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Abstract

A magnetoinductive (Darwin) particle simulation model developed for examining low frequency plasma behavior with large time steps is presented. Electron motion perpendicular to the magnetic field is treated as massless keeping only the guiding center motion. Electron motion parallel to the magnetic field retains full inertial effects as does the ion motion. This model has been implemented in two and three dimensions. Computational tests of the equilibrium properties of the code are compared with linear theory and the fluctuation dissipation theorem. This code has been applied to the problems of Alfvén wave resonance heating and twist-kink modes.

1. Introduction

Particle simulation models are valuable tools for understanding complex plasma behavior. However, the large number of time steps needed to simulate low frequency effects in particle models can be very expensive. Our computational model suitable for some applications combines Darwin's magnetoinductive model [1] for the electric and magnetic fields with a guiding center scheme [2] for electron motion perpendicular to the magnetic field. The fully relativistic electromagnetic codes often have excessive bremsstrahlung radiation because particle discreteness often results in larger current fluctuations than is physically realistic. The magnetoinductive model is nonradiative in character and thus has a lower background noise level while retaining self-consistent electric and magnetic fields of low frequencies. The application of the Darwin model to particle codes is discussed by Nielson and Lewis [3]. The method of solution of the field equations is similar to those described by Busnardo-Neto et al. [4]. A similar version of this model, which eliminates compressional Alfvén waves, has been used by Lee et al. [5] to investigate anomalous transport due to shear Alfvén waves.

We implemented this algorithm in a gridded two-and-one-half ($2\frac{1}{2}D$) dimensional configuration (two spatial and three velocity components) and in a three-dimensional code which uses mode expansion [6] for the third dimension. The basic physics of the model and its implementation into an algorithm is described in the next section. The expected dispersion and fluctuation characteristics of the homogeneous plasma model are compared with computational test results in Sec. III. Results from applications to Alfvén wave resonance heating and twist kink modes are presented in Sec. IV. The final section summarizes the paper.

II. Model

The time scale of the electron gyromotion is typically one of the shortest time scales in a plasma. When phenomena evolve on time scales much longer than this, it is convenient to treat this motion by the guiding center model[7]. The guiding center model assumes that the scale lengths of spatial variations of electric and magnetic fields are much longer than the gyroradius $\rho (k\rho \ll 1)$ and that the frequencies are much less than the gyrofrequency Ω $(\omega \ll \Omega)$. The leading terms dropping terms of $O(\omega/\Omega)$ for the drift motion perpendicular to the magnetic field are

$$\mathbf{v}_{D\perp} = \frac{c\mathbf{E} \times \hat{b}}{B} - \left[\frac{v_\parallel^2}{\Omega}(\hat{b} \cdot \boldsymbol{\nabla})\hat{b} + \frac{v_\perp^2}{2B\Omega}\boldsymbol{\nabla} B \right] \times \hat{b}, \tag{1}$$

where $\hat{b}$ is the unit vector in the direction of the magnetic field. The first term on the right-hand side of Eq. (1) is identified with the $\mathbf{E} \times \mathbf{B}$ drift, the second term with the curvature drift, and the third term with the $\boldsymbol{\nabla} B$ drift. Parallel kinetic effects resulting from finite electron inertia are retained. For the results presented in the applications (Sec. IV), the following equations of motion for electrons are used

$$\frac{d\mathbf{v}_{e\parallel}}{dt} = -\frac{e}{m}\mathbf{E}_\parallel, \tag{2}$$

$$\mathbf{v}_{e\perp} = \frac{c}{B_0^2}\mathbf{E} \times \mathbf{B}, \tag{3}$$

$$\frac{d\mathbf{x}_e}{dt} = \mathbf{v}_e, \tag{4}$$

where $\mathbf{E}_\parallel(\mathbf{x}) = \hat{b}(\mathbf{x}) \left[\hat{b}(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x})\right]$ and where $\hat{b}(\mathbf{x}) = \mathbf{B}(\mathbf{x})/|B(\mathbf{x})|$. The ion motion in this model is governed by the full nonrelativistic Newton-Lorentz equations

$$\frac{d\mathbf{v}_i}{dt} = \frac{e}{M}\mathbf{E} + \frac{e}{Mc}\mathbf{v}_i \times \mathbf{B}, \tag{5}$$

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i. \tag{6}$$

This model contains the essential ingredients of particle motion necessary for simulation of processes such as Alfvén wave heating. Electron Landau damping, for example, is included in our model. Finite ion Larmor radius effects are also included because of the full ion dynamics.

The Darwin model [1] neglects the transverse displacement current. The original intention of the Darwin model is to obtain the most accurate Lagrangian of the field-particle system as a function of the instantaneous velocities and positions of the particles. Such a Lagrangian must necessarily exclude radiation since its structure implies instantaneous action-at-a-distance. The explicit form of the Lagrangian is not required for present purposes, however.

The full electromagnetic equations are

$$\nabla \times \mathbf{E}_T = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{7}$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}_T + \frac{1}{c}\frac{\partial \mathbf{E}_T}{\partial t} \tag{8}$$

$$\nabla \cdot \mathbf{E}_L = 4\pi\rho \tag{9}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{10}$$

where the subscript T denotes the divergence-free (transverse) and where the subscript L denotes the curl-free (longitudinal) components of a vector. With these definitions, $\mathbf{E}_L$ and $\mathbf{E}_T$ satisfy

$$\nabla \times \mathbf{E}_L = \nabla \cdot \mathbf{E}_T = 0.$$

The Darwin approximation neglects the displacement current term in Eq. (8).

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}_T. \tag{11}$$

The magnetoinductive field equations differ by only one term from the full set of Maxwell's equations. This alteration changes the character of the field equations from an hyperbolic to an elliptic set of partial differential equations. The ellipticity of the reduced electromagnetic equations requires a different method of solution from that for the full Maxwell equations. The solution for $\mathbf{E}_T$ proceeds by eliminating the magnetic field from Eqs. (7) and (11) to obtain

$$\nabla^2 \mathbf{E}_T = \frac{4\pi}{c^2}\frac{\partial \mathbf{J}_T}{\partial t}. \tag{12}$$

The numerical instability [3] is avoided by expressing $\dot{\mathbf{J}}$ in a way that uses only present values of particle positions and velocities.

The electric and magnetic fields are computed on the grid using the Fast Fourier transform (FFT) technique. For the longitudinal electric field, the charge density is accumulated on the grid from quasiparticle positions. The charge density for a simulation plasma is given by

$$\rho(\mathbf{x}) = \sum_{j\epsilon g} q_j S(\mathbf{x} - \mathbf{x}_j) \tag{13}$$

where $S(\mathbf{x})$ is the particle shape factor and q_j is the charge of the quasiparticles. The simulation model uses Gaussian-shaped particles, whose explicit form in two dimensions is

$$S(\mathbf{x}) = \frac{1}{2\pi a_x a_y} \exp\left(-\frac{x^2}{2a_x^2} - \frac{y^2}{2a_y^2}\right)$$

and in three dimensions is given by

$$S(\mathbf{x}) = \frac{1}{(2\pi)^{3/2} a_x a_y a_z} \exp\left(-\frac{x^2}{2a_x^2} - \frac{y^2}{2a_y^2} - \frac{z^2}{2a_2^2}\right).$$

If an eigenmode expansion is used, for example in the z direction in addition to an x, y grid, the charge density may be expressed as

$$\rho(x, y, z) = \sum_{n=-N}^{N} \rho_n(x, y) \exp(i2\pi nz/L_z) \tag{14}$$

where

$$\rho_n(x, y) = \frac{1}{2\pi a_x a_y L_z} \exp\left(-\frac{k_z^2 a_z^2}{2}\right) \sum_j q_j \exp(-ik_z z_j) \exp\left[-\frac{(x - x_j)^2}{2a_x^2} - \frac{(y - y_j)^2}{2a_y^2}\right].$$

After interpolation to a gridded representation,

$$\rho(\mathbf{x}) = e \sum_g S(\mathbf{x} - \mathbf{x}_g) G(\mathbf{x}_g), \tag{15}$$

the charge density is expressed as a sum of values on a grid rather than as a sum over individual particles. With the inclusion of an eigenfunction expansion in the z-direction, a similar result for the n-th mode is obtained

$$\rho_n(x, y) = eS(k_z) \sum_g S(\mathbf{x} - \mathbf{x}_g) H_n(\mathbf{x}_g),$$

where

$$H_n(\mathbf{x}_g) = \sum_{j\epsilon g} \exp(-ik_z z_j) G(\mathbf{x}_g).$$

with $\mathbf{x}_g = (x_g, y_g)$ and $k_z = 2\pi n/L_z$.

Once the charge density is accumulated on the grid, it is fast Fourier transformed to $\mathbf{k}$-space. The equation for the longitudinal electric field is given by

$$i\mathbf{k} \cdot \mathbf{E}_L(\mathbf{k}) = 4\pi\rho(\mathbf{k}). \tag{16}$$

using the Fourier transform of Poisson's equation, Eq. (9).

Ampere's law for the magnetic field is solved similarly using the FFT. For the case of an eigenfunction expansion in z of the current,

$$\mathbf{J}(x, y, z) = \sum_{n=-N}^{N} \mathbf{J}_n(x, y) \exp(i2\pi nz/L_z)$$

where

$$\mathbf{J}_n(x, y) = \frac{1}{2\pi a_x a_y L_z} \exp\left(-\frac{k_z^2 a_z^2}{2}\right) \sum_j q_j \mathbf{v}_j \exp(-ik_z z_j)$$
$$\times \exp\left[-\frac{(x - x_j)^2}{2a_x^2} - \frac{(y - y_j)^2}{2a_y^2}\right].$$

The accumulation of the current density proceeds in the same way as the accumulation of the charge density. The equation for the current density expressed in terms of grid values is given by

$$\mathbf{J}(\mathbf{x}) = e \sum_g S(\mathbf{x} - \mathbf{x}_g) \mathbf{U}(\mathbf{x}_g). \tag{17}$$

The variable $\mathbf{U}$ is a measure of the velocity of the quasiparticles in the vicinity of the grid points. With a z direction eigenfunction expansion, the n-th mode of the accumulated current density may be written

$$\mathbf{J}_n(x, y) = eS(k_z) \sum_g S(\mathbf{x} - \mathbf{x}_g) \mathbf{W}(\mathbf{x}_g),$$

where

$$W_n(\mathbf{x}_g) = \sum_j^{\text{all particles}} \exp(-ik_z z_j) U(\mathbf{x}_g).$$

The Fourier transform of Ampere's law yields

$$\mathbf{B}(\mathbf{k}) = i\frac{4\pi}{c}\frac{\mathbf{k}\times\mathbf{J}(\mathbf{k})}{k^2}.$$

In our three-dimensional model, B_z is assumed constant (incompressible).

Solving for the transverse electric fields proves to be more arduous than solving for the magnetic field and for the longitudinal electric field. The method of solution developed here parallels the techniques reported by Busnardo-Neto et al. [4]. Changing the model to guiding-center motion adds complexity to their approach. If an external current is allowed, the transverse electric field is calculated from

$$\nabla^2 \mathbf{E}_T = \frac{4\pi}{c^2}\left[\frac{\partial \mathbf{J}_p}{\partial t} + \frac{\partial \mathbf{J}_{\text{ext}}}{\partial t} + \frac{\partial \mathbf{J}_m}{\partial t}\right]_T \tag{18}$$

where $\mathbf{J}_p$ is the contribution from the plasma particles and where $\mathbf{J}_{\text{ext}}$ is the contribution from some external source such as an antenna and J_m is the contribution due to the magnetization current. The plasma contribution is given by

$$\frac{\partial \mathbf{J}_p}{\partial t} = \sum_j q_{pj}\left[\frac{d\mathbf{v}_j}{dt}S(\mathbf{x}-\mathbf{x}_j) - \mathbf{v}_j\mathbf{v}_j\cdot\nabla S(\mathbf{x}-\mathbf{x}_j)\right]. \tag{19}$$

The full time derivative is applied to the particle velocity because the time derivative is evaluated in the particle's frame of reference. The second term on the right side of Eq. (19) is a convective derivative over the current density of the particle.

The complexity in obtaining the transverse electric field is apparent from the particle acceleration term of Eq. (19). The acceleration contains the very same transverse electric field that is being solved for. Using the particle equations of motion for the acceleration, the plasma contribution to the time derivative of the current becomes

$$\begin{aligned}\frac{\partial \mathbf{J}_p}{\partial t} &= q_p\frac{e}{M}\sum_j^{\text{ions}} S(\mathbf{x}-\mathbf{x}_j)\int\left[\mathbf{E}(\mathbf{x}') + \frac{\mathbf{v}_j}{c}\times\mathbf{B}(\mathbf{x}')\right]S(\mathbf{x}'-\mathbf{x}_j)d\mathbf{x}' \\ &+ q_p\frac{e}{m}\sum_j^{\text{electrons}} S(\mathbf{x}-\mathbf{x}_j)\int\left\{\hat{b}(\mathbf{x}')\left[\hat{b}(\mathbf{x}')\cdot\mathbf{E}(\mathbf{x}')\right]\right\}S(\mathbf{x}'-\mathbf{x})d\mathbf{x}' \\ &- \nabla\cdot\sum_j^{\text{all particles}} q_{pj}\mathbf{v}_j\mathbf{v}_j S(\mathbf{x}-\mathbf{x}_j).\end{aligned} \tag{20}$$

The first summation on the right side of Eq. (20) includes the ion acceleration. The second summation includes the parallel electron acceleration. The last summation expresses a convective derivative concerned with the flux of the current density of the particles into or from the neighborhood of $\mathbf{x}$. An important difference between this expression for $\dot{\mathbf{J}}$ from the previous cases [4] is the absence of an electron acceleration term perpendicular to the magnetic field. The perpendicular acceleration of the electron guiding center responds to the temporal change of the electric field. This is the acceleration that produces the polarization drift and is one order smaller in the expansion parameter ω/Ω_e than is retained in the model.(A preliminary attempt to include the $\dot{\mathbf{E}} \times \mathbf{B}$ acceleration term was unstable and this term was not included).

The combination of Eqs. (18) and (20) forms an integro-differential equation for the transverse electric field. The equation for the transverse electric force $[\mathbf{F}(\mathbf{k}) = S(\mathbf{k})\mathbf{E}(\mathbf{k})]$ in Fourier space is given by

$$\frac{k^2c^2}{V^2|S(k)|^2}\frac{N_0}{N_g}\mathbf{F}_T(\mathbf{k})+\omega_{pi}^2\left[\sum_{\mathbf{k}'} n_i(\mathbf{k}')\mathbf{F}_E(\mathbf{k}-\mathbf{k}')\right]_T+\omega_{pe}^2\left[\sum_{\mathbf{k}'} n_e(\mathbf{k}')\mathbf{F}_{\|}(\mathbf{k}-\mathbf{k}')\right]_T=\mathbf{\Gamma}_T(\mathbf{k}) \tag{21}$$

where

$$\mathbf{\Gamma}(\mathbf{k}) = -\omega_{pi}^2\sum_{\mathbf{k}'}\frac{\mathbf{U}_i}{c}(\mathbf{k}')\times\mathbf{F}_B(\mathbf{k}-\mathbf{k}')$$
$$+\frac{m}{e}\omega_{pe}^2 i\mathbf{k}\cdot I\!P(\mathbf{k}) - \frac{4\pi}{\mathbf{V}S(\mathbf{k})}\frac{N_0}{N_g}\left(\frac{\partial}{\partial t}\mathbf{J}_{\text{ext}}(\mathbf{k}) + \frac{\partial}{\partial t}\mathbf{J}_m(\mathbf{k})\right)$$

where n_e and n_i are the electron and ion number densities, $I\!P$ is the tensor formed by accumulating the charge weighted velocity products $\mathbf{vv}$ of the particles on the grid, and $\mathbf{U}_i$ is the grid weighted ion velocity. The magnetization current contribution $\mathbf{J}_m$ is calculated as follows.

$$\mathbf{J}_m = c\nabla\times\sum_j \hat{b}\mu_j S(\mathbf{x}-\mathbf{x}_j),$$

where $\hat{b} = \mathbf{B}/|B|$ and $\mu_j = mv_{\perp j}^2/2|B|$. From this

$$\frac{\partial \mathbf{J}_m}{\partial t} = -c\nabla\times\left\{\nabla\cdot\sum_j\left[\mathrm{v}_j\mu_j S(\mathbf{x}-\mathbf{x}_j)\hat{b}\right]-\sum_j\left[\mu_j S(\mathbf{x}-\mathbf{x}_j)\mathrm{v}_j\cdot\nabla\hat{b}(x)\right]\right\}.$$

We use an iterative procedure to solve Eq. (21). The right-hand side of Eq. (21) consists of known quantities and is therefore a source term which we call $\mathbf{\Gamma}$. The source term $\mathbf{\Gamma}$ is accumulated on the grid from the particle velocities. Equation (21) can be written in matrix form as

$$[L][F_T] = [\Gamma_T]$$

and has a formal solution of

$$[F_T] = [L]^{-1}[\Gamma_T].$$

The iterative procedure involves keeping the k'=0 terms in the summations on the left-hand side of Eq. (21) and transposing the other $k' \neq 0$ terms to the right side. In a gridded representation, the convolutions are handled by a matrix multiply in real space. This is the typical procedure of the pseudo-spectral renormalization method. The alias instability associated with the pseudo-spectral method, combined with the renormalized propagator, is avoided naturally by the finite size particle effects that cut off the higher aliases couplings. The transverse electric force equation is solved again with the values from the previous iteration placed in the summations with $k' \neq 0$ terms on the source side of the equation. This process repeats itself, usually a few iterations, until a convergence criterion is satisfied. In our three-dimensional code, the incompressible assumption is made; thus only the z-component of the transverse electric force is solved for. In codes with an eigenfunction expansion in one or more of the coordinates, a gridded representation in that coordinate is not done. For each eigenmode a matrix multiply may be done; however, between eigenmodes a standard convolution is done. This is typical of the spectral method, in which the exact convolution is computed.

The time dependence expressly appears in the equations of particle motion. A stable and accurate method of time differencing must be devised for the successful application of the model. This task is complicated by the differing structures of the equations governing the electron parallel motion and ion motion versus the electron perpendicular motion. The perpendicular equations of motion for electrons use a predictor-corrector [2] scheme while the ion and electron parallel equations of motion use a leapfrog scheme. By eliminating modes in a cone away from the magnetic field direction we enhance the time step because the numerical stability associated with high frequency waves improves.

A time cycle of the code is described below. From the previous time cycle, the particle positions and velocities $\mathbf{x}_j^n$, $\mathbf{x}_j^{n+1/2}$ and $\mathbf{v}_j^n$ are known where the time is given by $t = n\Delta t$. The field equations are solved to yield $\mathbf{E}_L^{n+1/2}$, $\mathbf{E}_T^n$, and $\mathbf{B}_T^n$. The leapfrog scheme for the ion and parallel electron motions advances the velocities to $\mathbf{v}_{e\|}^{n+1}$ and $\mathbf{v}_i^{n+1}$ and advances the positions to $\mathbf{x}_{e\|}^{n+3/2}$ and $\mathbf{x}_i^{n+3/2}$. To time center the velocity push, the magnetic fields and transverse electric fields are linearly extrapolated one-half time step forward. The perpendicular electron motion involves only a first derivative in time. The predictor-corrector scheme for the electrons provides a virtual time centered method of advancing the equations forward in time. A predicted electron position at time $(n+3/2)\Delta t$ is obtained from past and present information. A predicted longitudinal electric field $E_L^{*n+3/2}$ is solved from the predicted positions of the electrons and the positions of the ions. This information is used to predict advanced electron perpendicular velocities at the $(n+1)$-th time step. The modified form of Faraday's law is solved for a predicted transverse electric field after the predicted electron velocities are found. Finally, the predicted electric fields are used in a corrector step for the advanced electron velocity $\mathbf{v}_{e\perp}^{n+1}$ and position $\mathbf{x}_{e\perp}^{n+3/2}$. At this point the time cycle begins again.

III. Tests of Simulation Model

This section investigates the frequency, ω, and wavenumber, $\mathbf{k}$, characteristics of the thermal spectrum for a homogeneous simulation plasma. The comparison between the frequency characteristics of the fluctuation spectra produced by the simulation plasma with the predictions from linear homogeneous dispersion relations is an important test of the reactive characteristics of the numerical model. Another important test of the model's properties is the agreement between the field fluctuation levels and the theoretical predictions determined by statistical mechanics. For a homogeneous plasma in thermal equilibrium, the fluctuation dissipation theorem [8] can be used to predict the ensemble averaged (or time averaged) fluctuation levels of the electric and magnetic fields.

To obtain the dispersion relation, the electric and magnetic fields are assumed to have the phasor dependence $e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$. The electric field is expressed in terms of the current density when $\mathbf{B}$ is eliminated from Eqs. (8), (9), and (11) to yield

$$(n^2+1)\frac{\mathbf{k}(\mathbf{k}\cdot\mathbf{E})}{k^2} - n^2\mathbf{E} = \frac{-4\pi i}{\omega}\mathbf{J}$$

where $n^2 = k^2c^2/\omega^2$. In terms of indicial notation, we have

$$\gamma_{ij}E_j = \frac{-4\pi i}{\omega}J_i \tag{22}$$

where

$$\gamma_{ij} = (n^2+1)\frac{k_ik_j}{k^2} - n^2\delta_{ij}$$

and where δ_{ij} is the Kronecker delta function. Using linearized theory, the Fourier transformed plasma current of particle species α can be expressed as a linear function of the electric field through the susceptibility tensor χ^{α}_{ij};

$$J^{\alpha}_i(\omega,\mathbf{k}) = -i\omega\chi^{\alpha}_{ij}(\omega,\mathbf{k})E_j(\omega,\mathbf{k}). \tag{23}$$

The current densities of all species can be summed and combined with Eq. (22) to yield

$$\Lambda_{ij}(\omega,\mathbf{k})E_j(\omega,\mathbf{k}) = 0 \tag{24}$$

where Λ_{ij} is the dispersion tensor defined by

$$\Lambda_{ij} = \gamma_{ij} + 4\pi\chi_{ij}$$

and $\chi_{ij} = \sum_{\alpha} \chi_{ij}^{\alpha}$. The dispersion relation is obtained from setting the determinant of the dispersion tensor to zero

$$\Lambda \equiv \det |\Lambda_{ij}| = 0. \tag{25}$$

The roots of Eq. (25) for a given set $(\omega, \mathbf{k})$ determine the normal modes of oscillation.

The dispersion relation of a cold, homogeneous plasma assuming the magnetoinductive model with perpendicular guiding center electron motion is given by

$$\tan^2 \phi = \frac{-\left(1 + \frac{\omega_p^2}{\omega^2}\right)\left[\frac{k^2c^2}{\omega^2} - \frac{\omega_{pi}^2}{\Omega_i(\omega + \Omega_i)}\right]\left[\frac{k^2c^2}{\omega^2} + \frac{\omega_{pi}^2}{\Omega_i(\omega - \Omega_i)}\right]}{\left(\frac{\omega_p^2 + k^2c^2}{\omega^2}\right)\left[\left(1 - \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2}\right)\frac{k^2c^2}{\omega^2} + \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2}\left(1 + \frac{\omega_{pi}^2}{\Omega_i^2}\right)\right]} \tag{26}$$

where ϕ is the angle between $\mathbf{B}$ and $\mathbf{k}$, and $\omega_p^2 = \omega_{pe}^2 + \omega_{pi}^2$. Gaussian-shaped finite size particle effects are again included in Eq. (2.57) by simply replacing ω_{pi}^2 and ω_{pe}^2 by $\omega_{pi}^2 e^{-k^2a^2}$ and $\omega_{pe}^2 e^{-k^2a^2}$. The dispersion relation can be expressed as a cubic polynomial in ω^2. The roots of this polynomial determine the eigenfrequencies of the normal mode oscillations.

For propagation parallel to the magnetic field, the dispersion relation for electrostatic plasma oscillations is

$$\omega^2 = \omega_p^2 e^{-k^2a^2} + 3\frac{k^2T_e}{m}. \tag{27}$$

The first order thermal correction and finite size particle effects are included in Eq. (27). The spectrum of plasma oscillations has been investigated for a doubly periodic system and frequency peaks of the power spectra for the plasma frequency branch show good correspondence between the curve determined by Eq. (27) and points from the simulation results (not shown here). For propagation perpendicular to the magnetic field, the electrostatic spectrum exhibits peaks in the vicinity of the lower hybrid frequency, $\omega_{LH}^2 = \omega_{pi}^2 + \Omega_i^2$, and of harmonics of the cyclotron frequency. These peaks correspond to the ion Bernstein modes, which are predicted from warm plasma theory. The analysis of the electron Bernstein modes for the full dynamics electron model has been investigated by Kamimura et al. [9]. The dispersion relation for perpendicular propagation of the ion Bernstein modes is given by [10]

$$1 - \frac{e^{-k^2a^2}}{k^2\lambda_{Di}^2} \sum_{n=-\infty}^{\infty} \psi_n(\beta_i) \frac{n\Omega_i}{\omega - n\Omega_i} = 0, \tag{28}$$

where $\lambda_{Di}^2 = T_i/M\omega_{pi}^2$, and $\psi_n(\beta_i) = I_n(\beta_i)e^{-\beta_i}$ where I_n is the modified Bessel function of the first kind. The parameters for the run are: $L_x \times L_y = 128\Delta \times 320\Delta$, $a_x = 1.5\Delta$, $a_y = 15\Delta$, $v_{Te} = 1.0\omega_{pe}\Delta$, $M/m = 1600$, $T_i/T_e = 1.0$, $\Omega_i/\omega_{pi} = v_A/c = 1/3$ and $\theta = 3.1°$. The largest peaks of the spectrum are located at the lower hybrid frequency $\omega_{LH} \sim 3.2\Omega_i$. The spectrum has a zero frequency peak which is a zero frequency "convective mode" (Chu et al. [11]). The frequency peaks of the power spectra for the $(k_x, k_y = 0)$ modes plotted as a function of $k\Delta$ in Fig. 1 are in good agreement with Eq. (28).

For purely parallel propagation, the two electromagnetic modes predicted by the cold plasma analysis are the circularly polarized Whistler waves and the shear Alfvén waves. The Whistler wave were rotates in the same direction as the electron cyclotron motion whereas the shear Alfvén wave rotates in the same direction as the ion cyclotron motion. The shear Alfvén wave frequency is given by

$$\omega_s = \frac{k^2v_A^2}{2\Omega_i} e^{k^2a^2} \left[\left(1 + \frac{4\Omega_i^2}{k^2v_A^2} e^{-k^2a^2} \right)^{1/2} - 1 \right]. \tag{29}$$

For small k, the frequency ω has an approximately linear relationship to k, $\omega = kv_A$ and, for large k, the frequency approaches the ion cyclotron frequency $\omega = \Omega_i$. The peaks of the simulation frequency spectrum for $(k_x = 0, k_y)$ modes have reasonable correspondence between the simulation points and the theory (not shown here). The measured dispersion relation of the shear Alfvén wave for oblique propagation is displayed in Fig. 2. The $(k_x = 0, k_y)$ modes are plotted from a simulation run with parameters: $L_x \times L_y = 128\Delta \times 320\Delta$, $a_x = 1.5\Delta$, $a_y = 15\Delta$, $v_T = 1.0\omega_{pe}\Delta$, $M/m = 1600$, $T_i/T_e = 1.0$, $\Omega_i/\omega_{pi} = v_A/c = 1/3$, and $\theta = 3.1°$. The agreement between the simulation results and theory in Fig. 2 is good. The curve in Fig. 2 is the prediction from the full cold plasma dispersion relation, Eq. (25). The Whistler wave is not observed in the thermal spectrum. Tests with an antenna indicated however that the code displays the correct dispersion characteristics.

In the Vlasov theory the plasma dielectric may be written in the form,

$$\epsilon(\mathbf{k}, \omega) = 1 - \frac{k_\parallel^2c^2}{\omega^2} + \frac{\omega_{pe}^2}{\omega^2} \frac{\omega}{\sqrt{2}k_\parallel v_{Te}} Z\left(\frac{\omega + \Omega_e}{\sqrt{2}k_\parallel v_{Te}}\right) + \frac{\omega_{pi}^2}{\omega^2} \frac{\omega}{\sqrt{2}k_\parallel v_{Ti}} Z\left(\frac{\omega - \Omega_i}{\sqrt{2}k_\parallel v_{Ti}}\right).$$

For a strongly magnetized plasma, $\omega_{ce} \gg \omega$ and in the magnetoinductive approximation $k_{\|}^2/\omega^2 \gg 1$. This reduces the dielectric to the form

$$\epsilon(\mathbf{k},\omega) = \frac{-k_{\|}^2 c^2}{\omega^2} - \frac{\omega_{pe}^2}{\omega\Omega_e} + \frac{\omega_{pi}^2}{\omega^2}\frac{\omega}{\sqrt{2}k_{\|}v_{Ti}} Z\left(\frac{\omega-\Omega_i}{\sqrt{2}k_{\|}v_{Ti}}\right) .$$

The dispersion relation is given by the equation for the zeros of the dielectric $\epsilon(\mathbf{k},\omega)$=0. The shear Alfvén dispersion is shown in Fig. 3 along with the results of the three-dimensional simulation with parameters: $L_x \times L_y \times L_z = 32\Delta \times 32\Delta \times 1833\Delta$, $a_x = a_y = 1.5\Delta$, $a_z = 60\Delta$, $v_{Te} = 1.0\omega_{pe}\Delta$, $\Omega_i/\omega_{pi} = v_A/c = .4$, $M/m = 625$, $T_i/T_e = 1.0$, $\Omega_e/\omega_{pe} = 10.0$, and $\theta = 1°$.

For a periodic system consisting of Gaussian shaped particles, the fluctuation spectra for the longitudinal electric fields is predicted by Langdon and Birdsall [12] to be

$$V\frac{\langle E_L^2\rangle_{\mathbf{k}}}{4\pi} = \frac{T}{1+k^2\lambda_{De}^2 e^{k^2a^2}} \tag{30}$$

where $\langle\ \rangle$ denotes an ensemble averaging and V is the simulation volume. For this test run, the simulation is doubly periodic with the following parameters: $L_x \times L_y = 128\Delta \times 32\Delta$, $a_x = a_y = 1.5\Delta$, $V_T = 1.0\omega_{pe}\Delta$, $M/m = 40$, $T_i/T_e = 1.0$, $\Omega_e/\omega_{pe} = 2$, and $\theta = 0°$. The spectra were time averaged over approximately 16 ion cyclotron periods. The correspondence between the simulation observations and the expected values is shown in Fig. 4.

The level of the magnetic field fluctuations is a function of the angle ϕ between $\mathbf{k}$ and the magnetic field. The magnetic field fluctuations are obtained in terms of the current density fluctuations. The fluctuations of the current density are related to the linear response function using the fluctuation dissipation theorem [8]. Generalizing the fluctuation-dissipation theorem, A.G. Sitenko [13] has calculated the current density fluctuations in an infinite, nonisothermal plasma in terms of the imaginary (dissipative) part of the response function to be

$$\langle J_iJ_j\rangle_{\mathbf{k},\omega} = i\omega\left[\gamma_{ik}\Lambda_{km}^{-1}\right]^* \times \left[\gamma_{j\ell}\Lambda_{\ell n}^{-1}\right]\sum_{\alpha} T^{\alpha}\left(\chi_{mn}^{*\alpha} - \chi_{nm}^{\alpha}\right) \tag{31}$$

where γ_{ij} and Λ_{ij} are given by Eqs. (22) and (24), where $\sum_\alpha$ denotes the sum over different species, and where T^α is the temperature of species α.

Following Sitenko [13], let us introduce the tensor λ_{ij} whose elements are the cofactors of the dispersion tensor Λ_{ij},

$$\Lambda_{ij}\lambda_{jk} = \Lambda\delta_{ik}.$$

The current density fluctuations from Eq. (31) can then be expressed as

$$\langle J_i J_j \rangle_{\mathbf{k},\omega} = \frac{i\omega}{|\Lambda|^2}\Big\{ T_\parallel^e \gamma_{ik}\lambda_{k3}^* \gamma_{j\ell}\lambda_{\ell 3}\left(\chi_{33}^{e*} - \chi_{33}^e\right) + T^i \gamma_{ik}\lambda_{kn}^* \gamma_{j\ell}\lambda_{\ell n}\left(\chi_{mn}^{i*} - \chi_{nm}^i\right)\Big\}, \tag{32}$$

where the e and i superscripts refer to electrons and ions respectively. Inspection of the above formula indicates that the fluctuation spectrum has sharp peaks near the normal modes of plasma oscillation because of zeroes of the dispersion relation in the denominator, given by $\Lambda = 0$. For simplicity, we adopt the cold plasma theory to evaluate magnetic field fluctuations. This should be a reasonable approximation as long as the cold plasma theory can predict those weakly damped modes that contain the most energy. From Eq. (17), the magnetic field fluctuation spectrum can be computed by

$$\frac{\langle B^2 \rangle_{\mathbf{k},\omega}}{8\pi} = \frac{2\pi}{k^2c^2}\left[\langle J^2 \rangle_{\mathbf{k},\omega} - \frac{\langle k_i J_i k_j J_j \rangle_{\mathbf{k},\omega}}{k^2}\right] \tag{33}$$

once the current density fluctuation spectrum is known. The integration of Eq. (33) over all frequencies yields the time averaged fluctuation spectrum of the magnetic field

$$\frac{\langle B^2 \rangle_{\mathbf{k}}}{8\pi} = \int \frac{\langle B^2 \rangle_{\mathbf{k},\omega}}{8\pi}\frac{d\omega}{2\pi}. \tag{34}$$

The dispersion tensor contains the information for determining the current density fluctuations which subsequently can yield the magnetic field fluctuation spectrum. We shall examine the special cases where wave propagation is either completely parallel or perpendicular to the background magnetic field.

Let us first investigate fluctuation spectra for propagation parallel to the magnetic field, which is assumed to lie in the z-direction. The magnetic field fluctuation spectrum in Eq. (33) reduces to

$$\frac{\langle B^2 \rangle_{\mathbf{k},\omega}}{8\pi} = \frac{2\pi}{k^2c^2}\left[\langle J_x^2 \rangle_{\mathbf{k},\omega} + \langle J_y^2 \rangle_{\mathbf{k},\omega}\right]. \tag{35}$$

Here we evaluate the elements of the susceptibility tensor using the particle equations of motion in a cold fluid approximation. This gives a reasonable answer, because the fluctuation level for the resonant Alfvén waves is correctly described by the model. The coordinate axes are defined such that

$$\mathbf{B}_0 = B_0 \hat{k}$$

$$\mathbf{k} = k_x \hat{i} + k_z \hat{k}.$$

From Eqs. (2) and (3), the electron conductivity tensor elements are given by

$$4\pi\chi^e_{ij} = \begin{pmatrix} 0 & \dfrac{-i\omega_{pi}^2}{\Omega_i \omega} & 0 \\ \dfrac{i\omega_{pi}^2}{\omega\Omega_i} & 0 & 0 \\ 0 & 0 & -\omega_{pe}^2/\omega^2 \end{pmatrix} \tag{36}$$

Using Eq. (5), the ion conductivity tensor is given by

$$4\pi\chi^i_{ij} = \begin{pmatrix} \dfrac{-\omega_{pi}^2}{\omega^2 - \Omega_i^2} & -i\dfrac{\Omega_i}{\omega}\dfrac{\omega_{pi}^2}{\omega^2 - \Omega_i^2} & 0 \\ \dfrac{i\Omega_i}{\omega}\dfrac{\omega_{pi}^2}{\omega^2 - \Omega_i^2} & \dfrac{-\omega_{pi}^2}{\omega^2 - \Omega_i^2} & 0 \\ 0 & 0 & -\dfrac{\omega_{pi}^2}{\omega^2} \end{pmatrix} \tag{37}$$

in the cold plasma limit. Finite size particle effects can be included in Eqs. (36) and (37) by replacing ω_{pi}^2 and ω_{pe}^2 by $\omega_{pi}^2 e^{-k^2a^2}$ and $\omega_{pe}^2 e^{-k^2a^2}$ respectively.

Using the cold-plasma equations, the expression for the magnetic field fluctuation is

$$\frac{\langle B^2 \rangle_{\mathbf{k},\omega}}{8\pi} = \frac{iT^i}{2\omega}\frac{k^2c^2}{\omega^2}\left[\frac{\lambda_{11} + \lambda_{22}}{\Lambda} - c.c.\right]. \tag{38}$$

When ϕ=0, it can be seen that

$$\frac{\lambda_{11} + \lambda_{22}}{\Lambda} = \frac{\Lambda_{11} + \Lambda_{22}}{\Lambda_{11}\Lambda_{22} + \Lambda_{12}^2}.$$

We have

$$\frac{\langle B^2 \rangle_{\mathbf{k},\omega}}{8\pi} = \frac{iT^i}{2\omega}\frac{k^2c^2}{\omega^2} \times \left\{ \frac{2\left(\dfrac{k^2c^2}{\omega^2} + \dfrac{\omega_{pi}^2}{\omega^2 - \Omega_i^2}\right)}{\left[\dfrac{\omega_{pi}^2}{\Omega_i(\omega + \Omega_i)} - \dfrac{k^2c^2}{\omega^2}\right]\left[\dfrac{\omega_{pi}^2}{\Omega_i(\omega - \Omega_i)} + \dfrac{k^2c^2}{\omega^2}\right]} - c.c \right\}. \tag{39}$$

where c.c. is used to denote the complex conjugate

To obtain the time averaged spectrum, Eq. (39) is integrated over ω as described by Eq. (34). The resulting integral has the form $\frac{f(\omega)}{\omega - \omega'} d\omega$ where ω' is the resonant frequency of a normal mode. The undamped ideal normal modes have purely real frequencies so that the above type of integral over frequency encounters a singularity along the real axis of ω. Since the cold plasma undamped modes are a limiting case of the weakly damped modes, it is conventional to introduce a small imaginary frequency for damping to the (complex) frequency upon integration. By analytical continuation from this finite damping case (the Plemelj formula), we find that

$$\lim_{\nu \to 0} \int_{-\infty}^{\infty} \left[\frac{f(\omega)}{\omega - \omega' + i\nu} - c.c \right] \times d\omega = -2\pi i \int_{-\infty}^{\infty} f(\omega)\delta(\omega - \omega')d\omega; \tag{40}$$

since the principal parts of the integrals cancel.

The observed power spectra for the magnetic field reveal that, for parallel propagation, most of the energy is deposited in the shear Alfvén mode. Therefore the Plemelj formula is applied only to the poles representing the shear Alfvén resonance of Eq. (39). The magnetic field fluctuation spectrum is found to be

$$\frac{\langle B^2 \rangle_{\mathbf{k}}}{8\pi} = T^i \left(1 + \frac{4\Omega_i^2}{k^2 v_A^2} \right)^{-1/2} \left[\frac{\Omega_i^2}{\omega_s^2} - \left(1 + \frac{\Omega_i^2}{k^2 v_A^2} \right) \right]. \tag{41}$$

The inclusion of finite size particle effects and the periodicity of the simulation plasma modifies Eq. (41) to

$$V \frac{\langle B^2 \rangle_{\mathbf{k}}}{4\pi T^i} = 2 \left[1 + \frac{4\Omega_i^2}{k^2 v_A^2} e^{-k^2 a^2} \right]^{-1/2} \left[\frac{\Omega_i^2}{\omega_s^2} - \left(1 + \frac{\Omega_i^2 e^{-k^2 a^2}}{k^2 v_A^2} \right) \right]. \tag{42}$$

where ω_s is given by Eq. (29). There is good agreement between the simulation results in Fig. 5(a) and the prediction from Eq. (42). If the contribution from the pole representing the Whistler wave is evaluated along with the shear Alfvén wave, the fluctuation spectra is given by

$$\frac{\langle B^2 \rangle_{\mathbf{k}}}{8\pi} = T^i.$$

Let us now investigate the magnetic field fluctuation spectrum for wave propagation perpendicular to the magnetic field. The magnetic field fluctuations are given by

$$\frac{\langle B^2 \rangle_{\mathbf{k},\omega}}{8\pi} = \frac{2\pi}{k^2 c^2} \left[\langle J_y^2 \rangle_{\mathbf{k},\omega} + \langle J_z^2 \rangle_{\mathbf{k},\omega} \right] \tag{43}$$

from Eq. (33).

For the test results displayed here, the electron temperature is equal to the ion temperature such that $T_{\parallel}^{e} = T^{i} = T$. In this instance, the magnetic field fluctuation spectrum for perpendicular propagation is given by

$$\frac{\langle B^2 \rangle_{\mathbf{k},\omega}}{8\pi} = \frac{iT}{2\omega}\frac{k^2c^2}{\omega^2}\left[\Lambda_{22}^{-1} + \Lambda_{33}^{-1} - c.c\right]. \tag{44}$$

The observations of the power spectra indicate that most of the energy is deposited in a zero-frequency mode when $k_{\parallel} = 0$. The zero-frequency wave is not one that is derived from a collisionless model. This mode is proposed as a mechanism responsible for electron diffusion across the magnetic field by Chu, Chu, and Ohkawa [14]. When the electron parallel momentum equation, Eq. (2), has dissipation included as follows [14]

$$\frac{dv_{ez}}{dt} = \frac{-e}{m}E_z + \mu\nabla^2 v_{ez} - \nu\, v_{ez}$$

where ν is the electron-ion collision frequency and μ is the kinematic electron viscosity, the zz-element of the electron susceptibility tensor, χ_{33}, is given by

$$4\pi\chi_{33} = \frac{-\omega_{pe}^2}{\omega(\omega + i\eta)},$$

where $\eta = \nu + \mu k^2$.

The magnetic field fluctuation spectrum from Eq. (44) becomes

$$\frac{\langle B^2 \rangle_{\mathbf{k},\omega}}{8\pi} = T\frac{k^2c^2}{\omega_p^2 + k^2c^2}\frac{\eta - \gamma}{\omega^2 + \gamma^2} \tag{45}$$

where

$$\gamma = \frac{\eta(k^2c^2 + \omega_{pi}^2)}{k^2c^2 + \omega_p^2}.$$

Integrating over frequency, the time averaged fluctuation spectrum is given by

$$\frac{\langle B^2 \rangle_{\mathbf{k}}}{8\pi} = \frac{T}{2}\frac{k^2c^2\omega_{pe}^2}{(k^2c^2 + \omega_{pi}^2)}\frac{1}{(\omega_{pe}^2 + k^2c^2 + \omega_{pi}^2)}. \tag{46}$$

If the ion response is ignored, Eq. (46) reduces to the more familiar form

$$V\,\frac{\langle B^2 \rangle_{\mathbf{k}}}{4\pi T} = \frac{1}{1 + \dfrac{k^2c^2e^{k^2a^2}}{\omega_{pe}^2}} \tag{47}$$

when finite size particle effects and the periodicity of the simulation volume are included. Simulation results are depicted in Fig. 5(b) for the parameters: $L_x \times L_y = 128\Delta \times 32\Delta$, $a_x = a_y = 1.5\Delta$, $v_{Te} = 1.0\omega_{pe}\Delta$, $M/m = 40.0$, $T_i/T_e = 1.0$, $\Omega_e/\omega_{pe} = 2.0$, and $\theta = 0°$. The spectrum is averaged over approximately 16 ion cyclotron periods.

IV. Applications

This computational model has been successfully applied to problems of current interest in plasma physics. We present two examples of application of the present code: the first is the $2\frac{1}{2}D$ model to examine the resonant interaction of driven Alfvén waves and the second is a three-dimensional-study of nonlinear twist-kink modes.

(a). Kinetic Alfvén wave heating

An application to coupling of radio-frequency energy to the shear Alfvén resonance layer is presented. In a nonuniform plasma (in the x-direction), there exists a singularity in the ideal MHD equations associated with the shear Alfvén wave [15-17]. The MHD equations can be combined to [17]

$$\frac{d}{dx}\left[\frac{\varepsilon\alpha B^2}{\alpha B^2 k_\perp(x) - \varepsilon}\frac{dv_x}{dx}\right] - \varepsilon v_x = 0 \tag{48}$$

where

$$\varepsilon(x) = \omega^2 4\pi\rho(x) - k_\parallel^2(x)B^2(x)$$

$$\alpha(x) = 1 + \frac{\omega^2 c_s^2(x)}{v_A^2(x)[\omega^2 - k_\parallel^2 c_s^2(x)]}.$$

An electromagnetic wave with frequency ω will satisfy the resonance condition at a point x_s when $\varepsilon(x_s) = 0$ at $\omega = k_\parallel(x_s)v_A(x_s)$. Because of the existence of a continuous spectrum due to the singularity of the differential equation, Eq. (48), at $\varepsilon(x) = 0$, the wave energy may be dissipated at the resonance layer.

The physical absorption mechanisms were studied by Hasegawa and Chen [15,17]. The inclusion of kinetic effects changes the properties of the shear Alfvén wave, often termed the kinetic Alfvén wave, from the MHD solutions. For kinetic Alfvén waves, electron Landau damping [17] is expected to be the dominant heating mechanism for the parameters relevant to current experimental devices. Our investigations examine a homogeneous magnetized plasma slab in an unbounded system driven by a radio frequency antenna. The plasma behavior for an example when the wave generated by the antenna does not resonate with the plasma and for an example when the antenna driven wave satisfies the Alfvén wave resonance conditions is investigated in this subsection. The nonresonant case displays weak coupling between the plasma and the antenna, while the resonant case shows strong coupling between the plasma and the excited wave.

The simulation model is $2\frac{1}{2}$ dimensional (2 spatial and 3 velocity coordinates) and is periodic in both the x-direction and the y-direction. The antenna is located at approximately the centerline of the x-direction. The antenna current flows in the z-direction preserving the charge neutrality of the antenna. A sinusoidal travelling wave is launched by the antenna current of the form

$$\mathbf{J}_A(\mathbf{x}) = J_A \; S(x - x_A) \sin(\omega_A t - k_A y)\hat{k}$$

where J_A, ω_A and k_A are the amplitude, frequency, and wavenumber of the antenna and where x_A is the x-coordinate of the antenna current sheet. The strength of the antenna is normalized in terms of a parameter W, the energy of the source integrated over the simulation volume divided by the initial thermal energy of all the particles in the simulation,

$$W = \frac{\displaystyle\int_v \frac{B_A^2 + E_A^2}{8\pi} d\mathbf{x}}{\displaystyle\sum_j^{\text{all particles}} \frac{1}{2} m_{pj} v_j^2}. \tag{49}$$

where $\mathbf{B}_A$ and $\mathbf{E}_A$ are the antenna wave fields. The antenna couples directly to the transverse electric field and the magnetic field. The pump strength W is less than 1% for the results presented here. The ambient magnetic field direction is constrained to lie in the $y - z$ plane. With θ defined as $\theta = \tan^{-1}(B_y/B_z)$, the parallel wavenumber is given by $k_{\|} = k_y \sin\theta$. Thus the antenna launches a single coherent mode propagating obliquely to the magnetic field. For the results presented in this chapter, the condition $B_{0z} \gg B_{0y}$ will apply such that the angle θ is small. All of the results presented in this section were extracted from a system size $L_x \times L_y = 64\Delta \times 320\Delta$. The finite particle size is held fixed at $a_x = 1.5\Delta$ and $a_y = 15\Delta$. The results are obtained from plasmas with equal ion and electron temperatures, $T_i = T_e$. There are an average of four electrons and four ions within one grid cell. In magnetic confinement experiments the frequencies of the kinetic Alfvén wave spectrum are less than the ion cyclotron frequency. To clearly separate the Alfvén wave resonance effects from ion cyclotron resonance effects, it is desirable to place the kinetic Alfvén wave resonance frequency well below the ion gyrofrequency. It should also be mentioned that with the present guiding-center electron algorithm, it became possible to fulfill these constraints. For this nonresonant run, the plasma β is low, $\beta = 2.7 \times 10^{-4}$. The antenna strength is $W = 8.6 \times 10^{-3}$. The parallel phase velocity of the wave is $(v_{ph})_{\parallel} \equiv \omega_A/(k_A \sin\theta) = 2.79\omega_{pe}\Delta$ which is significantly larger than the Alfvén speed. The total electromagnetic field energy and the ion and electron kinetic energies demonstrate that the antenna couples only weakly to the plasma in this case. The electromagnetic field energy and the total ion kinetic energy do not show an appreciable increase in value.

For the resonant case, on the other hand, the parallel phase velocity of the wave is equal to the electron thermal velocity. It is expected that this parameter range will maximize the collisionless electron heating. The plasma β for this case is higher than the previous case but is still fairly low with $\beta = 1.7 \times 10^{-3}$. The plasma parameters for this case are: $M/m = 1600$, $v_{Te} = 1.0\omega_{pe}\Delta$, $\Omega_i = 8.3 \times 10^{-3}\omega_{pe}$, $\rho_i = 3.0\Delta$, $v_A = 1.0\omega_{pe}\Delta$, $\theta = 3.1^\circ$ and $\omega_{pe}\Delta t = 15$. The antenna parameters have the values: $\omega_A = 1.08 \times 10^{-3}\omega_{pe}$, $k_A\Delta = 1.96 \times 10^{-2}$, and $W = 3.1 \times 10^{-3}$. The parallel phase velocity of the antenna driven wave is therefore $(v_{ph})_{\parallel} = 1.0\omega_{pe}\Delta$. The simulation runs to time $\omega_{pe}t = 36000$ which is approximately 6.2 periods of the antenna wave. The amplitudes of the field components are at least a factor of four above their initial values. The field energy increases by over a factor of seventy above the initial wave energy in contrast to the nonresonant case where the increase was negligible. The ions increase in kinetic energy by 21% over the length of the run appearing to saturate by the end of the run. The electron kinetic energy keeps increasing over the full length of the run. The electrons only gain energy from motion parallel to the magnetic field indicating that Landau damping is responsible for the increase in electron energy. The collisionless electron damping also generates an electron current.

The net drift velocity of the electrons by the end of the run is $\langle v_{De} \rangle = .39\omega_{pe}\Delta$ which is 39% of the initial thermal velocity. These findings are consistent with the theoretical predictions [18,19].

(b). Twist-Kink Modes in Three Dimensions

A three-dimensional version of the model has been applied to the study of a plasma column twist. A plasma in a tandem mirror, for example, experiences differential azimuthal rotation due to a different amount of charge separation at different axial locations [20]. A plasma in a solar coronal loop is believed to get similarly twisted by the photospheric rotational motion at the feet [21]. We model these rotations by imposing an external radial electric field on the plasma column. The plasma is assumed to be strongly magnetized so that B_z is constant (incompressible). The twisting electric fields, constant in time, are given as

$$E_r = E_{r0} \sin\left(\frac{\pi}{r_0} r\right) \cos\left(\frac{2\pi}{L_z} z\right).$$

This radial field gives rise to an azimuthal rotation of both electrons and ions with $\mathbf{v}_\theta(r,z) = c\mathbf{E}_r(r,z) \times \mathbf{B}_z/B_z^2$. This sheared azimuthal flow $v_\theta(dv_\theta/dr \neq 0)$ peaks at $r = r_0/2$. The plasma experiences no externally imposed azimuthal flow beyond $r \geq r_0$. The largest amount of differential rotation takes place at $2\pi z/L_z = \pi/2$ and $3\pi/2$. We expect that the twisting of the plasma induces a field aligned current (J_z) due to the injected helicity, a pinching of the plasma column, and an eventual kink instability due to the large induced axial current.

The simulation is set up with the following parameters: $M/m = 125$, $v_{Te} = \omega_{pe}\Delta$, $\Omega_i/\omega_{pi} = v_A/c = 0.4$, $\Omega_i = 8 \times 10^{-2}\omega_{pe}$, $\rho_i = 1.1\Delta$, $v_A = 1.6\omega_{pe}\Delta$, $L_x = L_y = 32\Delta$, $L_z = 3200\Delta$, $\theta = 0°$, $\omega_{pe}\Delta t = 10$. The plasma β is 1.25×10^{-3} and thus the use of our low beta model is appropriate. The applied radial electric field has peak magnitude $E_{r0} = 0.5\frac{m}{e}\Delta\omega_{pe}^2$ with maximum radial distance $r_0 = 14\Delta$. The plasma is initially uniform in density and contains only a constant B_z field. A clear pattern of plasma rotation due to the external twisting $\mathbf{E}_r \times \mathbf{B}_z$ is seen in Fig. 7(a) at $200\omega_{pe}^{-1}$. As the plasma twist continues, the plasma is pinched and the density increases near the r=0 axis and the field aligned current, J_z, is induced, which produces the poloidal magnetic field, B_θ as seen in Fig. 7(b). These are not inconsistent with our conjecture and theory by Zweibel and Boozer [21].

The current is peaked near r=0 and so is B_θ. Thus the total magnetic field $\mathbf{B}$ starts to acquire shear in the radial direction. The sheared magnetic field structure may be best illustrated by the analysis of the magnetic fields in terms of the rotational transform and its associated so-called safety factor locally defined as

$$q = \frac{rB_z}{RB_\theta(\mathbf{x})},$$

where $R = L_z/2\pi$. Since the twist is a function of z, the "rotational transform" and the safety factor are functions of z and are thus local (z) quantities. When q is, for example, 3 at z=z_0, the magnetic field is spiraling in the azimuthal direction with a pitch of $3L_z$. This would amount to a winding in the poloidal direction of the particular field line once while winding three times in the toroidal direction (in the periodicity of z) if this local q=3 was held for all z. Such a local q is depicted in Fig. 7(c). From Shafranov's theory [22] the kink instability is expected when q becomes less than unity. In Figs. 7(d) and (e) we show the flow pattern and B_θ at t=400ω_{pe}^{-1}. As the twisting continues, the magnetic field lines become more wrapped showing a wider area with $q < 1$ (Fig. 7(f)). Figure 7(e) shows an m=1 distortion as exemplified by a crescent-shaped island and by a dipole structure. At this point it turns out that the system has achieved its maximum twist. It seems, in fact, that at this time the strong anticipated kink mode sets in, although the above conjecture is based on the analysis of the locally defined q. This strong kink mode makes the plasma unstable and shows a turbulent plasma motion. At t=800ω_{pe}^{-1}, the flow pattern is re-established and the central structure of B_θ indicate that the plasma has somewhat relaxed. The time histories (Fig. 8) correlate well with the magnetic field energy undergoing rapid growth until about time 400ω_{pe}^{-1} and then decreasing such that the energy stored in the magnetic field at time 800ω_{pe}^{-1} is nearly the same as at the earlier time of 200ω_{pe}^{-1}. The growth of the electron kinetic energy is seen to occur at the same times as the growth of the magnetic field energy.

V. Conclusions

The computational model presented here represents an attempt to extend the application of particle simulation to the low frequency regimes of plasma behavior. This model incorporates two approximations from a complete classical treatment: Darwin's radiationless formulation of the full electromagnetic field equations and electron guiding center motion perpendicular to the magnetic field. Tests of the code when it is configured as a homogeneous equilibrium plasma confirm theoretical expectations of its properties. In particular, the behavior of the field fluctuation levels exhibits a dependence on the angle of propagation with respect to the magnetic field not previously reported which is explained here. The code was routinely operated with a time step two orders of magnitude larger than what was otherwise feasible with explicit particle codes containing full dynamics electrons and self-consistent electric and magnetic fields. The investigations of Alfvén wave resonance heating and twist-kink modes are examples of problems that are treated successfully with the present model.

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Figure Captions

Fig. 1: Comparison of simulation results with theory for ion Bernstein modes.

Fig. 2: Dispersion curves for the shear Alfvén wave propagating obliquely to the ambient magnetic field.

Fig. 3: Dispersion curve for the shear Alfvén wave with θ=1° in the three-dimensional model.

Fig. 4: Longitudinal electric field fluctuation spectrum.

Fig. 5: Time-averaged magnetic field fluctuation spectrum for propagation (a) purely parallel to the magnetic field, (b) purely perpendicular to the magnetic field.

Fig. 6: Time evolution of (a),(d) total electromagnetic field energies, (b),(e) total ion kinetic energies, and (c),(f) total electron kinetic energies ending at $\omega_{pe}t$=36000 for the non-resonant case (a)-(c) and for the resonant case (d)-(f).

Fig. 7: Simulation results at time = $200\omega_{pe}^{-1}$ (a) (b) (c) and time = $400\omega_{pe}^{-1}$ (d) (e) (f): (a)(d) Plasma flow with n=1, (b)(e) Contours of B_θ at z=400Δ, (c)(f) Contours of local q at z=400Δ.

Fig. 8: Time history of (a) magnetic field energy, (b) electron kinetic energy with times marked to correspond with Fig. 7.