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Application of Newton's Method to Lagrangian Dynamical Systems

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Abstract: An algorithm for Newton's method is presented to find periodic orbits for N degree-of-freedom Lagrangian mappings. The computation time of the method is proportional to n, the period of the orbit, and the required storage space is reduced to $o(n)$. The construction of the algorithm is based upon block-diagonalization of the Hessian matrix of the action function. The index of the action function, which is closely related to orbital stability, is obtained during execution of the algorithm.

§1. Introduction

A Lagrangian system is one whose dynamics is described by a Lagrangian function $L(\mathbf{q},\dot{\mathbf{q}},t)$ through a variational principle. Many systems are Lagrangian including magnetic field lines, geodesic flows, geometric optics, hydrodynamics of ideal fluids, etc. A state of the system is represented by a point $\mathbf{q}$ in an N-dimensional configuration manifold $\mathcal{M}$, together with a velocity in the tangent space $T\mathcal{M}$.

The Lagrangian formulation can be converted into a Hamiltonian one by Legendre transformation, introducing canonical momenta, $\mathbf{p}=\partial L(\mathbf{q},\dot{\mathbf{q}},t)/\partial\dot{\mathbf{q}}$. The dynamics is given by flow of the one parameter group of diffeomorphisms in the phase space $(\mathbf{p},\mathbf{q})$. Its behavior can be equivalently viewed by looking at successive intersections on a sectional surface in the phase space which is transverse to the flow (Poincare surface of section).[1] For conservative systems with a compact energy surface the existence of such a section is guaranteed by Poincare's recurrence theorem.[2] The induced map on the surface of section is symplectic since a Hamiltonian phase flow preserves symplectic structure. A formal conversion of a continuous time system to a discrete one by introducing a map is possible for autonomous systems and periodically time dependent systems. The corresponding domain of the map in the Lagrangian formulation is $\mathcal{M}\times\mathcal{M}$.

Much of the analysis of dynamical systems relies on periodic orbits and the behavior of flows in their neighborhood. For example, quasiperiodic orbits can be obtained by interpolating nearby periodic orbits, the frequencies of which constitute the sequence of approximants to the frequency of the quasiperiodic orbit.[3,4]

Finding an orbit of a map T for given period and parameters is equivalent to solving a system of nonlinear simultaneous equations. By definition, an orbit of period n is obtained by finding a point $\mathbf{z}_0$ such that its n-th iterate returns to itself: $\mathbf{z}_0=(\mathbf{p}_0,\mathbf{q}_0)$ for symplectic maps or $\mathbf{z}_0=(\mathbf{q}_1,\mathbf{q}_0)$ for a map of configuration space. Namely, the system to be solved is $\mathbf{f}(\mathbf{z}_0) \equiv T^n\mathbf{z}_0 - \mathbf{z}_0 = 0$. This requires a 2N-dimensional root finder for 2N-dimensional mappings. Note that the evaluation of the $\mathbf{f}(\mathbf{z}_0)$ involves iterating the map n times, the period

of the orbit. When the desired orbit is unstable, an initial numerical error grows exponentially upon iteration of the map: $|\delta \mathbf{z}_t| \approx \exp(\sigma t)\ |\delta \mathbf{z}_0|$ where $\sigma > 0$ is the Lyapunov exponent. Thus the desired precision on $\mathbf{z}_t$ is lost along the orbit. For moderate periods the numerical error $|\delta \mathbf{z}_t|$ can easily become of order one; therefore, any scheme based on iteration becomes impractical for long enough, highly unstable, periodic orbits.

Alternatively, one can search for stationary configurations of the action function according to the variational principle for periodic orbits presented in §2. This scheme is suitable for finding unstable orbits since numerical errors on the orbit do not evolve under direct mappings. In this case the set of equations to be solved is $\mathbf{g}(\mathbf{q}_0, \mathbf{q}_1, \ldots \mathbf{q}_{n-1}) \equiv \nabla W_n = \mathbf{0}$, where W_n is the action function of the orbit and $\mathbf{g}$ is a $N \times n$ dimensional vector function. The number of equations to be simultaneously solved increases linearly with n, the period of an orbit. Accordingly, the main disadvantage of this method is that the computation time grows rapidly as n is increased. With straightforward Newton or quasi-Newton methods, the computation time grows as $o(n^3)$.[5]

In this paper we present a scheme employing Newton's method to find periodic orbits of Lagrangian systems whose configuration space is N-dimensional. As a preliminary, the Lagrangian dynamical system is introduced in §2 with a derivation of a map as discretization of a continuous time flow. Section 3 presents the linear algorithm for Newton's method. It is shown that the computation time of the resulting algorithm is proportional to n, the period of the orbit. It is also shown that the algorithm occupies storage space of size $o(n)$. Section 4 includes discussions on the problems of obtaining a good initial guess and numerical overflow stemming from the determinant of the Jacobian matrix becoming small. The relation of the index of the action function to orbital stability is also discussed. We illustrate the method for the case N=2.

§2. Lagrangian dynamical systems

Consider a dynamical system represented by a Lagrangian $L(\mathbf{q},\dot{\mathbf{q}},t)$. Here $\mathbf{q}$ is a point in some N-dimensional configuration manifold $\mathcal{M}$. The action of a path $\mathbf{q}(t)$, $t_0 \le t \le t_n$ is defined as

$$W[\mathbf{q}(t)] = \int_{t_0}^{t_n} L(\mathbf{q},\dot{\mathbf{q}},t)\,dt \tag{1}$$

Extremal paths, determined by setting the first variation δW to zero for $\mathbf{q}(t_0)$ and $\mathbf{q}(t_n)$ fixed, satisfy the Euler-Lagrange equations of motion.

Conversion to a discrete time mapping can be done formally for autonomous and periodically time dependent systems. To this end, introduce a partition of time interval, $[t_0, t_1, \cdots, t_n]$; each step corresponds to successive return of a flow to the Poincare surface of section for autonomous systems, or the strobing period for periodically time dependent systems. The discrete Lagrangian can be defined by integrating L along an extremal orbit segment, $\mathbf{q}(t)$, which begins at the point q_i at $t=t_i$, and ends at q_{i+1} at $t=t_{i+1}$

$$F(q_i,q_{i+1}) \equiv \int_{\substack{t_i \\ \mathbf{q}(t)\ \text{extremal}}}^{t_{i+1}} L(\mathbf{q},\dot{\mathbf{q}},t)\,dt \tag{2}$$

The action of an orbit segment from $t=t_i$ to $t=t_j$ can now be written

$$W(q_i, q_{i+1}, \ldots q_j) = \sum_{t=i}^{j-1} F(q_t, q_{t+1}) \tag{3}$$

Variation of W with respect to the intermediate points yields the mapping equations

$$\frac{\partial}{\partial q_t}\left[F(q_{t-1}, q_t) + F(q_t, q_{t+1})\right] = 0 \tag{4}$$

This equation locally defines a unique mapping, T, from (q_{t-1}, q_t) to (q_t, q_{t+1}) providing $\det(\partial^2 F/\partial q \partial q')$ is never zero. We make a stronger assumption, the twist condition: the mixed partial derivative matrix is assumed to be uniformly negative definite, i.e. there exists a $B>0$ such that for any δq, q, q'

$$\delta q \cdot b(q,q') \cdot \delta q \geq B \left| \delta q \right|^2$$

$$b(q,q') \equiv - \partial^2 F(q,q')/\partial q \partial q' \qquad (5)$$

It can be seen that this globally implies the existence of T.[6] The twist condition is the analogue of the Legendre condition in continuous time.

An orbit is a doubly infinite sequence of configurations $\ldots q_t, q_{t+1}, q_{t+2}, \ldots$ such that every finite segment $\{q_t, q_{t+1}, \ldots q_s\}$ is a stationary point of the action $W(q_t, q_{t+1}, \ldots q_s)$ for given q_t and q_s. An orbit is periodic with period n if

$$q_{t+n} = q_t \qquad (6)$$

such that n is the least integer satisfying Eq. (6). The action for a periodic orbit is

$$W_n(q_0, q_1, \ldots q_{n-1}) \equiv W(q_0, q_1, \ldots q_{n-1}, q_n = q_0) \qquad (7)$$

Equation (7) gives a variational principle for periodic orbits: a periodic orbit of period n is a critical point of W_n where all points $q_0, q_1, \ldots q_{n-1}$ are varied freely. This is true because criticality of W_n implies Eq. (4) for $0<t<n$, and variation with respect to q_0 implies $F_1(q_0, q_1) + F_2(q_{n-1}, q_0) = 0$, which implies (6) when (5) is satisfied.

The linear stability of the orbit is determined by considering the tangent map for the orbit, which is obtained by linearizing Eq. (4):

$$-b_{t+1}\, \delta q_{t+1} + a_t\, \delta q_t - \tilde{b}_t\, \delta q_{t-1} = 0$$

$$b_t \equiv - F_{12}(q_{t-1}, q_t)$$

$$a_t \equiv F_{11}(q_t, q_{t+1}) + F_{22}(q_{t-1}, q_t) \qquad (8)$$

giving a linear second difference equation. Here "$\tilde{}$" designates transpose of a matrix. The multipliers of a period n orbit are determined by the eigenvalue problem

$$\delta q_{t+n} = \lambda\, \delta q_t \tag{9}$$

which closes the recurrence relation (8).

§3. Linear algorithm for Newton's method

The algorithm for Newton's method can be written briefly as follows.

problem : Solve $\mathbf{g}(\mathbf{x}^*) = 0$ where $\mathbf{g}: \mathcal{R}^n \to \mathcal{R}^n$

Definition : $\mathbf{g}_k = \mathbf{g}(\mathbf{x}_k)$, $\mathbf{s}_k = \mathbf{x}_k - \mathbf{x}_{k-1}$ and J_k is Jacobian of $\mathbf{g}_k$

Input : $\mathbf{x}_0$

Do while $|\mathbf{s}_k| >$ given tolerance

evaluate J_k (step 1)

solve $J_k \mathbf{s}_{k+1} = -\mathbf{g}_k$ (step 2)

The vector $\mathbf{s}_k$ is the correction to the approximate solution $\mathbf{x}_k$.

Newton's method is well known for its rapid quadratic convergence.[7] That is,

$$\lim_{k\to\infty} \frac{|\mathbf{x}_{k+1} - \mathbf{x}^*|}{|\mathbf{x}_k - \mathbf{x}^*|^2} = C$$

where C is a bounded constant and $\{\mathbf{x}_k\}$ is a converging sequence to a solution $\mathbf{x}^*$. One of its disadvantages is that it usually requires $o(n^2)$ function evaluations to calculate the Jacobian matrix J in the step 1 above. Furthermore, in step 2, one has to solve a linear system requiring $o(n^3)$ arithmetic operations at each stage of the iteration; this can be reduced to $o(rn^2)$ for the case that J is a sparse matrix with band width r.[8]

We want to employ Newton's method to solve $\mathbf{g}(\mathbf{q}) \equiv \nabla W_n(q_0, q_1, \ldots q_{n-1}) = 0$. The Nn×Nn Jacobian matrix, $J \equiv \nabla\nabla W_n$, is given by Eq. (8) with the periodicity condition Eq. (6):

$$J \equiv \nabla\nabla W_n = \begin{pmatrix} a_0 & -b_1 & 0 & .. & 0 & -\tilde{b}_0 \\ -\tilde{b}_1 & a_1 & -b_2 & 0 & .. & 0 \\ .. & .. & \tilde{..} & .. & .. & .. \\ .. & 0 & -\tilde{b}_{n-3} & a_{n-3} & -b_{n-2} & 0 \\ 0 & .. & 0 & -\tilde{b}_{n-2} & a_{n-2} & -b_{n-1} \\ -b_0 & 0 & .. & 0 & -\tilde{b}_{n-1} & a_{n-1} \end{pmatrix} \qquad (10)$$

In the following we present an analytic inversion of **J**. The explicit form of J^{-1} is used to calculate **s**. Namely, instead of solving the linear system $J\,s = -g$, one obtains **s** directly by calculating $s = -J^{-1}g$. The resulting algorithm requires $o(n)$ computation time at each iterative stage while retaining the quadratic convergence of Newton's method.

The presentation of this method is divided into two parts; (A) inversion of **J** via block-diagonalization and (B) calculation of **s**. Below, we will denote the components of N×n dimensional vectors by $\tilde{x} \equiv (\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_{n-1})$ such that each x_i is N dimensional. Similarly, each entry of an Nn×Nn matrix is denoted by an N×N submatrix, as in (10). The index k denoting k-th stage of iterative Newton algorithm is dropped to avoid confusion with subscripts for vectorial components. Uppercase characters are used for matrices with the exception of the given matrices a_i and b_i.

(A) Inversion of J

The primary goal of this part is to obtain a decomposition of J into a product of easily invertible matrices (see Eq.(15)). We first show that J can be block-diagonalized using a non-orthogonal transformation U, which is upper-triangular with unit diagonal entries. Inversion of a triangular matrix is straightforward. Finally J^{-1} is obtained as a product of the inverses of the decomposing matrices (Eq.(18)).

First, consider the quadratic form $\tilde{x}Jx$ where **x** is an arbitrary vector. Successively completing the squares yields

$$\tilde{x}\,J\,x = \overbrace{\left(x_0 - A_0^{-1}b_1x_1 - B_0^{-1}x_{n-1}\right)}\, A_0 \left(x_0 - A_0^{-1}b_1x_1 - B_0^{-1}x_{n-1}\right)$$

$$+ \overbrace{\left(x_1 - A_1^{-1}b_2x_2 - B_1^{-1}x_{n-1}\right)}\, A_1 \left(x_1 - A_1^{-1}b_2x_2 - B_1^{-1}x_{n-1}\right)$$

$$+ \cdots$$

$$+ \overline{\left(x_{n-2}-A_{n-2}^{-1}b_{n-1}x_{n-1}-B_{n-2}^{-1}x_{n-1}\right)} A_{n-2} \left(x_{n-2}-A_{n-2}^{-1}b_{n-1}x_{n-1}-B_{n-2}^{-1}x_{n-1}\right)$$

$$+ \tilde{x}_{n-1}A_{n-1}\, x_{n-1} \qquad (11)$$

where the A_i and B_i are defined recursively by

$$\begin{aligned} A_0 &= a_0 \\ A_i &= a_i - \tilde{b}_i A_{i-1}^{-1} b_i \qquad 1 \le i < n-1 \\ A_{n-1} &= a_{n-1} - \tilde{b}_{n-1}A_{n-2}^{-1}b_{n-1} - D_0 - \tilde{D}_{n-1} \end{aligned} \qquad (12a)$$

$$\begin{aligned} B_{-1} &= I \\ B_i &= B_{i-1}\tilde{b}_i^{-1}A_i \qquad 0 \le i < n-1 \\ B_{n-1} &= I \end{aligned} \qquad (12b)$$

and we have introduced matrices D_i defined by

$$\begin{aligned} D_{n-1} &= \tilde{b}_{n-1}B_{n-2}^{-1} \\ D_i &= D_{i+1} + \tilde{B}_i^{-1}A_iB_i^{-1} \qquad n-1 > i \ge 0 \end{aligned} \qquad (12c)$$

Note that A_i and D_i-D_{n-1} are symmetric.

Thus Eq. (11) can be re-expressed as

$$\tilde{x}\, J\, x = \tilde{y}\, \mathcal{D}\, y \qquad (13)$$

where $\mathcal{D}$ is a block-diagonal matrix with the submatrices A_i along the diagonal. The vector y is obtained from x by a non-orthogonal transformation, $y = Ux$, where U is the upper triangular matrix

$$U = \begin{pmatrix} I & -A_0^{-1}b_1 & 0 & \cdots & \cdots & -B_0^{-1} \\ 0 & I & -A_1^{-1}b_2 & 0 & \cdots & -B_1^{-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & I & -A_{n-3}^{-1}b_{n-2} & -B_{n-3}^{-1} \\ \cdots & \cdots & \cdots & 0 & I & -A_{n-2}^{-1}b_{n-1}-B_{n-2}^{-1} \\ 0 & \cdots & \cdots & \cdots & 0 & I \end{pmatrix} \tag{14}$$

Since $\mathbf{x}$ is arbitrary in Eq. (13), we obtain the desired decomposition of J;

$$J = \tilde{U}\, \mathfrak{D}\, U \tag{15}$$

The inverse of U is obtained by solving for x_i in terms of y_i;

$$x_{n-1} = y_{n-1} \tag{16a}$$

$$x_i = y_i + A_i^{-1}b_{i+1}x_{i+1} + B_i^{-1}y_{n-1} \qquad n-2 \geq i \geq 0 \tag{16b}$$

$$= y_i + A_i^{-1}b_{i+1}\left(y_{i+1} + A_{i+1}^{-1}b_{i+2}x_{i+2} + B_{i+1}^{-1}y_{n-1}\right) + B_i^{-1}y_{n-1}$$

$$= \cdots$$

$$= y_i + A_i^{-1}b_{i+1}y_{i+1} + A_i^{-1}b_{i+1}A_{i+1}^{-1}b_{i+2}y_{i+2} + \cdots + A_i^{-1}b_{i+1}\cdots A_{n-3}^{-1}b_{n-2}y_{n-2}$$
$$+ \left(B_i^{-1} + A_i^{-1}b_{i+1}B_{i+1}^{-1} + \cdots + A_i^{-1}b_{i+1}A_{i+1}^{-1}b_{i+2}\cdots A_{n-2}^{-1}b_{n-1}B_{n-1}^{-1}\right) y_{n-1}$$

Each coefficient of y_j corresponds to $\left(U^{-1}\right)_{ij}$. That is, after straightforward matrix manipulations by using (12),

$$(U^{-1})_{ij} = b_i^{-1}\tilde{B}_{i-1}\tilde{B}_{j-1}^{-1}b_j \qquad 0 \le i \le j \le n-2$$

$$(U^{-1})_{in-1} = b_i^{-1}\tilde{B}_{i-1}D_i \qquad 0 \le i \le n-1$$

$$(U^{-1})_{ij} = 0 \qquad i > j \qquad (17)$$

Now J^{-1} is obtained from the inverses of $\mathcal{D}$ and U:

$$J^{-1} = U^{-1}\mathcal{D}^{-1}\tilde{U}^{-1}\ , \text{ i.e.,}$$

$$(J^{-1})_{ij} = \sum_{l=\max(i,j)}^{n-1} (U^{-1})_{il}\ A_l^{-1}\ (\tilde{U}^{-1})_{lj} \qquad (18)$$

Inserting Eq.(17) into Eq.(18) and using Eq.(12), we obtain the final expression for J^{-1}:

$$(J^{-1})_{ij} = b_i^{-1}\tilde{B}_{i-1}\left(D_{\max(i,j)} - D_{n-1} + \tilde{D}_i A_{n-1}^{-1} D_j \right) B_{j-1}\tilde{b}_j^{-1} \qquad (19)$$

Thus J^{-1} has been obtained in analytic form. It is explicitly symmetric. The matrices A_i, B_i and D_i are obtained recursively as shown in Eq.(12). However, we do not evaluate J^{-1} because this would require $o(n^2)$ arithmetic operations and $o(n^2)$ storage space for entries of the J^{-1}.

(B) Calculation of $\mathbf{s} = -J^{-1}\mathbf{g}$

By virtue of Eq.(19), the analytic form of $\mathbf{s}$ can be directly obtained.

$$-s_i = (J^{-1})_{ij}\ g_j$$

$$= \tilde{Q}_i \sum_{j<i} B_{j-1}\tilde{b}_j^{-1} g_j + b_i^{-1}\tilde{B}_{i-1} \sum_{j\ge i} Q_j g_j + \tilde{P}_i A_{n-1}^{-1} \sum_{j=0}^{n-1} P_j g_j \qquad (20)$$

where the matrices P_i, Q_i are defined as follows:

$$
\begin{aligned}
&P_{n-1} = I \\
&P_i = P_{i+1}\,\tilde{b}_{i+1}A_i^{-1} + \tilde{B}_i^{-1} && n-2 \geq i \geq 0 \\
&Q_i = P_i - D_{n-1}B_{i-1}\tilde{b}_i^{-1} && n-1 \geq i \geq 0
\end{aligned}
\tag{21}
$$

Equation (20) can be further simplified by introducing two vectors $\mathbf{u}_i$ and $\mathbf{d}_i$:

$$
\begin{aligned}
&\mathbf{u}_0 = \mathbf{d}_{n-1} = 0 \\
&\mathbf{u}_i = \mathbf{u}_{i-1} + B_{i-2}\tilde{b}_{i-1}^{-1}\mathbf{g}_{i-1} && 1 \leq i \leq n-1 \\
&\mathbf{d}_i = \mathbf{d}_{i+1} + Q_i\mathbf{g}_i && n-2 \geq i \geq 0
\end{aligned}
\tag{22}
$$

so that the recursive relations determine $\mathbf{u}_i$ upwardly from $\mathbf{u}_0$ and $\mathbf{d}_i$ downwardly from $\mathbf{d}_{n-1}$. The final expression for $\mathbf{s}$ is

$$
-\mathbf{s}_i = \tilde{Q}_i\,\mathbf{u}_i + b_i^{-1}\tilde{B}_{i-1}\,\mathbf{d}_i + \tilde{P}_i\left[A_{n-1}^{-1}\sum_{j=0}^{n-1}P_j\,\mathbf{g}_j\right] \tag{23}
$$

Once the i-independent term inside the bracket of Eq.(23) is obtained, the evaluation of each component $\mathbf{s}_i$ involves only two additions and four multiplications of N dimensional vectors and/or N×N matrices. As shown in Eq.(12), (21) and (22), all matrices and vectors are obtained recursively. To obtain $\mathbf{D}_i$ from $\mathbf{D}_{i+1}$ for example, one addition and two multiplications of matrices are required. Thus 3n-2 matrix operations are required to obtain $\mathbf{D}_i$ for all i. Therefore, the total number of arithmetic operations involved in calculating $\mathbf{s}$ at each stage of Newton method is proportional to n, which is the period of the orbit, neglecting the factor involving N. The function evaluations are involved only in calculating $\mathbf{a}_i$ and $\mathbf{b}_i$ as well as $\mathbf{g}_i$. Furthermore, the number of N×N matrices and N-dimensional vectors which need to be stored is proportional to n. Consequently, the size of storage space for this algorithm, which could have been $o(n^2)$, is reduced to $o(n)$.

§4. Discussion

The method presented in this paper is suitable for finding strongly unstable periodic orbits for a wide range of Lagrangian systems. When N=1 (i.e., for area preserving maps), the method is equivalent in its efficiency to the Green function method proposed by B. Mestel and I.C. Percival.[9] To the same extent as theirs, however, it is critical to choose a good initial trial solution. Even though there have been various techniques devised to supply a good initial trial, such as an extrapolation from solutions of lower parameters etc.,[9] this is still a difficult problem.

In fact, there are many methods in the literature for iterative solution of a set of nonlinear simultaneous equations.[5] In general, however, for such numerical algorithms there is no sufficient test for the existence of solutions. Convergence of an algorithm to a solution is guaranteed only when the solution exists and the initial trial solution is within the basin of attraction for the desired solution. This basin typically has a very complicated structure. That is, the system of equations to be solved is so complicated that it is often hopeless to analyze the behavior of the function in the neighborhood of a solution. Thus one should always try several methods to find the one most suitable to any particular problem.

Beside the linear dependence on the period of the orbit, another benefit of the present method is that the A_i's give information on the index of the action function of the orbit : the index[10] of a function is defined as the number of convex directions at its extremum. The index has been shown to be closely related to the orbital stability even though it is not a sufficient test for stability for general N.[11] That is,

$$\prod_{i=1}^{N} R_i = \left(-\frac{1}{4}\right)^N \frac{\prod_{t=0}^{n-1} \det A_t}{\prod_{t=0}^{n-1} \det b_t} \qquad (24)$$

where R_i is the residue associated with the corresponding reciprocal pair of multipliers, i.e., $R_i = 1/4\ (\ 2 - \lambda_i - 1/\lambda_i\ )$. Since the b_i's

have been assumed to be positive definite (Eq. (5)), Eq. (24) implies that an action minimizing orbit is hyperbolic for N=1.[12]

Implementation of Newton's method can be confronted with an overflow problem when **J** is nearly singular. That is, det(**J**) can become extremely small, which could result in overflow when computing $\mathbf{J}^{-1}$, even though $\mathbf{s} = -\mathbf{J}^{-1}\mathbf{g}$ is finite. Indeed this is the situation when the action is near an inflection point: as can be seen from Eq. (24), one of residues is very close to zero. A typical example is when an orbit is about to undergo tangent bifurcation where it has a pair of multipliers at +1. Another case is when the perturbation is so small that orbits are very close to the configurations of uniform rotation.

This problem cannot be resolved without loss of efficiency. However, its effect seems to be greatly lessened in our method by using det($\mathbf{A}_i$) instead of det(**J**). Namely, each det($\mathbf{A}_i$) can be finite even when det(**J**) would underflow. In examples we observe that det($\mathbf{A}_i$) indeed tends to be finite with the exception of det($\mathbf{A}_{n-1}$) which becomes very small for the nearly integrable case. However, $\mathbf{A}_{n-1}$ is not involved in determining the other matrices so that the algorithm less often encounters such a problem. Thus our method works for the orbits whose residues are fairly close to zero as well as for strongly unstable orbits.

We illustrate the present method with an N=2 example for which the generating function is[11]

$$F(\mathbf{q},\mathbf{q}') = \frac{1}{2}(\mathbf{q}'-\mathbf{q})\cdot\mathbf{b}\cdot(\mathbf{q}'-\mathbf{q}) - V(\mathbf{q}) \qquad (25)$$

where $V(q_1,q_2) = \frac{-1}{(2\pi)^2}\{k_1\cos(2\pi q_1)+k_2\cos(2\pi q_2)+h\cos[2\pi(q_1+q_2)]\}$

and $\mathbf{b} = \mathbf{I}$. The system represented by the Lagrangian (25) is periodic, that is, $F(\mathbf{q}+\mathbf{l},\mathbf{q}'+\mathbf{l})=F(\mathbf{q},\mathbf{q}')$, where $\mathbf{l}$ is an arbitrary integer vector. In such a case, the rotation frequency of an orbit, if it exists, can be defined as

$$\omega = \lim_{t-t'\to\infty} \frac{q(t) - q(t')}{t-t'}$$

And the periodicity condition for a periodic orbit, Eq.(6), can be generalized to

$$\mathbf{q}_{t+n} = \mathbf{q}_t + \mathbf{m} \qquad \mathbf{m} \text{ is an integer vector.}$$

Therefore, every periodic orbit has a frequency $\boldsymbol{\omega} = \mathbf{m}/n$.

The resulting map for the system (25) is equivalent to two coupled standard maps. Since the system is periodic in $\mathbf{q}$, the orbits can be depicted on a torus. The figure presented shows the symmetric[11] orbit of frequency $\boldsymbol{\omega} = (12,16)/21$ with $\mathbf{q}_0=(1/2,0)$ along the path in parameter space: $k_1=\varepsilon$, $k_2=.6\varepsilon$, $h=.4\varepsilon$ and ε varies from 0 to 8.0 with a step size $\Delta\varepsilon=.16$. For $\varepsilon=0$ the configuration corresponds to a uniform rotation, $\mathbf{q}_j = \boldsymbol{\omega}\cdot j + \mathbf{q}_0$. The multipliers of the orbit at $\varepsilon=8.0$ are $\lambda_1\approx 10^{18}$ and $\lambda_2\approx -10^{12}$. The orbit at ε is used as the initial trial to find the orbit at $\varepsilon+\Delta\varepsilon$. The required number of iterations of the Newton routine for each ε ranges from 3 to 5. As can be noticed from the figure, the points on the orbit tend to cluster together as ε becomes larger. Analogous clustering occurs for the area preserving case and leads to the formation of cantori.

The construction of our method relies only on the band structure of $\mathbf{J}$. The particular structure of $\mathbf{J}$ with band width $r=3N$ in Eq.(10) originates from the nature of the second difference equation (8) or the equation of motion (4), for general Lagrangian systems introduced in §2. Therefore the method presented here is expected to be applicable to a wide range of dynamical systems.

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Figure caption

Evolution of the points on a periodic orbit for the mapping Eq. (25). The box represents (q_1,q_2,ε) space which is $\mathcal{T}^2\times\mathcal{R}$. The parameter ε varies from 0 to 8.0 in the vertical direction. The unperturbed orbit at the bottom of the box has a point at each intersection of the tilted grid lines whose slopes are determined from the frequency $\omega=(12,16)/21$.

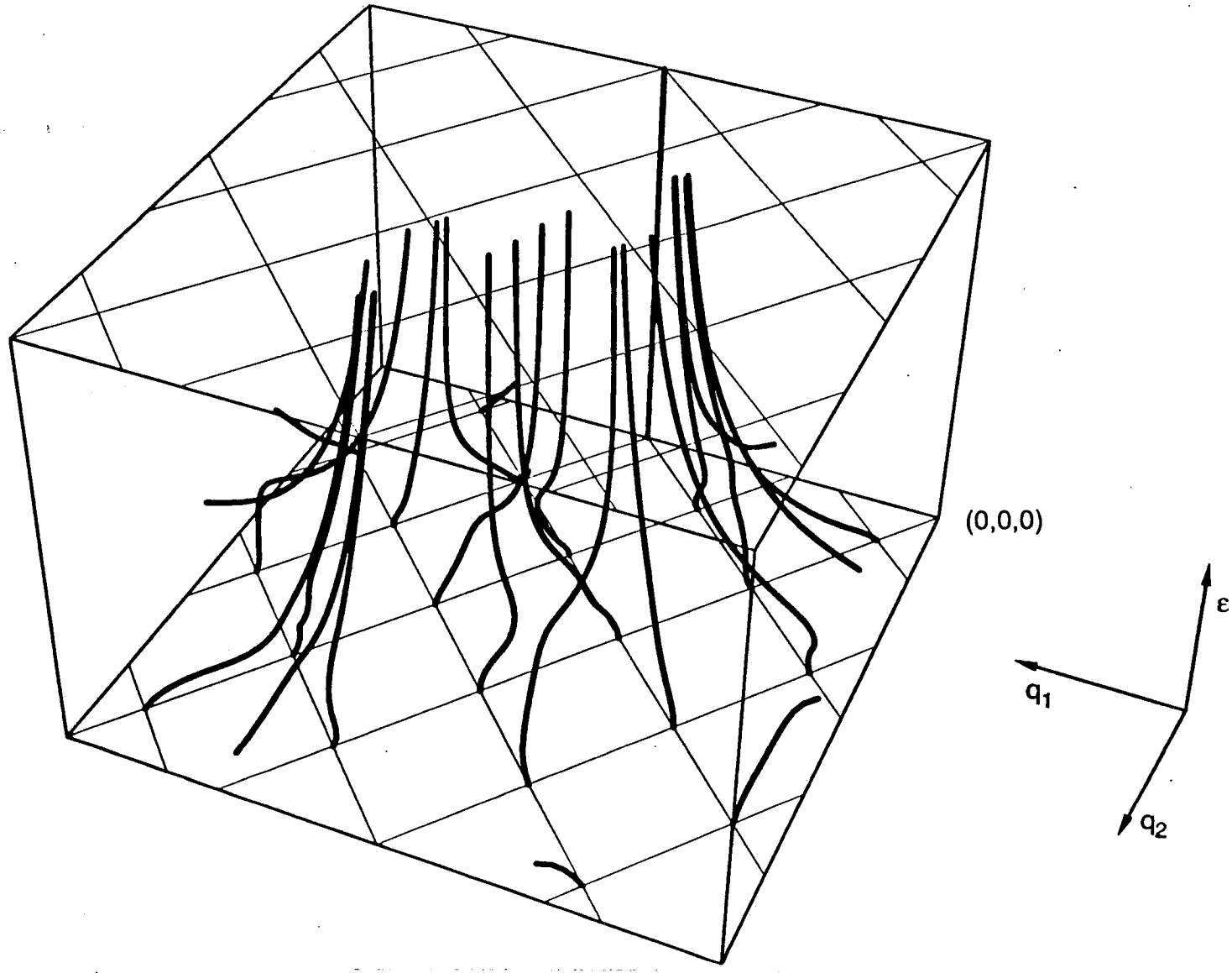

Fig. 1