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THEORY OF HOT PARTICLE STABILITY

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Abstract

The investigation of stabilization of hot particle drift reversed systems to low frequency modes has been extended to arbitrary hot beta, β_H for systems that have unfavorable field line curvature. Steep profile equilibria are considered where the thickness of the pressure drop, Δ, is less than plasma radius, r_p. The analysis describes layer modes which have $\frac{m\Delta}{r_p} < 1$, where m is the mode number, and radial structure larger than Δ. Stabilization is classified as either being "robust," where all excitations are positive energy, or "fragile," where stability criteria exist to magnetohydrodynamic (MHD)-like and drift compressional instabilities, but positive and negative energy waves are present (with the possibility that negative energy waves are destabilized by dissipative mechanisms). It is shown that in making a continuous transition from the fragile stability regime to the robust stability regime one must go through an unstable region. To bridge the unstable band in a physical manner one must either produce robust stability conditions very rapidly, or use transient stability techniques such as ponderomotive forces or transient minimum-B coils. The positive energy stabilization terms of the layer mode from wall stabilization terms and finite Larmor radius terms are explicitly exhibited. It is shown the robust stability can even be achieved with only wall stabilization, for all possible m-values of

the layer modes if $\beta_H > 2/3$. When robust stability conditions are fulfilled, the hot particles will have their axial bounce frequency less than their grad-B drift frequency. This allows for a low bounce frequency expansion to describe the axial dependence of the magnetic compressional response. Such as expansion provides for another negative energy source in the theory, with the system then being susceptible to the drift compressional instability if $\frac{d\beta_H}{dr} > \frac{d\beta_w}{dr}$ where β_w is the beta of the background plasma. If robust stabilization conditions are not fulfilled, the regions of fragile stability are extremely small if the MHD-like modes, the diamagnetic compressional and drift compressional modes are to be simultaneously avoided.

I. Introduction

There has been considerable work in the last few years in the investigation of whether hot particles in a symmetric mirror configuration can establish a stable configuration.[1–17] Interest was based on the hope that if the hot particles are sufficiently energetic, their dynamics would decouple from magnetohydrodynamic (MHD) motion and allow formation of a stable diamagnetic well. Under certain conditions the hot particle theory does predict a stabilization of the flute interchange mode.[1,2,6,9] However, a negative energy source[5,8] can still remain that causes destabilization under a variety of conditions such as: (1) a return to an unstable MHD theory when the background beta value exceeds a critical value;[3,4] (2) the core background density becomes too large and an unstable magnetic compressional mode is excited;[6,7] (3) destabilization of the negative energy precessional mode from positive dissipation mechanisms.[8,13,14]

More recent theory[12,15,16] has shown that the negative energy sources described in such references as 5,11,13,14 can be eliminated with the combination of wall stabilization and robust finite Larmor radius (FLR) stabilization effects. The basic assumptions of the theory in Ref. 15 and 16 were $1 \gg \beta_H, \frac{\omega_B}{\omega_b}, \frac{\Delta}{r_p}, \kappa\Delta/\beta_H$ where β_H is the hot particle beta, ω_b the hot particle bounce frequency, ω_B the grad-B drift, Δ the thickness of the pressure gradient region, r_p the plasma radius, and κ is the curvature (in Ref. 12. β_H was arbitrary but only flute modes of a z-pinch model were analyzed). We also point out that the deviation from an MHD response requires a hot species and that to have important FLR effects usually restricts the hot species to be ions. Equilibrium electric potentials are assumed to be order the background plasma temperature. In the ordering of this calculation, the equilibrium electric fields can be neglected in this analysis.

In this paper we would like to extend the analysis of Refs. 12, 15 and 16 to arbitrary beta and realistic geometry. The importance of arbitrary beta is demonstrated in Ref. 17, where it is shown that for displacement-like modes for MHD systems, wall stabilization becomes considerably stronger as beta approaches unity. It is necessary to retain the assumption $\Delta/r_p \ll 1$ to analytically solve for layer modes which satisfy the condition $m\Delta/r_p < 1$ where m is the mode number. It is desirable to consider system parameters that are stable to wall stabilization and robust finite Larmor radius stabilization effects. In order to achieve such parameters, we find that we require $\omega_B/\omega_b \gg 1$. Thus the relevant parameter regime is where hot particles should be treated in a low bounce frequency limit. In this case we find that a new negative energy source is introduced. This source has a magnetic compressional polarization and, in a slab analysis, gives rise to the drift compressional instability[18,19] when treated in a local WKB approximation. In this work we analyze the effect of this negative energy source in the long wavelength layer mode limit ($m\Delta/r_p < 1$). This new source can give rise to an unstable mode if too much background plasma density is present, or to new negative energy waves. The negative energy waves are destabilized by positive dissipation mechanisms of the background plasma and stabilized by the bounce resonances of the hot particles.

In order to calculate the FLR corrections properly, higher order corrections to the pressure tensor need to be calculated. In the hot particle limit the finite Larmor radius corrections are not the same as in the conventional finite Larmor radius treatment.[20,21] The formal derivation of the FLR terms is somewhat tedious, and the details of this calculation for a symmetric mirror geometry will be presented in another paper. For a z-pinch model the appropriate finite Larmor radius correction has been derived in Ref. 12.

II. Derivation of Equations

We consider an azimuthally symmetric mirror cell equilibrium where the magnetic field is $\mathbf{B}(\psi, s) = \nabla\psi \times \nabla\theta$, with ψ the magnetic flux, θ the azimuthal angle, and s the distance along the field line.

The perpendicular pressure balance equation for the equilibrium including FLR terms in the long thin approximation (paraxial limit) with equilibrium electric fields neglected is:

$$\frac{\partial}{\partial\psi}\left(B^2/2 + P_\perp\right) = \frac{\sigma B\kappa}{r} + \frac{1}{2}\frac{\partial}{\partial\psi}r^2\frac{\partial R}{\partial\psi} \tag{1}$$

where

$$P_\perp = \sum \frac{4\pi}{M^2}\int \frac{Bd\mu d\mathcal{E}}{|v_\parallel|}\mu B\bar{F} \qquad \text{(perpendicular pressure)},$$

$$P_\parallel = \sum \frac{4\pi}{M^2}\int \frac{Bd\mu d\mathcal{E}}{|v_\parallel|}Mv_\parallel^2\bar{F} \qquad \text{parallel pressure},$$

$$R = \sum \frac{4\pi}{M^2}\int \frac{Bd\mu d\mathcal{E}}{|v_\parallel|}\frac{\mu^2B^2}{2M\Omega^2}\bar{F},$$

$$\kappa = \frac{\nabla\psi\cdot(\mathbf{b}\cdot\nabla)\mathbf{b}}{|\nabla\psi|} \qquad \text{field line curvature},$$

$$\mathbf{b} = \frac{\mathbf{B}}{B} \quad \text{unit vector}, \quad \Omega = \frac{qB}{M} \quad \text{cyclotron frequency},$$

$$\sigma = 1 + \frac{(P_\perp - P_\parallel)}{B^2}, \qquad \frac{r^2}{2} = \int^\psi \frac{d\psi}{B},$$

$$\mathcal{E} = \frac{Mv^2}{2}, \text{ the particle energy}, \ \mu = \frac{Mv_\perp^2}{2B}, \text{ lowest order magnetic moment}$$

q is the charge and M the mass of a given species, and $v_\parallel^2 = \frac{2}{M}(\mathcal{E} - \mu B)$. $\bar{F}$ is that part of the equilibrium distribution function which is independent of the Larmor phase angle and is a solution of the time independent drift-kinetic equation up to second order in the Larmor radius. To lowest order, $\bar{F} = F_0(\mathcal{E}, \mu, \psi) + \cdots$ and it is assumed to be symmetric in $v_\parallel$. The summation $\sum$ is over all species.

In the limit of negligible perturbed parallel electric field $E_\parallel = \mathbf{E}\cdot\mathbf{b}$, the eigenmode equations for low frequency perturbation of the plasma equilibrium may be expressed in terms of the perpendicular components of the vector potential $\tilde{\mathbf{A}}_\perp = \sum_m \tilde{\mathbf{A}}_{\perp m}(\psi, s)\exp(-i\omega t + im\theta)$, or equivalently the magnetic field-line displacement vector $\boldsymbol{\xi}$, defined by $\tilde{\mathbf{A}}_\perp = \boldsymbol{\xi}\times\mathbf{B}$. ω is the frequency of the perturbation and m the azimuthal mode number.

The eigenmode equations are derivable from a quadratic variational form involving $\tilde{\mathbf{A}}_\perp$ and its adjoint $\tilde{\mathbf{A}}_\perp^\dagger$. It can be shown that the adjoint $\tilde{\mathbf{A}}_\perp^\dagger = \sum_m \tilde{\mathbf{A}}_{\perp m}(\psi, s)\exp(i\omega t - im\theta)$, that is, the quadratic form is self-adjoint.[22]

This quadratic form includes the conventional zero Larmor radius terms[23] with (1) the kinetic zero order Larmor radius terms that is derived in Ref. 24; (2) the appropriate finite Larmor radius (FLR) terms involving the perturbed $\tilde{B}_\parallel$ whose derivation in a symmetric mirror geometry will be presented in a forthcoming paper (for the z-pinch model, these FLR terms are derived in Ref. 12).

Before writing down this quadratic form, let us first establish an appropriate ordering of the plasma parameters. We use an ordering scheme based on the paraxial limit where the smallness parameter is $\epsilon \sim \kappa\Delta \ll 1$. To develop a method that is analytically tractable, we assume a steep equilibrium pressure gradient such that $\frac{m\Delta}{r_p} < 1$.

We intend to explore the limit where robust stabilization of low frequencies is possible through wall stabilization of the m=1 mode which requires $\beta_H \gtrsim P_{\parallel H}/\left(P_{\perp H} + P_{\parallel H}\right)$,[15–17] and FLR stabilization of the other m >1 modes which requires $\frac{a_H^2}{r_p^2}\beta_H \gtrsim \kappa\Delta$.[12] $P_{\parallel H}$ and $P_{\perp H}$ are the parallel and perpendicular pressure of the hot component, a_H is the hot particle Larmor radius.

We consider the case where $\beta_w \ll \beta_H$ and the FLR effects from the warm particles is negligible. For definiteness we take $\beta_w \approx \kappa\Delta$.

As a natural consequence of these considerations, we invoke the following ordering scheme

$$\beta_H \sim 1$$
$$\kappa\Delta \sim \frac{a_H^2}{r_p^2} \sim \frac{\beta_w}{\beta_H} \sim \epsilon \ll 1.$$

We also assume that

$$\frac{a_H}{\Delta} \sim \epsilon^p \quad , \quad p < 1/6$$

in order to formally validate the procedure we adopt to obtain solutions of the eigenmode equations. However, this restriction on the value of p is not severe, since other procedures are available when $p > 1/6$ and they yield essentially the same solutions.

We find it convenient to express the quadratic form in terms of the field amplitudes

ϕ and C defined by

$$\phi = \boldsymbol{\xi} \cdot \nabla\psi \tag{2}$$

$$\frac{C}{B^2}\frac{DP_{\perp H}}{D\psi} = \nabla_\perp \cdot \boldsymbol{\xi} \tag{3}$$

where

$$\frac{DP_{\perp H}}{D\psi} = \frac{\nabla\psi \cdot \nabla P_{\perp H}}{|\nabla\psi|^2}. \tag{4}$$

We have suppressed the subscript m on the field amplitudes.

The resulting quadratic form, up to order ϵ, may then be written as follows:

$$2\int_{\text{plasma}} \frac{d\psi ds}{B}\mathcal{L}(\phi, C) + \int_{\text{vacuum}} \frac{d\psi ds}{B}\mathbf{B}_1 \cdot \mathbf{B}_1 = 0 \tag{5}$$

where B_1 is the perturbed magnetic field in the vacuum and

$$\mathcal{L} = \mathcal{L}_I + \mathcal{L}_B + \mathcal{L}_c + \mathcal{L}_{\text{LVD}} + \mathcal{L}_{\text{CHG}} + \mathcal{L}_{\text{FLR}} + \mathcal{L}_{\text{KIN}}$$

$$\mathcal{L}_I = -\omega^2\rho\left\{\left(\frac{m\phi}{rB}\right)^2 + \left[r\frac{\partial\phi}{\partial\psi} + \frac{(\phi - C)r}{B^2}\frac{DP_{\perp H}}{D\psi}\right]^2\right\}, \text{ion inertia term} \tag{6}$$

$$\mathcal{L}_B = \sigma\left\{\frac{m^2}{r^2}\left(\frac{\partial\phi}{\partial s}\right)^2 + B^2r^2\left[\frac{\partial}{\partial s}\left(\frac{\partial\phi}{\partial\psi} + \frac{(\phi - C)}{B^2}\frac{DP_{\perp H}}{D\psi}\right)\right]^2\right\}, \text{line bending term} \tag{7}$$

$$\mathcal{L}_c = \frac{m^2\kappa}{Br}\left\{(\phi - C)^2\frac{D}{D\psi}\left(P_{\perp H} + P_{\parallel H}\right) + \phi^2\frac{D}{D\psi}\left(P_{\perp c} + P_{\parallel c}\right)\right\}, \text{curvature drive} \tag{8}$$

$$\mathcal{L}_{\text{LVD}} = m^2\left(\frac{D}{D\psi}P_{\perp w}/B^2\right)\frac{D}{D\psi}P_{\perp H}C^2, \text{Lee} - \text{Van Dam term} \tag{9}$$

$$\mathcal{L}_{\text{CHG}} = m\omega q_H B\left(\frac{D}{D\psi}\frac{n_H}{B}\right)C^2, \text{compressional charge uncovering term} \tag{10}$$

$$\mathcal{L}_{\text{FLR}} = m^2\left(\frac{DP_{\perp H}}{D\psi}\right)\frac{D}{D\psi}\overline{(P_{\perp H}a_H^2B^2)}$$
$$\left\{r^4\left[\frac{\partial}{\partial\psi}\left(\frac{C - \phi}{Br}\right)\right]^2 + \frac{(m^2 - 1)}{B^4r^2}(C - \phi)^2\right\}, \text{FLR term} \tag{11}$$

$$\int\frac{d\psi ds}{B}\mathcal{L}_{\text{KIN}} = \int d\psi\frac{2\pi q_H m}{M_H}\int d\mathcal{E}d\mu\frac{\partial F_H}{\partial\psi}$$
$$\int_{s^-}^{s^+} ds\int_{s^-}^{s^+} ds' K(s, s')\frac{\partial C(s)}{\partial s}\frac{\partial C(s')}{\partial s'}, \text{(kinetic term)} \tag{12}$$

where

$$
\begin{aligned}
K(s,s') &= \cos\left[\Omega(s^+,s^<)\right]\sin\left[\Omega(s^+,s^>)\right] \\
&\quad -\cot\left[\Omega(s^+,s^-)\right]\sin\left[\Omega(s^+,s)\right]\sin\left[\Omega(s^+,s')\right] \\
&= \frac{\sin\left[\Omega(s^<,s^-)\right]\sin\left[\Omega(s^+,s^>)\right]}{\sin\left[\Omega(s^+,s^-)\right]} \\
\Omega(s,s') &= -\int_{s'}^{s}\frac{ds''\omega_B(s'')}{\left|v_\parallel(\mathcal{E},\mu,s'')\right|} \\
v_\parallel(\mathcal{E},\mu,s^\pm) &= 0 \qquad s^+ > s^- \\
s^> &= \max(s,s') \\
s^< &= \min(s,s') \\
v_\parallel^2(\mathcal{E},\mu,s) &= \frac{2}{M}(\mathcal{E}-\mu B) \\
\omega_B &= -\frac{m\mu}{q_H B}\frac{DP_\perp}{D\psi} \\
\overline{P_{\perp H}a_H^2 B^2} &= \frac{MB^2}{8}\int d^3v\,\frac{F_H v_\perp^4}{\Omega^2} \\
\rho &\equiv \text{plasma mass density.}
\end{aligned}
$$

We also note the perturbations in the paralalel electric field has been neglected. This assumption is a standard assumption that can be justified if $a_s|\nabla_\perp\xi|/\xi \ll 1$ where a_s is the Larmor radius of the background plasma.

The ratio of the hot particle bounce frequency, $\omega_b \sim v_{thH\parallel}/L_p$ (L_p is the axial length of the plasma, $v_{th\parallel}$ the hot particle parallel thermal speed) to the hot particle grad-B drift, ω_B is

$$
\frac{\omega_b}{\omega_B} \approx P_\parallel^{1/2} r_p \Delta / P_\perp^{1/2}\beta_H L_p a_H = \frac{1}{(\mathrm{FLR})^{1/2}}\frac{1}{(\mathrm{WALL})^{1/2}}\left(\frac{\Delta}{r_p}\right)^{1/2}\left(\frac{P_\parallel^{1/2}}{P_\perp^{1/2}}\frac{L_\kappa}{L_p}\right)\frac{L_\kappa}{L_p}\frac{1}{m}
$$

where

$$
\mathrm{FLR} \equiv \frac{\beta_H a_H^2 L_\kappa^2}{\Delta r_p^3}, \qquad \mathrm{WALL} \equiv \frac{\beta_H L_\kappa^2}{L_p^2}, \qquad \kappa \approx \frac{1}{2}\frac{r_p}{L_\kappa^2}.
$$

We will observe that to eliminate negative energy sources by finite Larmor radius effects and wall stabilization, both "FLR" and "WALL" have to be bigger than unity. Further, $(P_\parallel/P_\parallel)^{1/2}(L_\kappa/L_p) \approx 1$ from equilibrium constraints. Thus, only the factor L_κ/L_p, which cannot be too large, can counteract the trend to have $\omega_b/\omega_B < 1$. Higher m aids the tendency to have $\omega_b/\omega_B < 1$ and for our theory $|m|$ can get as large as r_p/Δ. It is therefore appropriate to treat the ion response in the low bounce frequency limit. We can then evaluate the s and s' integration in $\mathcal{L}_{\rm KIN}$ by using the stationary phase approximation and obtain

$$\mathcal{L}_{\rm KIN} \approx -\frac{2q_H^2 B^2}{M_H}\left(\frac{\partial C}{\partial s}\right)^2 \int d^3v \frac{v_\parallel^2 \dfrac{\partial F_H}{\partial \psi}}{v_\perp^2 \dfrac{DP_{\perp H}}{D\psi}} \equiv \mathcal{L}_{\rm KIN1}. \tag{13}$$

In this form we will refer to $\mathcal{L}_{\rm KIN}$ as the drift compressional (negative energy) drive. Since we are interested in equilibria for which the plasma pressure gradient and the magnetic field gradient are in opposite direction inside the layer, the drift compressional term is negative definite and thereby constitutes a negative energy source which can result in the growth of instabilities. In order to avoid having this drive we need $\omega_b/\omega_B \gtrsim 1$. This may be possible by limiting β_H, not having Δ/r_p too small, and only barely satisfying the FLR stability condition.

There is also an imaginary contribution to $\mathcal{L}_{\rm KIN}$ due to the poles of $\cot[\Omega(s^+, s^-)]$ which occurs when ω (real) is equal to an integral multiple of the bounce frequency. Since these resonances are closely spaced, it is a good approximation to set $\cot[\Omega(s^+, s^-)] = -i$. Then for real ω, we can approximate the imaginary part of $\mathcal{L}_{\rm KIN}$ by

$$\mathrm{Im}\int \frac{d\psi ds}{B}\mathcal{L}_{\rm KIN} = \int d\psi \frac{2\pi q_H m}{M_H}\int d\mathcal{E} d\mu \frac{\partial F_H}{\partial \psi} \left\{\int_{s^-}^{s^+} ds \frac{\partial C}{\partial s}\sin\left[\Omega(s^+, s)\right]\right\}^2 . \tag{14}$$

This imaginary contribution is small and does not affect the lowest order solution of the eigenmode equation. However, through the quadratic form, this term produces a leading order negative energy "dissipative" response.

We note that with the present ordering in which $\beta_H \gg \kappa\Delta$, $\dfrac{\omega_\kappa}{\omega_B} \sim \mathcal{O}(\epsilon)$, where ω_κ is the curvature drift frequency of the hot species. We will, however, consider not only modes where $\dfrac{\omega}{\omega_\kappa} \sim \mathcal{O}(1)$, but also extend the frequency range to include $\omega_\kappa \lesssim \omega < \omega_B$ by an appropriate subsidiary ordering. In this range the drift compressional layer modes can be excited.

We have not included more conventional Larmor radius terms since these terms are always smaller than the terms we have retained if,

$$\frac{n_H}{n} < \beta_H$$

where $\frac{n_H}{n}$ is the ratio of the hot particle (ion) density to the total ion density.

To proceed with the analysis, we divide the plasma equilibrium into three regions as in Refs. 15 and 16. In region I, where $\psi_p^- > \psi > 0$, the pressure gradient is assumed to be negligible. In region II, where $\psi_p^+ > \psi > \psi_p^-$, there is a large pressure gradient due to the hot plasma component, with the hot plasma density decreasing from its peak value at $\psi = \psi_p^-$ to zero at $\psi = \psi_p^+$. Region III, where $\psi_w > \psi > \psi_p^+$, encompasses the volume between the hot plasma boundary and the conducting wall at $\psi = \psi_w$. The magnetic field $\mathbf{B}_1$ in this region is determined by the vacuum magnetic field equation together with the boundary condition that the normal component of the perturbed magnetic field is zero at $\psi = \psi_w$ and is continuous at $\psi = \psi_p^+$. The solution $\mathbf{B}_1$ can be explicitly found in the paraxial approximation.

Inside region II, hereafter referred to as the layer, the relative magnitudes of the terms in the quadratic form are:

$$r^2\Delta^2\mathcal{L}_{\mathrm{FLR}} \sim \frac{m^2 P_{\perp H}^2}{B^4}\frac{a_H^2}{\Delta^2}\left\{1 + \frac{(m^2-1)\Delta^2}{r^2}\right\}(C-\phi)^2$$

$$r^2\Delta^2\mathcal{L}_{\mathrm{KIN1}} \sim 2k_\parallel^2\frac{r^2\Delta^2}{a_H^2}\frac{P_{\parallel H}}{P_{\perp H}}C^2$$

$$r^2\Delta^2\mathcal{L}_B \sim \sigma k_\parallel^2 r^2\left(\phi - \frac{CP_{\perp H}}{B^2}\right)^2 + \sigma m^2 k_\parallel^2\Delta^2\phi^2$$

$$r^2\Delta^2\mathcal{L}_I \sim \frac{\omega^2\rho r^2}{B^2}\left\{\left(\phi - \frac{CP_{\perp H}}{B^2}\right)^2 + \frac{m^2\Delta^2}{r^2}\phi^2\right\}$$

$$r^2\Delta^2\mathcal{L}_C \approx \frac{m^2\kappa\Delta P_H}{B^2}(\phi - C)^2$$

$$r^2\Delta^2\mathcal{L}_{\mathrm{LVD}} \approx \frac{m^2 P_{\perp w}P_{\perp H}}{B^4}C^2$$

$$r^2\Delta^2\mathcal{L}_{\mathrm{CHG}} \approx \frac{mr\Delta q_H n_H}{B}\omega C^2.$$

The dominant term is $\mathcal{L}_{\mathrm{FLR}}$. In order to minimize $\mathcal{L}_{\mathrm{FLR}}$, we require

$$\frac{C-\phi}{Br} = \chi(s) \qquad (15)$$

where $\chi(s)$ is independent of ψ. Then the largest contribution to the finite Larmor radius term is annihilated in Eq. (11). The remaining finite Larmor radius term is assumed to compete with terms of the order of the curvature.

The next most dominant terms depend on whether we seek a high frequency ($\omega_\kappa < \omega < \omega_B$) or a low frequency solution ($\omega \lesssim \omega_\kappa$).

In the high frequency limit, the inertia term $\mathcal{L}_I$ is the next most dominant term. We minimize it in Eq. (6) by relating C to ϕ through the equation

$$\frac{\partial \phi}{\partial \psi} = -\frac{(\phi - C)}{B^2}\frac{DP_{\perp H}}{D\psi} \approx \frac{(\phi - C)}{B}\frac{\partial B}{\partial \psi} = -\chi r \frac{\partial B}{\partial \psi}. \tag{16}$$

Integrating in ψ, we obtain for ϕ:

$$\phi = -\chi B r + \hat{C}(s) + \mathcal{O}\left(\frac{\Delta}{r_p}\right) \tag{17}$$

where $\hat{C}(s)$ is independent of ψ.

Thus, in minimizing $\mathcal{L}_{\rm FLR}$ and $\mathcal{L}_I$, we have introduced two new field variables, namely $\chi(s)$ and $\hat{C}(s)$, which are functions only of the field-line coordinate s. C and ϕ are related to $\hat{C}(s)$ and $\chi(s)$ by:

$$\left.\begin{aligned} C &= \hat{C}(s) \\ \phi &= \hat{C}(s) - Br\chi(s) \end{aligned}\right\} \psi_p^+ > \psi > \psi_p^-. \tag{18}$$

It should be noted that in the high frequency limit, these solutions are independent of the order in which we minimize $\mathcal{L}_{\rm FLR}$ and $\mathcal{L}_I$. If $\frac{a_H}{\Delta} > \epsilon^{1/6}$, we can obtain the same answer for the high frequency mode by interchanging the order of our annihilation procedures.

If we seek a low frequency solution, the drift compressional term $\mathcal{L}_{\rm KIN1}$ is the next most dominant term. Minimization of $\mathcal{L}_{\rm KIN1}$ forces $C(s)$ to be independent of s. To next order, we then minimize the inertia $\mathcal{L}_I$ and (or) bending terms $\mathcal{L}_B$. The minimization of $\mathcal{L}_I$ and $\mathcal{L}_B$ proceeds as previously shown, the only difference being that $\hat{C}$ is now a constant independent of s.

Hence, whether we seek a high frequency or a low frequency solution, the original field variables C and ψ can be related to two new field variables $\hat{C}$ and $\chi(s)$ by Eq. (18). In the high frequency limit $\hat{C} = \hat{C}(s)$, while in the low frequency limit $\hat{C} =$ constant.

In region I, the Euler-Lagrange equation for ϕ is

$$\frac{\partial}{\partial\psi}\frac{\partial}{\partial s}\sigma Br^2\frac{\partial}{\partial s}\frac{\partial\phi}{\partial\psi}-\frac{\partial}{\partial s}\frac{\sigma m^2}{Br^2}\frac{\partial\phi}{\partial s}+\frac{\partial}{\partial\psi}\frac{\rho\omega^2r^2}{B}\frac{\partial\phi}{\partial\psi}-\frac{m^2\rho\omega^2}{B^3r^2}\phi=0. \tag{19}$$

Now using $Br^2\approx\psi/2$, and B, ρ, σ is independent of ψ, we find that the solution for ϕ is,

$$\phi=\hat{\phi}(s)\left(\frac{\psi}{\psi_p^-}\right)^{\frac{|m|}{2}}\quad,\quad\psi_p^->\psi>0. \tag{20}$$

In region III, the solution for the vacuum magnetic field perturbation $\mathbf{B}_1$ in the paraxial limit is,

$$\mathbf{B}_1=\frac{\partial\hat{\phi}_{\text{ex}}(s)}{\partial s}\frac{e^{im\theta}}{\left\{\left(\frac{\psi_p^+}{\psi_w}\right)^{\frac{m}{2}}-\left(\frac{\psi_w}{\psi_p^+}\right)^{\frac{m}{2}}\right\}}$$
$$\cdot\left\{\left[\left(\frac{\psi}{\psi_w}\right)^{\frac{m}{2}}-\left(\frac{\psi_w}{\psi}\right)^{\frac{m}{2}}\right]\frac{\nabla\psi}{Br^2}+i\left[\left(\frac{\psi}{\psi_w}\right)^{\frac{m}{2}}+\left(\frac{\psi_w}{\psi}\right)^{\frac{m}{2}}\right]\nabla\theta\right\} \tag{21}$$

where $\mathbf{B}_1\cdot\nabla\psi=0$ at the conducting wall $\psi=\psi_w(s)$.

We relate the coefficients $\hat{\phi}(s)$ and $\frac{\partial}{\partial s}\hat{\phi}_{\text{ex}}(s)$ to $\hat{C}(s)$ and $\chi(s)$ by requiring continuity of ϕ at $\psi=\psi_p^-$ and of the normal component of the perturbed magnetic field at $\psi=\psi_p^+$.

$$\begin{aligned}\hat{\phi}(s)&=\hat{C}-\chi B^*r^-\\ \frac{\partial}{\partial s}\hat{\phi}_{\text{ex}}(s)&=\frac{\partial}{\partial s}\left(\hat{C}-\chi B_vr^+\right)\end{aligned} \tag{22}$$

where $B^*=B\Big|_{\psi=\psi_p^-}$, $B_v=B\Big|_{\psi=\psi_p^+}$, and r^-, r^+ are the inner, outer edge of the layer.

If we substitute these expressions for ϕ, C, and $\mathbf{B}_1$ in Eq. (4), we obtain the following quadratic form in the variables $\hat{C}$ and χ:

$$\begin{aligned}\int\frac{d\psi ds}{B}\Bigg\{&-\frac{\omega^2\rho m^2}{r^2B^2}\left(\hat{C}-\chi Br\right)^2+\frac{\sigma m^2}{r^2}\left[\frac{\partial}{\partial s}\left(\hat{C}-\chi Br\right)\right]^2-D_c\left(\hat{C}-\chi Br\right)^2\\ &-D_H\chi^2B^2r^2+m^2\left(\frac{D}{D\psi}P_{\perp H}\right)\overline{\frac{D}{D\psi}P_{\perp H}a_H^2B^2\frac{(m^2-1)}{B^2}}\chi^2\\ &+\left(\frac{\partial\hat{C}}{\partial s}\right)^2\frac{2q_H^2B}{M_H}\int d^3v\frac{v_\parallel^2\frac{\partial F_H}{\partial\psi}}{v_\perp^2\frac{\partial B}{\partial\psi}}\Bigg\}+\int ds|m|\sigma_0\left[\frac{\partial}{\partial s}\left(\hat{C}-\chi B^*r_p\right)\right]^2\\ &+\int\frac{d\psi ds}{B}D_H\left(\tilde{\beta}_w+\frac{\omega}{\omega_\kappa}\right)+\int ds|m|Z_{|m|}\left\{\frac{\partial}{\partial s}\left(\hat{C}-\chi B_vr_p\right)\right\}^2\\ &-\int ds\frac{\omega^2\rho_0|m|}{B^{*2}}\left(\hat{C}-\chi B^*r_p\right)^2=0\end{aligned} \tag{23}$$

where

$$
\begin{aligned}
D_H &= \frac{m^2\kappa}{Br}\frac{D}{D\psi}\left(P_{\perp H}+P_{\parallel H}\right) \\
D_c &= \frac{m^2\kappa}{Br}\frac{D}{D\psi}\left(P_{\perp w}+P_{\parallel w}\right) \\
\tilde{\beta}_w &= \frac{m^2}{D_H}\left(\frac{D}{D\psi}P_{\perp w}/B^2\right)\frac{D}{D\psi}P_{\perp H} \\
\frac{1}{\tilde{\omega}_k} &= \frac{mq_H B}{D_H}\frac{D}{D\psi}n_H/B.
\end{aligned}
$$

and the last three terms are the largest terms in a $m\Delta/r_p$ expansion.

The Euler-Lagrange equations derivable from the quadratic form are the eigenmode equations.

In Sec. III and IV, we will substitute appropriate trial functions in the quadratic form to obtain approximate dispersion relations. We discuss separately the low frequency limit and the high frequency limit.

As we have already mentioned, in the low frequency limit, where ω is of the order of the curvature drift frequency $\omega_{\kappa H}$ or typical MHD growth rates γ_{MHD}, the dominant drift compression term in Eq. (23) is extremized by choosing $\hat{C}$ to be constant, independent of s. The Euler-Lagrange equation relating $\hat{C}$ to χ is readily derived. After substituting for $\hat{C}$, we obtain a quadratic form in χ which represents a generalization of that previously derived by Berk et al.,[15,16] for $\beta_H < 1$ and by Pearlstein and Kaiser[17] who analyzed the case $\hat{C}$=0, m=1 at arbitrary beta.

In the high frequency limit, the frequency dependent terms in Eq. (23) balance the drift compressional term. We shall explore the occurrence of instabilities due to the presence of the negative definite drift compressional term.

III. Low Frequency

In the low frequency limit, we assume

$$\frac{2r\Delta}{a_H^2}\frac{P_{\parallel H}}{P_{\perp H}} > \left|\frac{m^2\kappa}{k_\parallel^2 r}\left(\tilde{\beta}_w + \frac{\omega}{\tilde{\omega}_\kappa}\right)\right| \quad , \quad \frac{\omega^2\rho|m|}{k_\parallel^2 B^{*2}}.$$

We take $\hat{C}$ to be constant in order to annihilate the dominant $\mathcal{L}_{\mathrm{KIN1}}$ mode term in Eq. (23). Then, keeping only lowest order terms in $m\Delta/r$ and extremizing with respect to $\hat{C}$ yields the relation:

$$\hat{C}\left[\int \frac{d\psi ds}{B} D_H\left(\tilde{\beta}_w + \frac{\omega}{\tilde{\omega}_\kappa}\right) - \int ds \frac{\rho_0|m|\omega^2}{B^{*2}}\right] + \int ds \frac{\rho_0|m|\omega^2 r_p \chi}{B^*} = 0. \tag{24}$$

We also note that the range of the integrals, $\int ds\rho_0|m|\omega^2/B^{*2}$ and the last integral in Eq. (24) are only where there are hot particles. This is important in systems where the hot particles are only in a fraction of a flux tube, like in the plug of a tandem mirror.

Substituting this expression for $\hat{C}$ in Eq. (23), we obtain:

$$\begin{aligned}
&\int \frac{d\psi ds}{B}\left[-D_H \frac{B^2 r^2}{B_v^2 r_p^2}\hat{\phi}_0^2 + m^2\left(\frac{D}{D\psi}P_{\perp H}\right)\overline{\frac{D}{D\psi}P_{\perp H}a_H^2 B^2}\frac{(m^2-1)}{B^2 B_v^2 r_p^2}\hat{\phi}_0^2\right] \\
&\quad + \int ds|m|\left[\sigma_0\left(\frac{\partial}{\partial s}\frac{B^*}{B_v}\hat{\phi}_0\right)^2 + Z_{|m|}\left(\frac{\partial}{\partial s}\hat{\phi}_0\right)^2\right] \\
&\quad - \int ds \frac{\omega^2\rho_0|m|}{B_v^2}\hat{\phi}_0^2 \\
&\quad - \frac{\left\{\int ds \frac{\omega^2\rho_0|m|}{B^* B_v}\hat{\phi}_0\right\}^2}{\left[\int \frac{d\psi ds}{B}D_H\left(\tilde{\beta}_w + \frac{\omega}{\tilde{\omega}_\kappa}\right) - \int ds\frac{\omega^2\rho_0|m|}{B^{*2}}\right]} = 0
\end{aligned} \tag{25}$$

where

$$\hat{\phi}_0 = \chi B_v r_p,$$

and we note that the integral $|m|\omega^2 \int \frac{ds\rho_0}{B_v^2}\hat{\phi}_0^2$ is over the entire flux tube.

It is useful to rewrite the term proportional to curvature $\left(\kappa = \frac{\partial^2 r}{\partial s^2}\right)$:

$$\begin{aligned}
-\int \frac{d\psi \partial s}{B} D_H \frac{B^2 r^2}{B_v^2 r_p^2}\hat{\phi}_0^2 &= -\int d\psi ds\, r\frac{\partial^2 r}{\partial s^2}\left\{\frac{D}{D\psi}\left(P_{\perp H} + P_{\parallel H}\right)\right\}\hat{\phi}_0^2/B_v^2 r_p^2 \\
&\approx \int ds m^2 \frac{1}{r_p}\frac{\partial^2 r_p}{\partial s^2}\frac{\left(P_{\perp H}^* + P_{\parallel H}^*\right)}{B_v^2}\hat{\phi}_0^2
\end{aligned}$$

in terms of the vacuum curvature. To do this, we note that in the long thin approximation

$$\psi_p^- = \frac{B^* r_p^2}{2}.$$

Now define

$$r_v^2 = 2\psi_p^- / B_v^{(s)}$$

where $B_v(s)$ is the magnetic field in the vacuum region. Then, defining $\eta^2 \equiv \frac{B_v}{B^*} = \frac{r_p^2}{r_v^2}$, the curvature at $\psi = \psi_p^-$ is given by:

$$\frac{1}{r_p}\frac{\partial^2 r_p}{\partial s^2} = \frac{1}{r_v}\frac{\partial^2 r_v}{\partial s^2} + \frac{2}{\eta r_v}\frac{\partial \eta}{\partial s}\frac{\partial r_v}{\partial s} + \frac{1}{\eta}\frac{\partial^2 \eta}{\partial s^2}.$$

Thus

$$\int ds\ m^2 \frac{1}{r_p}\frac{\partial^2 r_p}{\partial s^2}\frac{\left(P_{\perp H}^* + P_{\parallel H}^*\right)}{B_v^2}\hat{\phi}_0^2$$
$$= m^2 \int ds \hat{\phi}_0^2 \beta^* \frac{1}{r_v}\frac{\partial^2 r_v}{\partial s^2} - m^2 \int ds \hat{\phi}_0^2 \left(\frac{1}{\eta}\frac{\partial \eta}{\partial s}\right)^2 \left\{2 - 3\beta^*\right\}$$
$$- m^2 \int ds \frac{2\hat{\phi}_0}{\eta}\frac{\partial \hat{\phi}_0}{\partial s}\frac{\partial \eta}{\partial s}\beta^*$$

where

$$\beta^* \equiv \frac{P_{\perp H}^* + P_{\parallel H}^*}{B_v^2}$$

and we have made use of parallel pressure balance (lowest order)

$$\frac{\partial P_\parallel}{\partial s} = \frac{(P_\parallel - P_\perp)}{B}\frac{\partial B}{\partial s}$$

and perpendicular pressure balance (lowest order):

$$\frac{\partial}{\partial s}\frac{2P_\perp}{B^2} = \frac{\partial}{\partial s}\frac{B_v^2}{B^2}.$$

We may also conveniently rewrite:

$$\int ds |m| \sigma_0 \left(\frac{\partial}{\partial s}\frac{B^*}{B_v}\hat{\phi}_0\right)^2 = \int ds |m| \sigma_0 \left(\frac{\partial}{\partial s}\hat{\phi}_0 / \eta^2\right)^2$$
$$= \int ds |m| \left(1 - \beta^*\right) \left[\left(\eta \frac{\partial}{\partial s}\frac{\hat{\phi}}{\eta}\right)^2 - \frac{2\hat{\phi}_0}{\eta}\frac{\partial \hat{\phi}_0}{\partial s}\frac{\partial \eta}{\partial s} + \frac{3\hat{\phi}_0^2}{\eta^2}\left(\frac{\partial \eta}{\partial s}\right)^2\right]. \quad (26)$$

The quadratic form can then be expressed as follows:

$$\int ds\hat{\phi}_0^2\beta^*\left\{\frac{1}{r_v}\frac{\partial^2 r_v}{\partial s^2}+\frac{(m^2-1)}{r_p}\frac{\partial^2 r_p}{\partial s^2}\right\}$$
$$+\int ds\int\frac{d\psi}{B}m^2\left(\frac{D}{D\psi}P_{\perp H}\right)\overline{\frac{D}{D\psi}P_{\perp H}a_H^2B^2\frac{(m^2-1)}{B^2B_v^2r_p^2}}\hat{\phi}_0^2$$
$$+\int ds\,(|m|-1)\,\sigma_0\left(\frac{\partial}{\partial s}\hat{\phi}_0/\eta^2\right)+\int ds\,(|m|Z_{|m|}-1)\left(\frac{\partial}{\partial s}\hat{\phi}_0\right)^2$$
$$+\int ds(2-\beta^*)\left(\eta\frac{\partial}{\partial s}\hat{\phi}_0/\eta\right)^2-\int ds\frac{\omega^2\rho_0|m|}{B_v^2}\hat{\phi}_0^2$$
$$-\frac{\left\{\int ds\frac{\omega^2\rho_0|m|}{B^*B_v}\hat{\phi}_0\right\}^2}{\left[\int\frac{d\psi ds}{B}D_H\left(\tilde{\beta}_w+\frac{\omega}{\tilde{\omega}_\kappa}\right)-\int ds\frac{\omega^2\rho_0|m|}{B^{*2}}\right]}=0. \tag{27}$$

This quadratic form illustrates that the drive for the m=1 unstable mode (the first term) is due only to the vacuum field-line curvature $\frac{1}{r_v}\frac{\partial^2 r_v}{\partial s^2}$. For $m\geq 2$, the second term which contains contributions from the total curvature, contributes to instability.

An alternative quadratic form may be obtained by substituting for $\int ds|m|\sigma_0\left(\frac{\partial}{\partial s}\hat{\phi}_0/\eta^2\right)^2$ (instead of Eq. (26)):

$$\int ds|m|\sigma_0\left(\frac{\partial}{\partial s}\hat{\phi}_0/\eta^2\right)=\int ds|m|\left(1-\beta^*\right)\left[\left(\eta^{2-|m|}\frac{\partial}{\partial s}\hat{\phi}/\eta^{2-|m|}\right)^2\right.$$
$$\left.-2|m|\frac{\hat{\phi}_0}{\eta}\frac{\partial\hat{\phi}_0}{\partial s}\frac{\partial\eta}{\partial s}+\frac{(4|m|-|m|^2)}{\eta^2}\left(\frac{\partial\eta}{\partial s}\right)^2\right]. \tag{28}$$

We then obtain:

$$\int ds\beta^*\frac{m^2}{r_v}\frac{\partial^2 r_v}{\partial s^2}\hat{\phi}_0^2+\int ds\int\frac{d\psi}{B}m^2\left(\frac{DP_{\perp H}}{D\psi}\right)\overline{\frac{D}{D\psi}P_{\perp H}a_H^2B^2D\psi\frac{(m^2-1)}{B^2B_v^2r_p^2}}\hat{\phi}_0^2$$
$$-\int ds|m|\left(Z_{|m|}-1\right)\left(\frac{\partial}{\partial s}\hat{\phi}_0\right)^2$$
$$-\int ds|m|\left[\frac{2|m|(|m|-1)}{\eta^2}\left(\frac{\partial\eta}{\partial s}\right)^2-\left(\eta^{|m|}\frac{\partial}{\partial s}\hat{\phi}_0/\eta^{|m|}\right)^2\right.$$
$$\left.-\left(\eta^{2-|m|}\frac{\partial}{\partial s}\hat{\phi}_0/\eta^{2-|m|}\right)\right]+\int ds|m|\beta^*\left[\frac{|m|(|m|-1)}{\eta^2}\left(\frac{\partial\eta}{\partial s}\right)^2\hat{\phi}_0^2\right.$$

$$- \left(\eta^{2-|m|}\frac{\partial}{\partial s}\hat{\phi}_0/\eta^{2-|m|}\right)^2\Bigg] - \int ds\frac{\omega^2\rho_0|m|}{B_v^2}\hat{\phi}_0^2$$

$$- \frac{\left\{\int ds\frac{\omega^2\rho_0|m|}{B^*B_v}\hat{\phi}_0\right\}^2}{\left[\int\frac{d\psi ds}{B}D_H\left(\tilde{\beta}_w+\frac{\omega}{\tilde{\omega}_\kappa}\right) - \int ds\omega^2\rho_0\frac{|m|}{B^{*2}}\right]} = 0. \tag{29}$$

Equation (29) shows somewhat more clearly than Eq. (27) the influence of the field-line bending terms on the higher $m > 1$ modes. It may be noted that the local (total) curvature does not appear explicitly.

The extremization of Eq. (29) leads to a differential integral equation that needs to be studied numerically. Considerable simplification occurs if the conducting wall at $\psi = \psi_w$ is close to the plasma surface $\psi = \psi_p^+$, the magnitude of $Z_{|m|}$ is large. For this case, we can obtain an approximate dispersion relation by choosing $\hat{\phi}_0$ to be a constant independent of s in order to annihilate the large line bending term proportional to $Z_{|m|} \gg 1$. With this choice of $\hat{\phi}_0$, the dispersion relation is:

$$\frac{\omega^2\left\langle\gamma^2_{\text{MHD}}\right\rangle\left\langle\tilde{\beta}_w+\frac{\omega}{\tilde{\omega}_\kappa}\right\rangle - a_0\omega^4}{\left\langle\gamma^2_{\text{MHD}}\right\rangle\left\langle\tilde{\beta}_w+\frac{\omega}{\tilde{\omega}_\kappa}\right\rangle - \omega^2} + \left\langle\gamma^2_{\text{MHD}}\right\rangle g_{|m|} = 0 \tag{30}$$

where

$$g_{|m|} = 1 - \left\langle\frac{(m^2-1)\beta_H a_H^2}{2\kappa_v\Delta r^2}\right\rangle - \left\langle\frac{\left[(4|m|-2m^2)-(4|m|-3m^2)\beta_H\right]_{r_v}}{16\kappa_v\beta_H\left(1-\beta_{\perp H}\right)^2}\left(\frac{\partial\beta_{\perp H}}{\partial s}\right)^2\right\rangle \tag{31}$$

$$a_0 = 1 - \frac{\left[\int_h ds\frac{\rho_0}{B^*B_v}\right]^2}{\left(\int_h ds\frac{\rho_0}{B_v^2}\right)\left(\int_h ds\frac{\rho_0}{B^{*2}}\right)} \tag{32}$$

$$\left\langle\gamma^2_{\text{MHD}}\right\rangle = -\frac{\int dsm^2\beta^*\frac{1}{r_v}\frac{\partial^2 r_v}{\partial s^2}}{\int_c ds\frac{|m|\rho_0}{B_v^2}} \tag{33}$$

$$\left\langle \frac{(m^2-1)\beta_H a_H^2}{2\kappa_v \Delta r^2} \right\rangle = \frac{-\int \frac{ds d\psi}{B} \left(\frac{D}{D\psi} P_{\perp H} \right) \left(\frac{D}{D\psi} P_{\perp H} a_H^2 B^2 \right) \frac{(m^2-1)}{B^2 B_v^2 r_p^2}}{\int ds \beta^* \frac{1}{r_v} \frac{\partial^2 r_v}{\partial s^2}} \tag{34}$$

$$\left\langle \frac{\left[(4|m|-2m^2)-(4|m|-3m^2)\beta_H\right] r_v}{16\kappa_v \beta_H \left(1-\beta_{\perp H}\right)^2} \left(\frac{\partial \beta_{\perp H}}{\partial s} \right)^2 \right\rangle$$
$$= \frac{-\int ds \frac{(4|m|-2m^2)-(4|m|-3m^2)\beta^*}{16\left(1-\beta_\perp^*\right)^2} \left(\frac{\partial \beta_\perp}{\partial s} \right)^2}{\int ds \beta^* \frac{1}{r_v} \frac{\partial^2 r_v}{\partial s^2}} \tag{35}$$

$$\beta^* = \frac{P_{\perp H}^* + P_{\parallel H}^*}{B_v^2}$$

$$\beta_\perp^* = \frac{2P_{\perp H}^*}{B_v^2}$$

$$\left\langle \tilde{\beta}_w + \frac{\omega}{\tilde{\omega}_\kappa} \right\rangle \left\langle \gamma_{\text{MHD}}^2 \right\rangle = \frac{\int \frac{d\psi ds}{B} D_H \left(\tilde{\beta}_w + \frac{\omega}{\tilde{\omega}_\kappa} \right)}{\int ds |m| \rho_0 / B^{*^2}}.$$

The subscript "h" refers to where there is hot particles, while the subscript "c" refers to entire cell where there is plasma. To evaluate a_0 approximately, we can take $P_\perp(s) = P_{\perp 0}\left(1 - s^2/L_p^2\right)$ and assume $1/L_p > \frac{1}{B_v}\frac{dB_v}{ds}$. Then, with $\beta_0 = \frac{2P_{\perp 0}}{B^2(s=0)}$, we find

$$a_0 \doteq 1 - \frac{\left[\int_0^1 \frac{dx}{\left[1-\beta_0(1-x^2)\right]^{1/2}}\right]^2}{\int_0^1 \frac{dx}{1-\beta_0+\beta_0 x^2}} \xrightarrow[\beta_0 \ll 1]{} \frac{\beta_0^2}{45}.$$

Equation (30) may conveniently be rewritten as

$$Q(\Omega) \equiv \frac{(1-a_0)\Omega^4}{\left(\Omega^2 + g_{|m|}\right)\left(\Omega^2 - \frac{\Omega}{\Omega_\kappa} - \tilde{\beta}_w\right)}$$
$$= \frac{(1-a_0)\Omega^4}{(\Omega-\Omega_1)(\Omega+\Omega_1)\left(\Omega-\Omega_2^+\right)\left(\Omega-\Omega_2^-\right)} = 1 \tag{36}$$

where $1 > a_0 > 0$ by the Schwartz inequality and

$$\Omega^2 = \omega^2 / \langle \gamma^2_{\mathrm{MHD}} \rangle$$

$$\Omega_\kappa = \frac{\tilde{\omega}_\kappa}{\langle \gamma^2_{\mathrm{MHD}} \rangle^{1/2}}$$

$$\Omega_1 = \left(-g_{|m|}\right)^{1/2}$$

$$\Omega_2^\pm = \frac{1}{2\Omega_\kappa} \pm \left[\frac{1}{4\Omega_\kappa^2} + \tilde{\beta}_w\right]^{1/2} .$$

$Q(\Omega)$ is sketched schematically in Fig. 1 as a function of Ω for the case $g_{|m|} < 0$, and it will be observed that there exists four real roots for Ω. Thus, the condition for robust stability is $g_{|m|} < 0$, and it can be achieved with wall stabilization and finite Larmor radius stabilization at finite values of β_H.

For m=1, wall stabilization is achieved when

$$\frac{(2-\beta_H)}{16\beta_H\left(1-\beta_{\perp H}\right)^2} \frac{r_v}{\kappa_v} \left(\frac{\partial \beta_{\perp H}}{\partial s}\right)^2 > 1.$$

For $m > 1$, FLR stabilization requires

$$\frac{(m^2-1)\beta_H a_H^2}{2\kappa_v \Delta r^2} > 1.$$

Note that Eq. (35) is positive definite if $\beta_H > 2/3$. Thus, if the curvature is small, $\beta_H > 2/3$ is the condition that wall effects stabilize all layer modes, independent of m. Previously, it was noted that $\beta_H > 1/2$ can stabilize the m=1 mode in the limit of weak external curvature.[25]

If $g_{|m|} > 0$, the system is susceptible to instability. In this limit, if we look for a root where $\left|\frac{\Omega}{\tilde{\Omega}_\kappa}\right| \ll 1$, we find the solution

$$\Omega^2 = -\frac{\left(g_{|m|} - \tilde{\beta}_w\right)}{2a_0} \pm \left[\frac{\left(g_{|m|} - \tilde{\beta}_w\right)^2}{4a_0^2} + \frac{g_{|m|}\tilde{\beta}_w}{a_0}\right]^{1/2} \tag{37}$$

and a negative value of Ω^2 can always be found.

When $a_0 \ll 1$ (for small β_h, $a_0 \sim O(\beta_H^2)$), we find that the unstable Ω is given by

$$\Omega^2 = -\frac{g_{|m|}\tilde{\beta}_w}{\left(g_{|m|} - \tilde{\beta}_w\right)} \quad , \quad \text{if} \quad \tilde{\beta}_w > g_{|m|} \tag{38a}$$

$$\Omega^2 = -\frac{\left(g_{|m|} - \tilde{\beta}_w\right)}{a_0} \quad , \quad \text{if} \quad \tilde{\beta}_w < g_{|m|} \tag{38b}$$

Equation (38a) is a generalized flute interchange mode that arises for $\tilde{\beta}_w > g_{|m|}$, while Eq. (38b) is a generalized compressional mode instability.

We now calculate the stability conditions more precisely when $a_0 \ll 1$ (i.e., $\beta_H \ll 1$). For the low frequency interchange mode where we can neglect a_0, the dispersion relation simplifies to

$$\Omega^2 \approx \frac{g_{|m|}\left(\dfrac{\Omega}{\Omega_\kappa} + \tilde{\beta}_w\right)}{g_{|m|} - \dfrac{\Omega}{\Omega_\kappa} - \tilde{\beta}_w}. \tag{39}$$

The interchange mode with

$$\Omega \approx i g_{|m|}^{1/2} \Big/ \left(1 - \frac{g_{|m|}}{\left(\tilde{\beta}_w + i g_{|m|}^{1/2}/\Omega_\kappa\right)}\right)^{1/2} \tag{40}$$

is unstable if

$$\left|\tilde{\beta}_w + \frac{i g_{|m|}^{1/2}}{\Omega_\kappa}\right| > g_{|m|} \quad \text{or roughly} \quad g_{|m|} < \Omega_\kappa^{-2} \quad \textit{and} \quad \tilde{\beta}_w < g_{|m|}.$$

(If $\tilde{\beta}_w \ll g_{|m|}$, the precise instability condition is $\Omega_\kappa^2 < \dfrac{4}{|g_{|m|}|}$). Hence, as $g_{|m|} \to 0$, the condition for instability is always satisfied. Thus, even if a fragile instability can be achieved with $g_{|m|} > 0$, the route to robust instability $\left(g_{|m|} < 0\right)$ as β_H increases, requires the bridging of an unstable band $0 < g_{|m|} \lesssim 4/\Omega_\kappa^2$ where the interchange instability exists.

Even if $\Omega_\kappa^2 > \dfrac{4}{g_{|m|}}$, there still exists the possibility of an unstable compressional mode. At higher frequencies where $\Omega^2 > g_{|m|}$, the dispersion relation (again assuming $a_0 \ll 1$) may be approximated by

$$a_0\Omega^4 - \frac{\Omega^3}{\Omega_\kappa} + \Omega^2\left(g_{|m|} - \tilde{\beta}_w\right) = 0. \tag{41}$$

When $\Omega_\kappa^2 > \dfrac{1}{4a_0\left(g_{|m|} - \tilde{\beta}_w\right)}$, we obtain an unstable mode with

$$\Omega \approx i\left(g_{|m|} - \tilde{\beta}_w\right)^{1/2} \Big/ a_0^{1/2}. \tag{42}$$

Thus, for $\tilde{\beta}_w \ll g_{|m|}$, the range of Ω_κ^2 where stability is possible is

$$\frac{1}{4a_0 g_{|m|}} > \Omega_\kappa^2 > \frac{4}{g_{|m|}}. \tag{43}$$

The existence of this stability band requires

$$a_0 \equiv \alpha \beta_H^2 < \frac{1}{16}$$

which limits the hot particle beta to

$$\beta_H < \beta_{H\,\mathrm{max}} = \frac{1}{4\alpha^{1/2}}. \tag{44}$$

After Eq. (35) we estimated α and found when $\beta_H \ll 1$, $\alpha^{1/2} \approx .15$. Thus there is not a very stringent limitation on β_H for the existence of a stability band. However, the stability band in Ω_κ does narrow appreciably as $\beta_H \to 1$.

IV. Higher Frequency Drift Compressional Modes

In the higher frequency limit where the terms proportional to the frequency ω become comparable with the compressional drift term, it is convenient to use $\hat{C}(s)$ and $\hat{\phi}(s) = \hat{C} - \chi B^* r_p$ as the field variables in place of $\hat{C}(s)$ and $\chi(s)$.

Equation (23) can then be written as follows:

$$\int \frac{d\psi ds}{B}\left\{ -\left(\frac{\omega^2 \rho m^2}{r^2 B^2} + D_c\right)\left[\hat{C}\left(1 - \frac{Br}{B^* r_p}\right) + \hat{\phi}\frac{Br}{B^* r_p}\right]^2 \right.$$
$$+ \frac{\sigma m^2}{r^2}\left[\frac{\partial}{\partial s}\left(\hat{C}\left(1 - \frac{Br}{B^* r_p}\right) + \hat{\phi}\frac{Br}{B^* r_p}\right)\right]^2$$
$$+ \left[m^2\left(\frac{D}{D\psi}P_{\perp H}\right)\overline{\frac{D}{D\psi}P_{\perp H} a_H^2 B^2 \frac{(m^2-1)}{B^4}} - D_H\right]\frac{B^2 r^2}{{B^*}^2 r_p^2}(\hat{C} - \hat{\phi})^2$$
$$\left. + D_H\left(\tilde{\beta}_w + \frac{\omega}{\tilde{\omega}_\kappa}\right)\hat{C}^2 + \frac{2 q_H^2 B}{m_H}\int d^3 v \frac{v_\parallel^2 \frac{\partial F_H}{\partial \psi}}{v_\perp^2 \frac{\partial B}{\partial \psi}}\left(\frac{\partial \hat{C}}{\partial s}\right)^2 \right\}$$
$$+ \int ds \left\{ |m|\sigma_0\left(\frac{\partial}{\partial s}\hat{\phi}\right)^2 + |m| Z_{|m|}\left[\frac{\partial}{\partial s}\left(\hat{C}\left(1 - \frac{B_v}{B^*}\right) + \hat{\phi}\frac{B_v}{B^*}\right)\right]^2 \right.$$
$$\left. - \frac{\omega^2 \rho_0 |m|}{{B^*}^2}\hat{\phi}^2 \right\} = 0. \qquad (45)$$

When

$$\frac{\omega^2 a_H^2 m^2}{v_A^2 \beta_H^2} \sim 2 k_\parallel^2 r^2 \frac{P_\parallel}{P_\perp} > \frac{k_\parallel^2 \beta_H^2}{4} |m| Z_{|m|} \frac{a_H^2}{\Delta},$$

the coupling of $\hat{\phi}$ to $\hat{C}$ is small so that the quadratic form may be approximated by,

$$\omega^2 \mathcal{A}_0(\hat{C},\hat{C}) + \omega \mathcal{A}_1(\hat{C},\hat{C}) + \mathcal{A}_2(\hat{C},\hat{C}) = 0 \qquad (46)$$

where

$$\mathcal{A}_0(\hat{C},\hat{C}) \equiv \int \frac{d\psi ds}{B}\frac{\rho m^2}{r^2 B^2}\left(1 - \frac{Br}{B^* r_p}\right)^2 \hat{C}^2$$
$$\mathcal{A}_1(\hat{C},\hat{C}) \equiv -\int \frac{d\psi ds}{B} m q_H \left(\frac{D}{D\psi}\frac{n_H}{B}\right)\hat{C}^2$$
$$\mathcal{A}_2(\hat{C},\hat{C}) \equiv -\int \frac{d\psi ds}{B}\left\{ m^2\left(\frac{D}{D\psi}\frac{P_{\perp w}}{B^2}\right)\left(\frac{D}{D\psi}P_{\perp H}\right)\hat{C}^2 + 2\frac{q_H^2 B}{m_H}\int d^3 v \frac{v_\parallel^2 \frac{\partial F_H}{\partial \psi}}{v_\perp^2 \frac{\partial B}{\partial \psi}}\left(\frac{\partial \hat{C}}{\partial s}\right)^2 \right\}.$$

We have neglected line bending terms and terms proportional to field-line curvature and finite Larmor radius effect.

In this limit, the perturbation is predominantly magnetic compressional in character. It is stable if

$$\mathcal{A}_1^2(\hat{C},\hat{C}) > 4\mathcal{A}_0(\hat{C},\hat{C})\mathcal{A}_2(\hat{C},\hat{C}). \tag{47}$$

This criterion imposes an upper limit on the mass density $\rho_0 \equiv n_w m_w$ of the warm plasma.

In order to obtain a crude estimate of the stability criterion, we make the substitution $\dfrac{\partial \hat{C}}{\partial s} \to k_{\|}\hat{C}$, and we obtain from Eq. (47):

$$\frac{v_{\mathrm{thH}}^2}{\beta_H v_A^2} \approx \frac{n_w M_w}{n_H M_H} < \frac{3}{2\beta_H\left\{\dfrac{k_{\|}^2 P_{\|H}}{P_{\perp H}}\Delta^2 - \dfrac{m^2\beta_H a_H^2 \beta_w}{8r_p^2}\right\}} \tag{48}$$

where $k_{\|}$ is bounded by the requirement that the bounce frequency be less than the gradient-B drift frequency, that is

$$k_{\|}L_p < \frac{\omega_{DH}L_p}{v_{thH\|}} \approx \frac{|m|\beta_H}{2}\frac{a_H}{\Delta}\frac{L_p}{r_p}\left(\frac{P_{\perp H}}{2P_{\|H}}\right)^{1/2}. \tag{49}$$

For $k_{\|}L_p \sim O(1)$, this restriction on $k_{\|}$ is satisfied for moderate β_H as indicated in the arguments after Eq. (12). Our estimate on bounds of $k_{\|}$ is admittedly crude, as $k_{\|}v_{thH\|} \sim \omega_B$ requires more complicated analysis to determine precise stability. Further, as indicated in Refs. 18 and 19, remnant instabilities exist when $k_{\|}v_{thH\|} \gtrsim \omega_B$.

Let us now assess the band of stability that is available. First neglecting the higher frequency drift compressional branch, we have found that the condition for low frequency stability to interchange and compressional modes, namely Eq. (43), can be written approximately as:

$$\frac{8r_p}{|m|g_{|m|}\kappa a_H^2} < \frac{n_w M_w}{n_H M_H} < \frac{r_p}{2|m|g_{|m|}\kappa a_H^2 a_0}. \tag{50}$$

However, the need to stabilize the high frequency compressional branch (Eq. (48)) imposes an additional upper limit on $\dfrac{n_w M_w}{n_H M_H}$ which is more stringent than that imposed by the right-hand side of Eq. (50). Thus, when $g_{|m|} > \tilde{\beta}_w \sim \frac{\beta_w}{2\kappa\Delta}$, and if $\tilde{k}_\parallel^2 \equiv k_\parallel^2 - \frac{m^2\beta_w\beta_H a_H^2}{8r_p^2\Delta^2}\frac{P_{\perp H}}{P_{\parallel H}}$. The stability band is determined by

$$\frac{8r_p}{|m|g_{|m|}\kappa a_H^2} < \frac{n_w M_w}{n_H M_H} < \frac{3}{2\beta_H \tilde{k}_\parallel^2 \Delta^2}\frac{P_{\perp H}}{P_{\parallel H}} \sim \frac{12 r_p^2}{\beta_H^3 a_H^2 m^2}\frac{1}{\left(1 - \frac{\beta_w}{\beta_H}\right)}. \tag{51}$$

The last approximate equality is obtained by substituting for $k_\parallel^2$ its largest possible value allowable by our ordering (Eq. (49)) in order to determine the most restrictive condition.

Our analysis also constrains m to be $|m| < \frac{r_p}{\Delta}$. Thus we may finally write the condition for the existence of a stability band to be

$$\left(1 - \frac{\beta_w}{\beta_H}\right) < \frac{4}{3} < \frac{g_{|m|} r_p \kappa}{|m|\beta_H^3}. \tag{52}$$

It should be noted that $g_{|m|} > \frac{\beta_w}{2\kappa\Delta}$ implies that the left-hand side is greater than zero and hence with the exception of the case where β_H is fairly small, Eq. (52) cannot be satisfied. The unstable low frequency interchange mode and the drift compressional mode may rule out a stable operating regime when $g_{|m|} > \frac{\beta_w}{2\kappa\Delta}$.

Even if the regime $g_{|m|} < 0$ can somehow be reached with auxiliary stability methods, so that the low frequency modes are then stabilized by wall and FLR effects, $\left(\beta_H > \frac{2}{3}\right)$, the unstable drift compressional mode can be unstable unless the beta of the warm background plasma satisfies the condition (Eq. (51)):

$$\left(1 - \frac{\beta_w'}{\beta_H'}\right)\frac{\beta_w}{\beta_H} < \frac{12\Delta^2}{\beta_H^3 a_H^2}\frac{T_w}{T_H}, \tag{53}$$

where the largest m consistent with our orderings is $|m| = \frac{r_p}{\Delta}$. We have introduced β_w' and β_H', the spatial gradient of the warm and hot plasma beta respectively, since this more correctly denotes the parameter dependences in the inequality. The magnitude of β_w is restricted only by the formal ordering introduced at the beginning of the analysis. However, as will be discussed in Appendix A, an eikonal analysis of the high frequency compressional mode yields a local stability criterion with the same structure as Eq. (53) and this result is valid for finite values of β_w. Thus, stability of the high frequency drift compressional mode can be achieved if $\frac{\beta_w'}{\beta_H'} > 1$.

Drift compressional mode instabilities have been discussed previously by Hasegawa[18] and Migliuolo.[19] However, this limit with $\omega_B \gg k_\parallel v_{th\parallel}$ was not emphasized by Hasegawa, and our analysis results serve to elucidate the numerical studies of Migliuolo.

When Eq. (47) is satisfied, the drift compressional mode separates into a positive energy wave (ω^+) and a negative energy wave (ω^-) with real frequencies

$$\omega^\pm = \mp \frac{1}{2\mathcal{A}_0} \left\{ \left[\mathcal{A}_1^2 - \mathcal{A}_0 \mathcal{A}_2\right]^{1/2} \pm \mathcal{A}_1 \right\}. \tag{54}$$

If $\beta'_H \gg \beta'_w$,

$$\omega^+ \sim -\frac{12}{\beta_H} \frac{v_A^2}{v_{th}^2} \frac{r}{m\Delta} \Omega$$

$$\omega^- \sim -\frac{2k_\parallel^2 r \Delta}{m \beta_H} \frac{P_\parallel}{P_\perp} \Omega.$$

The positive energy wave is however destabilized by the negative energy dissipative effect due to resonant particle interactions if the magnitude of its frequency is less than the diamagnetic drift frequency, $|\omega^+| < |\omega^*|$ (see Appendix A), while the negative energy wave is damped.

The presence of this negative energy dissipation can also destabilize other positive energy waves present in the system, for example shear Alfvén waves. It is however possible (we will not discuss it here) that the simultaneous presence of both positive energy and negative energy dissipation in appropriate proportions can stabilize all positive energy and negative energy waves.

V. Discussion

We have examined the stability of a plasma with hot particles in a system with unfavorable curvature. For a steep radial pressure profile, we have reduced the stability analysis to a relatively simple quadratic form. The low frequency modes ($\omega \lesssim \omega_\kappa$) are analyzed analytically if the plasma edge is close to a conducting wall and all criteria obtained have made use of this assumption. More general analysis will require extensive numerical investigations to solve an integro-differential equation. In the high frequency limit ($\omega_\kappa < \omega < \omega_B$), rough WKB-like solutions are used to obtain the relevant dispersion relation for the compressional instability. We have shown that numerical solutions (not reported here) closely agree with the WKB method of evaluation.

Particular focus is on the stability properties of tandem mirrors with symmetric end-plugs when hot particles are used to attempt stabilization. The investigation examines the stability of steep profile equilibria to long wavelength layer modes where $m\Delta/r_p < 1$ with Δ the pressure gradient width (for m=1 the analysis is often applicable independent of the steepness assumption). These modes usually produce the more pessimistic stability conditions than the shorter wavelength eikonal modes and nonlinearly they should be more deleterious. The investigation has generalized previous treatments of hot particle stability in that arbitrary beta and finite Larmor radius effects are investigated for realistic mirror geometry which is modelled in the paraxial approximation.

In order to establish parameters of interest we have assumed that it is important to achieve robust stability condition where MHD perturbations are positive energy (as enough energy is expended either from line bending in the presence of conducting walls or from plasma distortion from the magneto-compressional finite Larmor radius term). Because finite Larmor radius is crucial, the hot component would usually have to consist of ions rather than an electron component. We also find that the requirements for achieving simultaneously finite Larmor radius and wall stabilization, as well as having a sharp profile equilibrium, usually requires that the hot ions be treated in the low bounce frequency approximation compared to the grad-B drift. We find that fulfilling these MHD-like stability requirements makes the system susceptible to a hot particle drift compressional instability, which can cause a further deterioration of stable operating regimes, even when robust stability is achieved.

As was found in previous studies, if one doesn't achieve robust stabilization, there are parameters where stability exists in the absence of dissipation. If one first of all neglects the effects of wall stabilization, finite Larmor radius, the background plasma beta, β_w, and the drift compressional mode, the stability that comes from the analysis of Eq. (30) if $\beta_H < 1$ is found from Eq. (43) to be roughly given by (here the overall scaling is given and more precise criteria require the evaluation of the integrals given in the text).

$$\frac{10}{\beta_H^2} > \frac{|m|a_H^2}{4L_p L_\kappa}\frac{n_w M_w}{n_H M_H} > 4q, \tag{55}$$

where $|m|$ is the mode number, a_H the hot particle Larmor radius, L_p is the hot plasma axial length, L_κ, the axial length of curvature variation ($\kappa_v \approx r_p/2L_\kappa^2$, $r_p \equiv$ plasma radius) n_ω the core plasma density, n_H the hot plasma density, β_H the mid-plane hot plasma beta, M_H and M_w is the hot particle mass and the background ion mass respectively,$q \approx 1$ for a single cell and in tandem mirrors $q \approx \dfrac{B_c^2}{B_a^2}\dfrac{L_\kappa}{L_c}$ with L_c the central cell length, B_c the magnetic field in the central cell and B_a the magnetic field where the hot particles are located. The mode number is restricted to $|m| < r_p/\Delta$, where Δ is the thickness of the pressure gradient length. For higher mode numbers, eikonal analysis applies, with results reported in Refs. 9 and 11. The right-hand inequality in Eq. 55 is contained in Refs. 15 and 16 while the left-hand inequality, due to the excitation of the surface magnetic compressional instability, was not reported in previous work. For example, Refs. 15 and 16 did not retain high enough terms in beta to extract the surface compressional mode. Somewhat similar stability bands have been obtained in the eikonal limit.[6,9]

The stability band defined by Eq. (55) erodes as the core beta increases. Further, the negative energy precessional mode is present in this band and is subject to destabilization by positive dissipation mechanisms. We will discuss this type of interaction later in the discussion. For now we note that the stability band disappears when the core beta exceeds the value

$$\tilde{\beta}_w \equiv \beta_w \frac{L_\kappa^2}{2\Delta r_p} \approx 1 \tag{56}$$

(Lee-Van Dam condition) whereupon instability, similar to the conventional MHD modes is predicted.

As the warm beta threshold given occurs at a relatively low value, and because of the possibility of destabilizing the negative energy precessional mode, it is important to consider if the negative energy sources can be eliminated. In the text we demonstrate that through a combination of wall stabilization and finite Larmor radius stabilization, it is possible to eliminate the negative energy band (if the drift-compressional branch is ignored). However, one cannot move to the positive energy band continuously, as one must go though a band of parameters where conventional unstable MHD theory is applicable as one tries to increase the positive energy sources. Before the negative energy sources are eliminated, the stability band is found to be (Eq. (50)):

$$\frac{10}{\beta_H^2 g_{|m|}} > \frac{|m| a_H^2 n_w M_w}{4 L_p L_\kappa n_H M_H} > \frac{4q}{g_{|m|}} \tag{57}$$

if $g_m > 0$ If a conducting wall is near the plasma $g_{|m|}$ is very roughly given by

$$g_{|m|} \approx 1 - \frac{(m^2-1)\beta_H a_H^2 L_\kappa^2}{2\Delta r_p^3} - \frac{\left[|m| - m^2/2 + \left(\frac{3}{4}m^2 - |m|\right)\beta_H\right] L_\kappa^2 \beta_H}{4L_p^2(1-\beta_H)^2}.$$

As the FLR and wall stabilization is increased $g_{|m|}$ goes to zero, and the right-hand inequality must eventually fail. Thus, if an experiment is to reach the robust stabilization regime where $g_{|m|} < 0$ where the negative energy sources are eliminated as well as the Lee-Van Dam limit, one must pass through an unstable MHD-like band. For this to be possible one must provide for auxiliary stabilization such as (1) ponderomotive stabilization, (2) transient quadrupole stabilization, (3) or somehow change the stability properties rapidly enough so that the plasma does not have a chance to disrupt. Once $g_{|m|}$ is made negative for all m numbers, the modes discussed thus far are stable in a robust manner. Dissipation mechanisms will cause damping of modes and the core beta can be increased without a Lee-Van Dam limit.

In the above discussion we considered modes of the MHD type. In that discussion the same criteria resulted whether the hot particles are in the high bounce frequency or the low bounce frequency limit. It is explained in the text, after Eq. (12), that the conditions for eliminating negative energy sources through wall stabilization, nearly forces the plasma parameters to lie in the low bounce frequency regime, especially when higher m-modes are considered. However, in the low bounce frequency regime the plasma is subject to the magnetic drift compressional mode, which produces more stringent stability conditions than the magnetic compressional mode which gave the left-sided constraint in Eq. (55) and Eq. (57). Instead, the constraint for an operating regime is (Eq. (51)),

$$\frac{3r_p^2}{\beta_H^3|m|L_\kappa^2} > \frac{|m|a_H^2 n_w M_w}{4L_p L_\kappa n_H M_H} > \frac{4q}{g|m|}. \tag{58}$$

Except when β_H or q are fairly small, Eq. (58) cannot be satisfied.

If one can establish $g_{|m|} < 0$, the only instability constraint is the drift compressional mode. If in this regime one tries to increase the beta of the background species, from the left-hand side of Eq. (58), its beta value is limited to (Eq. (53))

$$\left(1 - \frac{\beta'_w}{\beta'_H}\right)\frac{\beta_w}{\beta_H} << \frac{12T_w\Delta^2}{T_H a_H^2 \beta_H^3} \quad (\text{if } \beta'_w < \beta'_H). \tag{59}$$

The satisfaction of the stability constraints in Eqs. (57)-(59) give rise to real frequency positive and negative energy waves, which are destabilized by negative and positive energy dissipation mechanism respectively. If a system has an appropriate mix of both dissipation mechanisms, even these modes can be stabilized.

Appendix A – Drift Compressional Mode-Eikonal Approximation

If we neglect $E_\parallel$ perturbation, line-bending, terms of the order of the field line curvature, and FLR terms,t he quadratic form describing compressional modes may be approximated by

$$\int \frac{d\psi ds d\theta}{B}\Bigg[\rho\omega^2 \boldsymbol{\xi}\cdot\boldsymbol{\xi}^+ + \tau Q_L Q_L^+ - \int_{\text{hot}} d^3v \frac{\partial F_{0H}}{\partial\varepsilon}\frac{(\omega-\omega_*)\mu^2 Q_L Q_L^+}{(\omega-\omega_D-k_\parallel v_\parallel)}$$
$$-\sum\int_{\text{warm}} d^3v \frac{\partial F_{0w}}{\partial\varepsilon}\mu^2 Q_L Q_L^+\Bigg] = 0 \qquad (A1)$$

where

$$\mathbf{A}_\perp \to \mathbf{A}_\perp e^{-i\omega t + im\theta + i\int^s k_\parallel ds + i\int^r k_r dr}$$
$$\boldsymbol{\xi} = \frac{\mathbf{b}\times\mathbf{A}_\perp}{B}$$
$$Q_L = -B\nabla_\perp\cdot\boldsymbol{\xi}$$
$$\omega_D \approx m\frac{1}{q}\mu\frac{\partial B}{\partial\psi}$$
$$\omega^* = -m\frac{1}{q}\left(\frac{\partial F_0}{\partial\psi}\right)\Big/\left(\frac{\partial F_0}{\partial\varepsilon}\right)$$
$$\tau = \left(1+\frac{1}{B}\frac{\partial P_\perp(B,\psi)}{\partial B}\right)$$

We have assumed

$$k_r \gg \frac{1}{P}\frac{\partial P}{\partial r} \quad , \quad k_r \gg k_\parallel \gg \frac{1}{P}\frac{\partial P}{\partial s},$$

and have represented the perturbations in eikonal form.

Let $\boldsymbol{\xi}$ be expressed in terms of the field amplitudes ϕ and C:

$$\boldsymbol{\xi} = -\frac{\mathbf{b}}{im}\times\nabla\frac{\phi}{B} - \frac{c}{B^2}\frac{\mathbf{b}\times\nabla B}{im}.$$

The quadratic form may then be approximated as follows

$$\int\frac{d\psi ds}{B}\Bigg\{-\frac{\rho\omega^2}{m^2}\left[\frac{m^2}{r^2B^2}\phi\phi^+ + \left(\frac{ik_r\phi}{B}+\frac{Cr}{B}\frac{\partial B}{\partial\psi}\right)\left(-\frac{ik_r\phi^+}{B}+\frac{C^+r}{B}\frac{\partial B}{\partial\psi}\right)\right]$$
$$+\left[\frac{\omega}{mc}B\frac{D}{D\psi}\frac{q_H n_H}{B} - B\frac{\partial B}{\partial\psi}\frac{D}{D\psi}P_{\perp w}/B^2\right]CC^+$$
$$-\frac{q_H}{m}\int d^3v\frac{\partial F_{0H}}{\partial\psi}\left(1-\frac{\omega}{\omega_*}\right)\frac{(\omega-k_\parallel v_\parallel)^2}{(\omega-k_\parallel v_\parallel-\omega_D)}CC^+\Bigg\} = 0 \qquad (A2)$$

where

$$\frac{1}{\omega - k_\| v_\| - \omega_D} = -\frac{1}{\omega_D} - \frac{(\omega - k_\| v_\|)}{\omega_D^2} + \frac{(\omega - k_\| v_\|)^2}{\omega_D^2(\omega - k_\| v_\| - \omega_D)}$$

and we have made use of the equilibrium pressure balance relation

$$\frac{DP_{\perp H}}{D\psi} + \frac{DP_{\perp w}}{D\psi} + \frac{B\partial B}{\partial \psi} = 0.$$

Varying with respect to ϕ, ϕ^+, we obtain

$$k_\perp^2 \phi = -ik_r r \frac{\partial B}{\partial \psi} C$$
$$k_\perp^2 \phi^+ = ik_r r \frac{\partial B}{\partial \psi} C^+$$

where

$$k_\perp^2 \equiv \frac{m^2}{r^2} + k_r^2.$$

Equation (A2) then simplifies to the local dispersion relation at ψ and s:

$$\alpha_0 \omega^2 + \omega \alpha_1 + \alpha_2 = 0 \qquad (A3)$$

where

$$\alpha_0 \equiv \frac{\rho}{k_\perp^2} \left(\frac{1}{B} \frac{\partial B}{\partial \psi} \right)^2 \qquad (A4)$$

$$\alpha_1 \equiv -\frac{q_H B}{m} \frac{D}{D\psi} \frac{n_H}{B} \qquad (A5)$$

$$\alpha_2 \equiv B \frac{\partial B}{\partial \psi} \frac{D}{D\psi} P_{\perp w}/B^2 + \frac{q_H}{m} \int d^3v \frac{\partial F_{0H}}{\partial \psi} \frac{\left(1 - \frac{\omega}{\omega^*}\right)(\omega - k_\| v_\|)^2}{(\omega - k_\| v_\| - \omega_D)}. \qquad (A6)$$

In the limit $|\omega_D| \gg k_\| v_\| \gg \omega$, $\omega \ll \omega^*$, for the hot particles,

$$\alpha_2 \approx B \frac{\partial B}{\partial \psi} \frac{D}{D\psi} P_{\perp w}/B^2 - \frac{q_H^2 k_\|^2}{m^2 \left(\frac{\partial B}{\partial \psi} \right)} \int d^3v \frac{v_\|^2}{\mu} \frac{\partial F_{0H}}{\partial \psi}$$

and Eq. (A3) is similar in structure to Eq. (46) derived in the "layer" approximation if $k_\perp^2 \Delta^2 \approx 3$ where Δ is the thickness of the region of finite pressure gradient. If we neglect particle resonances, the local dispersion relation predicts stability when

$$\left(\frac{q_H}{m}\frac{D}{D\psi}\frac{n_H}{B}\right)^2 > 4\frac{\rho}{k_\perp^2}\left(\frac{1}{B}\frac{\partial B}{\partial \psi}\right)\left\{B\left(\frac{\partial B}{\partial \psi}\right)^2\frac{D}{D\psi}P_{\perp w}/B^2 - \frac{q_H^2}{m^2}k_\parallel^2\int d^3v\frac{v_\parallel^2}{\mu}\frac{\partial F_{0H}}{\partial \psi}\right\} \qquad (A7)$$

Since $\left(\frac{D}{D\psi}P_{\perp w}/B^2\right) < 0$ and $\frac{\partial F_0}{\partial \psi} < 0$, a sufficient condition for stability is

$$B\left(\frac{\partial B}{\partial \psi}\right)^2\left|\frac{D}{D\psi}P_{\perp w}/B^2\right| > \left|\frac{q_H^2}{m^2}k_\parallel^2\int d^3v\frac{v_\parallel^2}{\mu}\frac{\partial F_{0H}}{\partial \psi}\right| \qquad (A8)$$

If we model the distribution function D_{0H} by

$$F_{0H}(\varepsilon,\mu,\psi) \propto \begin{cases} (\mu B_m - \varepsilon)^{1/2} e^{-\frac{\mu B_m}{T_H}}, & \mu B_m \geq \varepsilon \\ 0 & \mu B_m < \varepsilon \end{cases}$$

we obtain

$$\frac{q_H^2}{m^2}\frac{k_\parallel^2}{\left(\frac{\partial B}{\partial \psi}\right)}\int d^3v\frac{v_\parallel^2}{\mu}\frac{\partial F_{0H}}{\partial \psi} = \frac{2}{m_H}\frac{q_H^2}{m^2}\frac{k_\parallel^2}{\left(\frac{\partial B}{\partial \psi}\right)}\frac{B}{4}\left(\frac{\partial}{\partial \psi} - \frac{\partial B}{\partial \psi}\frac{\partial}{\partial B}\right)\left(\frac{B_m}{B} - 1\right)n_H$$

where

$$n_H \propto \frac{B}{B_m}\left(1 - \frac{B}{B_m}\right)T_H^2$$

$$P_{\parallel H} = n_H\left(1 - \frac{B}{B_m}\right)T_H$$

$$P_{\perp H} = 2\frac{n_H T_H B}{B_m}.$$

$B_m(\psi)$ is the maximum of $B(\psi, s)$. For this model, we obtain the following sufficient condition for stability of the compressional mode (Eq. (A8))

$$B\frac{\partial B}{\partial \psi}\left|\frac{D}{D\psi}P_{\perp w}/B^2\right| > \frac{2}{m_H}\frac{q_H^2}{m^2}\frac{k_\parallel^2}{\left(\frac{\partial B}{\partial \psi}\right)}\frac{B}{4}\left|\left(\frac{\partial}{\partial \psi} - \frac{\partial B}{\partial \psi}\frac{\partial}{\partial B}\right)\left(\frac{B_m}{B} - 1\right)n_H\right|. \qquad (A9)$$

If we substitute for $k_{\|}$ its largest value possible in our ordering:

$$\frac{k_{\|}^2 P_{\| H}}{m_H n_H} \approx \omega_D^2 \approx \frac{m^2}{q_H^2}\left(\frac{P_{\perp H}}{n_H}\right)^2\left(\frac{1}{B}\frac{\partial B}{\partial \psi}\right)^2.$$

Equation (A9) for the stability criterion is

$$\left|\frac{D}{D\psi}P_{\perp w}/B^2\right| > \frac{P_{\perp H}}{B^2}\left\{\frac{\partial}{\partial \psi} - \frac{\partial B}{\partial \psi}\frac{\partial}{\partial B}\right\}\log\left(\frac{B_m}{B}-1\right)n_H. \qquad (A10)$$

When Eq. (A7) is satisfied, the compressional mode separates into a positive energy wave

$$\omega^+ = -\frac{1}{2\alpha_0}\left\{\left[\alpha_1^2 - 4\alpha_0\alpha_2\right]^{1/2} + \alpha_1\right\}$$

and a negative energy wave

$$\omega^- = \frac{1}{2\alpha_0}\left\{\left[\alpha_1^2 - 4\alpha_0\alpha_2\right]^{1/2} - \alpha_1\right\}.$$

If we now include the small resonant particle contribution from the kinetic term in Eq. (A2), we obtain the following corrections to $\omega^+ \to \omega^+ + \delta\omega^+$, $\omega^- \to \omega^- + \delta\omega^-$

$$\delta\omega^+ \approx \frac{i\pi}{|k_{\|}|}k\frac{1}{(\omega^+ - \omega^-)}\frac{2\pi m}{q_H m_H}\int\frac{d\psi ds}{B}\int_0^{\mu_{\max}^+} B d\mu \frac{\partial F_{0H}}{\partial \psi}\left(1-\frac{\omega^+}{\omega *}\right)\mu^2\left(\frac{\partial B}{\partial \psi}\right)^2\Bigg|_{\varepsilon=\varepsilon_0^+}$$

$$\delta\omega^- \approx \frac{i\pi}{k_{\|}|}\frac{1}{(\omega^- - \omega^+)}\frac{2\pi m}{q_H m_H}\int\frac{d\psi ds}{B}\int_0^{\mu_{\max}^-} B d\mu \frac{\partial F_{0H}}{\partial \psi}\left(1-\frac{\omega^-}{\omega *}\right)\mu^2\left(\frac{\partial B}{\partial \psi}\right)^2\Bigg|_{\varepsilon=\varepsilon_0^-}$$

where $\varepsilon_0^{\pm} = \frac{m_H}{2}\frac{(\omega^{\pm} - \omega_D)^2}{k_{\|}^2} + \mu B$ and $\mu_{\max}^{\pm}$ is the solution of $\varepsilon_0^{\pm} = \mu B_m$.

Since $\frac{q_H(\omega^- - \omega^+)}{m} > 0$, $\frac{\partial F_{0H}}{\partial \psi} < 0$, $\mathrm{Im}(\delta\omega^-) < 0$, and the negative energy mode is stable $\left(\frac{\omega^-}{\omega *} < 0\right)$. However, $\mathrm{Im}(\delta\omega^+) > 0$ when $1 > \frac{\omega^+}{\omega *}$, and the positive energy mode is unstable when the magnitude of its frequency is less than that of the diamagnetic frequency. The resonant particle contribution is effectively a negative energy dissipative effect.

Other positive energy waves present in the system, for example shear Alfvén waves, will be destabilized by these resonant particle interactions. It is however possible that the simultaneous presence of both positive and negative energy dissipation in appropriate proportions can stabilize all positive energy and negative energy waves.

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Figure Caption

Sketch of $Q(\Omega)$ as a function of Ω for the case $g_{|m|} < 0$ (robust stability) which gives rise to a quartic equation. As there exists four real roots for Ω, there can be no instability.

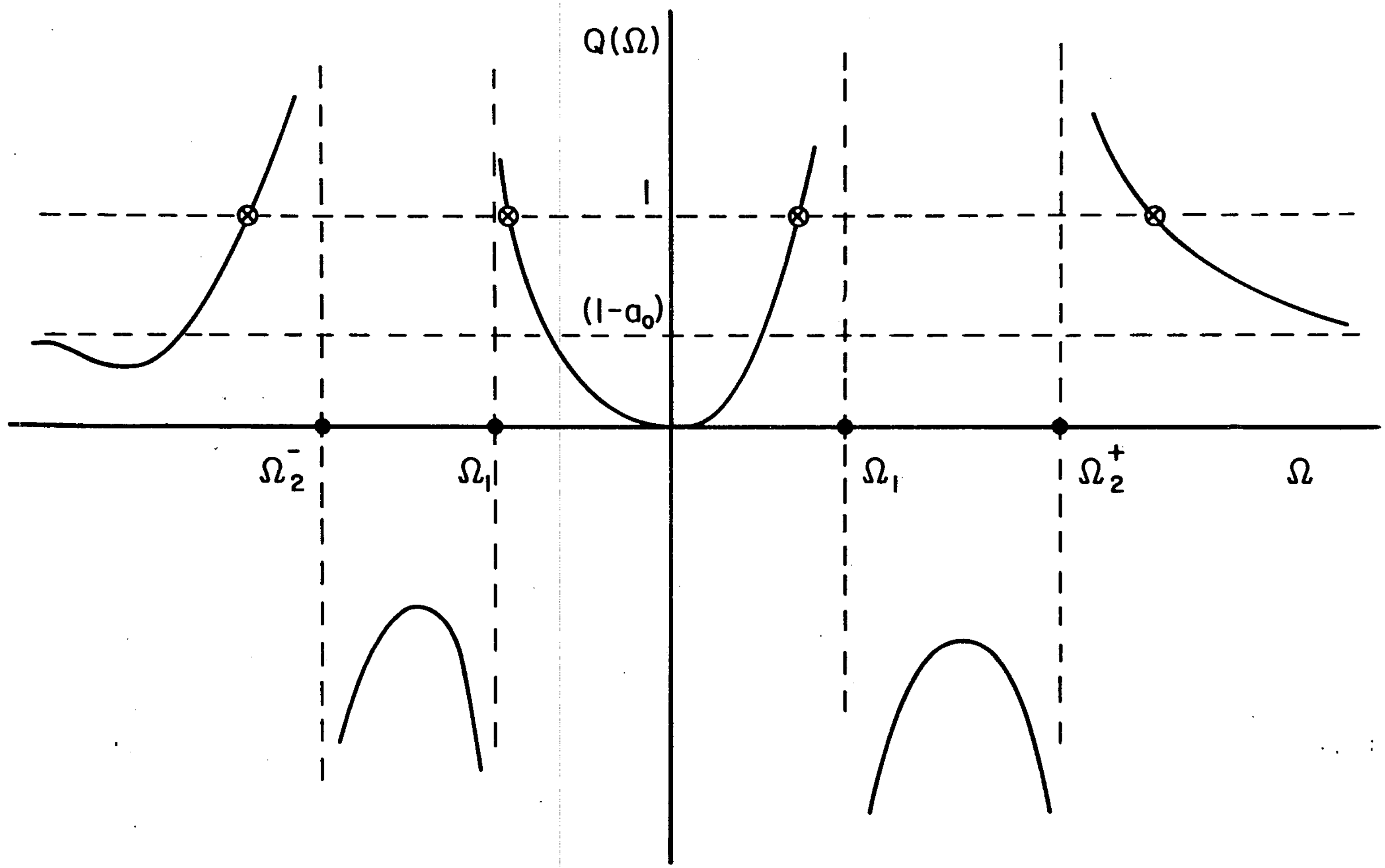

Fig. 1