Bigg\} A$$

$$\Big/\left\{\omega - 2\Gamma\tau\varepsilon_n k_y - \frac{\tau}{1+\tau}\eta_e k_y + i\chi_{\perp e}k_\perp^2 + i\chi_{\|e}k_z^2\right\} \;. \tag{11}$$

The anomalous heat flux from the radial component of the magnetic fluctuations is given by

$$K_{ex}^M = -\frac{3}{2}\chi_{\|e} N \frac{\beta_i}{2}\frac{cT_i}{eB}\frac{\rho_{ei}}{r_n}\frac{1}{\eta_e}\frac{dT_{e0}}{dx}\sum_{\mathbf{k}}|A_k|^2 k_y \Bigg\{\Bigg[k_y\omega_k\left(\frac{\beta_i}{2}\eta_e - \frac{k_\perp^2+\frac{\beta_i}{2}}{1+\tau}\left(\eta_e - 2\varepsilon_n(1+\tau)\right)(\Gamma-1)\right)$$

$$+\frac{\beta_i}{2}\tau k_y^2\left(\frac{\eta_e}{1+\tau} - 2\varepsilon_n(\Gamma-1) - 2\varepsilon_n(2\Gamma-1)\eta_e\right) + k_z^2 k_\perp^2(\Gamma-1)\Bigg]\left(\omega_k - \frac{\tau}{1+\tau}\eta_e k_y\right)$$

$$+\left[k_y\gamma_k\left(\frac{\beta_i}{2}\eta_e - \frac{k_\perp^2+\frac{\beta_i}{2}}{1+\tau}\left(\eta_e - 2\varepsilon_n(1+\tau)(\Gamma-1)\right)\right) + \chi_{\perp e}k_\perp^2 k_y\frac{\beta_i}{2}\eta_e\right]\left(\gamma_k + \chi_{\perp e}k_\perp^2 + \chi_{\|e}k_z^2\right)\Bigg\}$$

$$\Big/\left\{\left(\omega_k - \frac{\tau}{1+\tau}\eta_e k_y\right)^2 + \left(\gamma_k + \chi_{\perp e}k_\perp^2 + \chi_{\|e}k_z^2\right)^2\right\} \;. \tag{12}$$

It is evident that the magnetic fluctuations are small in the short wavelength region $(k^{-1} \sim \rho_e)$ where the η_e instability growth rate is maximum and thus the anomalous heat conduction from $|A_k|^2$ in this small scale is proportional to collision and negligible. But the numerical simulation study of Refs. 4, 5 and 6 shows that the magnetic fluctuation spectrum peaks at the lowest $k_x \sim k_y < \omega_{pe}/c$ in the system as the mode coupling terms terminate the exponential growth phase of the instability. The simulation also shows that the amplitude of the fluctuations in the saturated state are given to within numerical factors by the mixing length formulas obtained by balancing the mode coupling terms with the dominant linear terms driving the instability.

For $k_\perp < \omega_{pe}/c$ and $\left|\chi_{\|e}k_z^2\right| > |\omega|$, the growth rate is given by $\gamma_k \sim \sqrt{2\varepsilon_n\eta_e\tau(1+\tau)}\,|k_y|$, and from Eq. (12) with $K_{ex}^M = -\chi_e^M\frac{dT_{e0}}{dx}$, we obtain

$$\chi_e^M = \frac{3}{2}\frac{\tau}{1+\tau}\sum_{\mathbf{k}}\gamma_k\frac{k_y^2T_e^2}{k_z^2B^2e^2}\left|\frac{e}{T_e}A_{\|,k}\right|^2 \;. \tag{13}$$

The mixing length formula for the magnetic fluctuation is given by[2]

$$\frac{eA_{\|}}{T_e} \simeq \left|\frac{1-\omega_{*pe}/\omega}{1-\omega_{*pe}/\omega + c^2k_\perp^2/\omega_{pe}^2}\right| \left|\frac{ck_z}{\omega_{*e}}\right| \frac{1}{k_x r_n} \cong \left|\frac{ck_z}{\omega_{*e}}\right| \frac{1}{k_x r_n} \qquad \text{for} \qquad \frac{\omega_{pe}^2}{c^2} > k_\perp^2 \,. \tag{14}$$

The short wavelength drift wave model assumed in the simulations adds the condition $\left(\frac{\omega_{pe}}{c}\right)^2 > k_\perp^2 > \frac{m_e}{M_i}$ for consistency. With Eq. (14) and $\gamma_k \sim \sqrt{2\varepsilon_n\eta_e\tau(1+\tau)}\,|k_y|\left[\frac{v_{ei}}{r_n}\right]$, we obtain the anomalous electron thermal conductivity formula for c/ω_{pe} electromagnetic turbulence

$$\chi_e^M \cong c_0\,\sqrt{2\varepsilon_n\eta_e\tau(1+\tau)}\sqrt{\frac{\beta_i}{2}}\,\frac{v_{ei}}{r_n}\,\frac{c^2}{\omega_{pe}^2} \tag{15}$$

with $c_0 = \frac{3}{2}\frac{\tau^3}{1+\tau}$.

The transport given by Eq. (15) differs by the factor $c_0\,q\,\sqrt{\frac{\tau(1+\tau)}{\varepsilon_{T_e}}}\,\beta_i$ from the estimate using the simple Rechester-Rosenbluth formula[7] and $c_0\,\frac{qR}{r}\,\sqrt{\frac{\tau(1+\tau)}{\varepsilon_{T_e}}}\,\beta_i$ from the banana regime trapped electron transport given by Horton[8] *et al.* in Eq. (90). The Eq. (15) can be written in another form

$$\chi_e^M \cong c_0\,\sqrt{2\tau(1+\tau)\varepsilon_{T_e}}\,\frac{1}{r_{T_e}}\,\frac{cT_i}{eB}\,\frac{c}{\omega_{pe}} \tag{16}$$

and differs by the factor $c_0\,\sqrt{2\tau(1+\tau)\varepsilon_{T_e}}$ from the Zhang and Mahajan[9] formula arrived at from scaling arguments in the renormalized perturbation theory to the nonlinear electron gyrokinetic equation.

In this communication, we derive a reduced form of the electron thermal balance equation in Eqs. (3)–(8) and estimate the anomalous thermal conductivity formula for c/ω_{pe} electromagnetic turbulence problem in a self-consistent manner from the fluid equations which is typical of ∇T_e driven turbulence.

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