IFS #219

Destabilization of Magnetohydrodynamic Modes with Finite Larmor Radius Effects in Tandem Mirrors

M. Kotschenreuther and H.L. Berk

Institute for Fusion Studies

The University of Texas at Austin

Austin, Texas 78712-1060

Abstract

Difficulties in the stabilization of ideal magnetohydrodynamic (MHD) ballooning modes by finite Larmor radius (FLR) effects are considered, in tandem mirror geometry for azimuthal mode numbers $\ell > 1$. A kinetic formalism is used to obtain corrections to the long thin approximation, when keeping terms of quadratic order in the curvature. If $\eta_i = \partial \ell n T_i / \partial \ell n n_i \geq 0$, with T_i, n_i, the ion temperature and density, ion resonance effects eliminate absolute FLR stability, though the residual growth rates are substantially reduced from the MHD values. However, the residual modes are still important, and mixing length estimates of the confinement degradation from modes with $\ell > 1$ indicate they can still severely limit the achievement of reactor-grade operation near and above the threshold beta value predicted from the ideal MHD theory. This is most severe if the ion temperature decreases radially ($\eta_i > 0$), whereupon significant instabilities can even arise below the ideal threshold. However, if $-2/3 < \eta < 0$, the lowest order FLR theory suffices to produce stability.

I. Introduction

In tandem mirror reactor designs emphasis has been placed on finite Larmor radius (FLR) stabilization to allow operation at values of β above the ideal magnetohydrodynamic (MHD) instability threshold.[1,2,3] Such stability follows from the kinetic MHD equations in the long thin approximation,[4] where the magnetic field line curvature is taken to be weak; specifically, terms proportional to curvature squared are neglected. Here we include corrections from these terms, and find that the stability picture is greatly changed. In particular, modes which are stabilized by FLR in the long thin limit are unstable, though with reduced growth rates. This destabilization arises from parallel ion resonance effects qualitatively similar to those found by Freidberg and Seylor[5] in a pinch model, and Kotschenreuther[6] in tokamaks.

The instability problems are enhanced with choke coils in the plasma as the curvature drive is increased. One cannot use a flute mode to estimate FLR effects. As a result, above the ideal MHD threshold beta value, the lowest order FLR stability criterion in the long thin approximation, even <u>without</u> curvature square terms, may not be strongly satisfied as the ballooning nature of the instability localizes the mode so that it does not sample the favorable FLR influence of the full actual cell.

When lowest FLR stability is achieved, the residual instabilities from curvature squared effects are still significant. As a rough estimate for the potency of these instabilities we use mixing length arguments for the turbulent diffusion coefficient D (i.e., $D \sim \gamma/k_\perp^2 \sim \gamma r_p^2/\ell^2$, where ℓ is the mode number and r_p the plasma radius, and γ the growth rate, which gives a plasma lifetime $\tau_{\rm pl}$ that is roughly $\tau_{\rm pl} \approx \ell^2/\gamma$). In order to satisfy the Lawson criterion ($n\tau_{\rm pl} > 10^{14}\,\sec/\mathrm{cm}^3$) we find that it is necessary to have the axial length of the mirror region a substantial fraction of the central cell length, which sets limitations on the desire in reactor designs to maximize the ratio length of the central cell to mirror regions.

The analysis given here is for the eikonal approximation; we note that a non-eikonal analysis is also possible for a steep pressure profile for low ℓ-numbers, and for

Gaussian shaped pressure profiles as given in Refs. (7,8). For the sake of brevity, the sharp boundary calculation is not presented here. We also neglect the effects of radial electric fields.

The structure of the paper is as follows. In Sec. II we derive the dispersion relation. In Sec. III we analyze the dispersion relation for instability. In Sec. IV we discuss the implications of the instabilities to tandem mirror design.

II. Derivation of Dispersion Relation

The equations governing the eigenmode system satisfy a variational quadratic form that have been derived in Ref. (6,7). The equilibrium magnetic field is given by $\mathbf{B} = \boldsymbol{\nabla}\alpha \times \boldsymbol{\nabla}\theta = B\mathbf{b}$ and all perturbed field amplitudes are proportional to $f(s)\exp[iS]$, where $\boldsymbol{\nabla} S = \frac{\partial S}{\partial \alpha}\boldsymbol{\nabla}\alpha + \frac{\partial S}{\partial \theta}\nabla\theta = \mathbf{k}_\perp$, with $\ell \equiv \frac{\partial S}{\partial \theta}$ and $\lambda = \frac{\partial S}{\partial \alpha}$ independent of the field line s. If we assume $E_\| = 0$, the usual limit for MHD equations, the quadratic form for the field amplitudes ϕ and Q_L in a system without electric field flows is of the form

$$\mathcal{L} = \mathcal{L}_{\text{local}} + \mathcal{L}_{\text{kinetic}} = 0 \tag{1}$$

with

$$\begin{aligned}\mathcal{L}_{\text{local}} = \int_{-\infty}^{\infty} \frac{ds}{B} \Bigg\{ & \sigma k_\perp^2 \left(\frac{\partial \phi}{\partial s}\right)^2 + \tau \left[\ell Q_L - \frac{\sigma}{\tau} B\phi(\ell\kappa_\alpha - \lambda\kappa_\theta)\right]^2 \\ & - \phi^2 \frac{\sigma \ell}{\tau}(\ell\kappa_\alpha - \lambda\kappa_\theta)\left[\frac{DP_\perp(\alpha, B, \Phi)}{D\alpha} + \frac{DP_\|(\alpha, B, \Phi)}{D\alpha}\right] \\ & - \frac{m_i k_\perp^2 \phi^2}{B^2}\left[n_i\omega^2 - \omega\frac{\ell}{q_i B}\frac{DP_{\perp i}}{D\alpha} + \frac{\ell^2}{2q_j^2}\frac{1}{B}\frac{\partial B}{\partial \alpha}\frac{DL_i}{D\alpha}\right]\Bigg\}\end{aligned} \tag{2}$$

$$\begin{aligned}\mathcal{L}_{\text{kinetic}} = & -\sum_j \frac{4\pi}{m_j^2}\int d\varepsilon d\mu \left(\omega\frac{\partial F_j}{\partial \varepsilon} + \frac{\ell c}{q_j}\frac{\partial F_j}{\partial \alpha}\right) \\ & \int_{t^-}^{t^+} dt \int_{t^-}^{t^+} dt' K(t,t')\left[\ell\mu Q_L(t') + m_j v_\|^2(t')\phi(t')(\ell\kappa_\alpha(t') - \lambda\kappa_\theta(t'))\right] \times \\ & \left[\ell\mu Q_L(t) + m_j v_\|^2(t)\phi(t)(\ell\kappa_\alpha(t) - \lambda\kappa_\theta(t))\right]\end{aligned} \tag{3}$$

where

$$\sigma = 1 + (P_\perp - P_\parallel)/B^2$$

$$\tau = 1 + \frac{1}{B}\frac{\partial P_\perp}{\partial B}$$

$$\varepsilon = \frac{1}{2}m_j v^2 + q_j\Phi \equiv \text{particle energy}$$

$$\mu = \frac{1}{2}m_j v_\perp^2/B \equiv \text{magnetic moment}$$

$$\Phi \equiv \text{equilibrium electron potential}$$

$$\mathbf{k} = (\mathbf{b}\cdot\nabla)\mathbf{b} = \kappa_\alpha\nabla\alpha + \kappa_\theta\nabla\theta$$

$$v_\parallel^2 = 2\left[\varepsilon - \mu B - q_j\Phi_j\right]$$

$$\left\{\begin{matrix} P_{\perp j} \\ P_{\parallel j} \\ L_j \end{matrix}\right\} = 4\pi\int \frac{d\varepsilon d\mu F_j}{|v_\parallel|}\left\{\begin{matrix} \mu B \\ m_j v_\parallel^2 \\ \mu^2 B^2/m_j \end{matrix}\right\}$$

$$P_{\perp,\parallel} = \sum_j P_{\perp,\parallel j}$$

$$\frac{D}{D\alpha} \equiv \frac{\partial}{\partial\alpha} + \frac{\partial B}{\partial\alpha}\frac{\partial}{\partial B}$$

$$K_j(t,t') = \sin\left[\Omega_j(t^+,t^<)\right]\cos\left[\Omega_j(t^+,t^>)\right] + \cos\left[\Omega_j(t',t^+)\right]\cos\left[\Omega_j(t,t^+)\right]\cot\left[\Omega_j(t^+,t^-)\right]$$

$$\Omega_j(t',t) = \int_{t'}^{t}(\omega - \omega_{d_j}(t''))dt''$$

$$t^> = \max(t,t')$$

$$t^< = \min(t,t')$$

$$v_\parallel(t^+) = v_\parallel(t^-) = 0$$

$$\omega_{d_j} = \frac{\mu\mathbf{k}\times\mathbf{b}\cdot\nabla B}{m_j\omega_{cj}} + \frac{v_\parallel^2}{\omega_{cj}}\mathbf{k}\times\mathbf{b}\cdot\boldsymbol{\kappa}$$

$$\omega_{cj} = q_j B/m_j \equiv \text{ion cyclotron frequency}$$

Here, ϕ is the electrostatic potential and Q_L is the compressional magnetic perturbation. We also note the relation

$$\frac{\sigma}{\tau}\frac{\partial P_\perp}{\partial\alpha} + \frac{\partial P_\parallel}{\partial\alpha} = \frac{D(P_\perp + P_\parallel)}{D\alpha}(1 + O(\kappa r)) \tag{4}$$

In order to minimize the perturbed magnetic energy we choose

$$Q_L = \frac{\sigma}{\ell_T} B\phi(\ell\kappa_\alpha - \lambda\kappa_\theta)[1 + \mathcal{O}(\beta)]. \tag{5}$$

We now proceed by treating $\mathcal{L}_{\text{kin}}$ as a perturbation to $\mathcal{L}_{\text{local}}$. This is valid if $\kappa r_p \ll 1$ where r_p is the plasma radius, as one can estimate that $\mathcal{L}_{\text{loc}} \sim \beta\kappa r_p$, while $\mathcal{L}_{\text{kin}} \sim \beta(\kappa r_p)^2$.

To proceed further, we consider the following model for a tandem mirror system shown in Fig. 1. The system is taken as a symmetric central cell with unfavorable curvature in the region labelled L_k. The ions are electrostatically trapped by the ambipolar potential Φ_a, and where distribution is approximated by a Maxwellian with a temperature gradient,

$$F_i = \frac{n(\alpha) m_i^{3/2}}{(2\pi T(\alpha))^{3/2}} \exp\left[-\frac{\varepsilon}{T(\alpha)}\right]. \tag{6}$$

The anchor region is taken as an MHD stable region with large pressure weighting to give strong stabilization from this region. We also assume that the transition region between the anchor and the central cell is longer than the destabilizing region of the central cell (as it would be with a high field choke coil present as part of the central cell).

In order to excite an MHD instability we assume the eigenfunction, ϕ, is small in the anchor region so as to avoid weighting with good curvature, and ϕ is nearly constant in the region designated as L_k where there is an unfavorable contribution to the curvature. The line bending between the anchor and the central cell allows the system to be stable at low beta.

First, consider the contribution to $\mathcal{L}$ from the line bending and curvature drive terms to the right of $s = L_c$. We label this δW and it is given by,

$$\delta W = -2\ell^2\phi^2(L_c)\int_0^{L_{\max}} \frac{ds}{B}\kappa_\alpha \frac{DP}{D\alpha} + \int_{L_c}^{\infty} \frac{ds}{B} k_\perp^2 \left(\frac{\partial\phi}{\partial s}\right)^2 \tag{7}$$

where we have assumed pressure isotropy of ion and electrons in the central cell, we have used Eq. (5), and taken $\kappa_\theta = 0$ as the curvature drive comes primarily from the symmetric central cell where $\kappa_\theta = 0$. In the long thin approximation we have,

$$\kappa_\alpha = \frac{d^2}{B^{1/2} ds^2}\left(\frac{1}{B^{1/2}}\right) \quad , \quad Br^2 = B_0 r_0^2 \tag{8}$$

where $B_0 \equiv$ magnetic field at midplane, and $r_0 \equiv$ radius at midplane. With these relations we can rewrite Eq. (7) as

$$\delta W = \frac{-k_\perp^2 r^2 \phi^2(L_c)}{r_0^2 B_0 L_k} C_D(\beta - \beta_c) \tag{9}$$

where β is the central cell beta, β_c the critical beta for ideal MHD instability, r_0 the central cell minor radius, and C_D is a constant of order unity

$$C_D = \frac{2 r_0 B_0 L_k \ell^2}{\beta k_\perp^2 r^2} \int \frac{ds}{B} \kappa_\alpha \frac{Dp}{D\alpha} \tag{10}$$

and

$$\beta_c = \frac{\beta \int_{L_c}^{\infty} \frac{ds}{B} k_\perp^2 \left(\frac{\partial \phi_c}{\partial s}\right)^2}{2\ell^2 \phi^2(L_c) \int_0^{\infty} \frac{ds}{B} \kappa_\alpha \frac{Dp}{D\alpha}}, \tag{11}$$

where ϕ_c is the eigenfunction at marginal stability. We have assumed that in the anchor region $\phi_c \approx 0$, which follows if the anchor region is strongly stabilizing and there is a long length in the transition region so that ϕ_c can be small. Roughly, the critical beta will be $\beta_c \approx L_k^2 / L_{\text{tr}}^2$ where L_{tr} is the length in the transition region.

If we model the magnetic field by

$$B = \begin{cases} B_0 & |s| < L_c \\ B_0 \left(1 + \frac{(s - L_c)^2}{L_k^2}\right) & |x| > L_c \end{cases} \tag{12}$$

then

$$C_D = \frac{\pi}{16} \frac{r_0}{\beta} \frac{\ell^2}{k_\perp^2 r^2} \frac{\partial \beta}{\partial r_0}. \tag{13}$$

In the central cell region $|s| < L_c$, ϕ is determined from the variation of $\mathcal{L}$ with respect to ϕ in Eq. (1), which produces the Euler-Lagrange equation,

$$\frac{\partial}{\partial s} \frac{k_\perp^2}{B} \frac{\partial \phi}{\partial s} + \frac{k_\perp^2 \left[\omega(\omega - \omega_{*i}) + \frac{\beta}{2}\omega_{*i}\left(\omega - \frac{1 + 2\eta_i}{1 + \eta_i}\omega_{*i}\right)\right]\phi}{v_A^2 B} = 0 \tag{14}$$

where

$$\omega_i^* = \frac{\ell}{q_i n_i}\frac{DP_i}{D\alpha} \quad , \quad v_A^2 = B_0^2/n_i m_i$$

$$\beta = \frac{2(P_i + P_e)}{B_0^2} = \frac{2n_i(T_i + T_e)}{B_0^2} \quad , \quad \eta_i = \frac{d\ln\ T_i}{d\ln\ n_i}.$$

The solution of this equation for even or odd modes is

$$\begin{aligned} \phi &= \phi_0 \cosh(qs) \qquad \text{(even mode)} \\ \phi &= \phi_0 \sinh(qs) \qquad \text{(odd mode)} \end{aligned} \tag{15}$$

where

$$q^2 = -\left[\omega(\omega - \omega_{*i}) + \frac{\beta}{2}\omega_{*i}\left(\omega - \frac{1+2\eta_i}{1+\eta_i}\omega_{*i}\right)\right] \Big/ v_A^2. \tag{16}$$

At the interface $s = L_c$ we have,

$$\frac{1}{\phi(L_c)}\frac{\partial\phi(L_c)}{\partial s} = q\ \mathrm{trgh}(qL_c), \tag{17}$$

where

$$\mathrm{trgh}(x) = \begin{cases} \tanh(x); & \text{even mode} \\ \coth(x); & \text{odd mode}\ . \end{cases}$$

We now evaluate the quadratic form of Eq. (1) over the entire flux tube. In the region between $0 < |s| < L_c$, we integrate by parts and note that only end-point terms contribute when one uses the relation arising from the Euler-Lagrange equation. The endpoint contribution is given by $k_\perp^2 \phi(L_c)\dfrac{d\phi(L_c)}{ds}\Big/ B(L_c)$, and we use Eq. (17) to express $d\phi(L_c)/ds$ in terms of $\phi(L_c)$. For the contribution for $s > L_c$, we first neglect contributions of the inertia and FLR terms, which we assume is small relative to similar contributions in the central cell. Then we find

$$k_\perp^2 \frac{\phi^2(L_c)}{B_0 L_c}\left[\Omega\mathrm{trgh}\Omega - \frac{L_c}{L_k}C_D(\beta - \beta_c)\right] = -\frac{\mathcal{L}_{\mathrm{kin}}}{2} \tag{18}$$

where

$$\Omega^2 = \frac{-L_c^2}{v_A^2}\left[\omega(\omega - \omega_{*i}) + \frac{\beta}{2}\omega_{*i}\left(\omega - \omega_{*i}\frac{1+2\eta_i}{1+\eta_i}\right)\right]. \tag{19}$$

If one neglects $\mathcal{L}_{\mathrm{kin}}$ in Eq. (18), one obtains the usual result of the long-thin approximation, specifically, sufficiently large finite Larmor radius effects can stabilize MHD modes. Then the left-hand side of Eq. (16) is real; specifically, both ω and the eigenfunction ϕ are real. Corrections to this dispersion relation which can alter stability arise only from the imaginary part of $\mathcal{L}_{\mathrm{kin}}$, which is due to resonance effects.

To compute $\mathcal{L}_{\mathrm{kin}}$, we take $\Omega_e(t^-, t^+) \ll 1$, while for ions $\Omega_i(t^-, t^+) \gg 1$. Hence, electrons do not produce a significant resonant contribution and the dissipative contribution arises primarily from the ions. We also take the ion transit time through the region L_k to short compared to ω^{-1}. Thus we have assumed

$$\frac{\omega L_k}{v_{thi}} \ll 1 \ll \frac{\omega L_c}{v_{thi}}. \tag{20}$$

with $v_{thi} = (T_i/m_i)^{1/2}$. These assumptions can be verified *a-posteriori* if $\beta \ll 1$. Also recall that the eigenfunction is assumed nearly constant in the region L_k. Finally, we take $E_{\|} = 0$, an assumption that can be verified using that $\omega L_k / v_{thi} \ll 1$. Note some additional ion acoustic effects found in tokamak calculations,[6] do not arise in our model as $E_{\|} = 0$. One can also show that if one assumes

$$\frac{\omega L_c}{v_{the}} \equiv \frac{\omega L_c}{v_{thi}} \left(\frac{T_i}{T_e}\frac{m_e}{m_i}\right)^{1/2} \ll 1$$

that the ratio of the electron to ion dissipative terms is roughly,

$$\frac{\dfrac{\omega^2 L_c^2}{v_{thi}^2}\left(\dfrac{T_i}{T_e}\dfrac{m_e}{m_i}\right)^{1/2}}{\left(1 + \dfrac{\omega^2 L_k^2}{v_{thi}^2}\right)\left(1 - \dfrac{\omega_{*i}}{\omega}\right)}.$$

Thus, unless one is dealing with a high enough frequency wave, or if the frequency is close to ω_{*i}, the electron damping is small compared with the ion damping term, and thus it will be neglected in this analysis.

Now we take ω to be mainly real. When the ion transit time of the central cell is long compared to an oscillation time, the resonances of the equation,

$$\operatorname{Im} \cot \Omega(t^+, t^-) = -\pi \sum_n \delta\left(n\pi - \Omega(t^-, t^+)\right) \tag{21}$$

are closely spaced. Then we can sum over n, as in Ref. 9, and show that the summation can be taken as unity. Also note that as $\omega L_k / v_{thi} \ll 1$, the cosine factors in $K(t,t')$ can be taken as unity. With these approximations we can evaluate $\mathcal{L}_{\mathrm{kin}}$ and find that the dispersion relation can be written as,

$$\Omega \mathrm{trgh} \Omega - C_D(\beta - \beta_c)\frac{L_c}{L_k} = \frac{-B_0 L_c}{2k_\perp^2 \phi^2(L_c)}\mathcal{L}_{\mathrm{kin}}$$

$$= ig\beta_i^{1/2}\left(\frac{r_0}{L_k}\right)^2 \frac{\left[\omega - \omega_{*i}\left(1 + \frac{\eta_i}{2(1+\eta_i)}\right)\right] L_c}{v_A} \tag{22}$$

where g is a geometrical factor of order unity,

$$g \equiv \frac{2}{\sqrt{\pi}} L_k^2 \int_0^1 d\lambda \left\{ \int_0^{B_0/\lambda} dB \frac{ds}{dB} \frac{\left(2 - \lambda \frac{B}{B_0}\right)}{\sqrt{1 - \lambda B/B_0}} \left(\frac{B_0}{B}\right)^{1/2} \left[\frac{d^2}{ds^2}\left(\frac{B_0}{B}\right)^{1/2}\right] \right\}^2 .$$

For the model where the magnetic field in the region L_k is given by $B = B_0\left(1 + (s - L_c)^2/L_k^2\right)$,

$$g = \frac{2}{\sqrt{\pi}} \frac{3\pi^2}{40} \simeq .83.$$

Note that the kinetic term is small by a factor of roughly $\beta^{1/2}\left(\frac{r_0}{L_k}\right)^2 \sim \beta^{1/2} r\kappa$, so its treatment as a perturbation is justified.

III. Analysis of Dispersion Relation

We examine solutions to Eq. (20) in two limits 1) significantly unstable MHD modes, so that $\beta - \beta_c \gg L_k/L_c$ where β_c is the critical beta for ideal MHD stability, 2) MHD stable modes with $\beta_c - \beta \ll L_k/L_c$. The case $|\beta - \beta_c| \lesssim L_k/L_c$, gives the transition of the two regimes, but will not be discussed in detail. A detailed calculation would show that for $|\beta - \beta_c| \lesssim L_k/L_c$, the mode is flute-like and always unstable if $\eta_i > 0$, when $\beta > \beta_c$, and even unstable for $\beta < \beta_c$ unless η_i is very small or negative.

CASE 1: $\beta - \beta_c \gg L_k/L_c$

For this limit trgh$\Omega \approx 1$. The solution to Eq. (18) can be written as $\omega = \omega_0 + \delta\omega$ with ω_0 the lowest order solution obtained by neglecting $\mathcal{L}_{\text{kin}}$. Then $\omega_0 = \nu\omega_{*i}$, with

$$\nu = \frac{1}{2}(1-\beta/2) \pm \frac{1}{2}\left\{(1-\beta/2)^2 + 2\beta\left(1+\frac{\eta_i}{1+\eta_i}\right) - \frac{4\gamma_{\text{MHD}}^2}{\omega_{*i}^2}\right\}^{1/2} \tag{23}$$

where $\gamma_{\text{MHD}} = C_D \dfrac{v_A}{L_k}(\beta - \beta_c)$ is the growth rate without FLR effects, i.e. when $\omega_{*i} = 0$. Equation (23) gives the long thin limit FLR stabilization result, plus additional finite β corrections. For such an FLR stabilized mode ν is real and there are two roots with one greater and one less than $(1-\beta/2)/2$.

To next order we include $\mathcal{L}_{\text{kin}}$ and find,

$$\delta\omega\frac{\partial\Omega}{\partial\omega} = ig\beta_i^{1/2}\left(\frac{r_0}{L_k}\right)^2\left\{\omega_0 - \omega_{*i}\left[1+\frac{\eta_i}{2(1+\eta_i)}\right]\right\}\frac{L_c}{v_A}. \tag{24}$$

For the root with $\nu > 1/2(1-\beta/2)$,

$$\frac{\delta\omega}{\gamma_{\text{MHD}}} = 2ig\beta_i^{1/2}\left(\frac{r_0}{L_k}\right)^2\left[\frac{1}{2\nu - 1 + \beta/2}\right]\left[1+\frac{\eta_i}{2(1+\eta_i)} - \nu\right]. \tag{25}$$

For $\eta_i \geq 0$, which is the usual case, it follows that Eq. (25) is always unstable if $\beta < 2$. For a mode which is strongly FLR stabilized to lowest order, i.e., with $\omega_{*i}^2 \gg \gamma_{\text{MHD}}^2$, Eq. (25) simplifies to

$$\frac{\delta\omega}{\gamma_{\rm MHD}} = \frac{2ig\beta_i^{1/2}\left(\frac{r_0}{L_k}\right)^2}{1+\beta/2}\frac{\gamma_{\rm MHD}^2}{\omega_{*i}^2}, \quad |\eta_i| \ll \gamma_{\rm MHD}^2/\omega_{*i}^2$$
$$\frac{\delta\omega}{\gamma_{\rm MHD}} = \frac{ig\beta_i^{1/2}\left(\frac{r_0}{L_k}\right)^2\eta_i}{(1+\eta_i)}, \quad |\eta_i| \gg \gamma_{\rm MHD}^2/\omega_{*i}^2 \tag{26}$$

where we have taken $\beta \ll 1$.

CASE 2:

Now let us examine the case $C_D(\beta_c-\beta) \ll L_k/L_c$. The lowest order solutions corresponds to a stable Alfvén standing wave in the central cell with FLR corrections. Note that it can be shown that if $\beta_A^{1/2}L_A/L_c \ll 1$, with β_A the beta value in the anchor region, and L_A the length of the anchor region, then the eigenfunction is small in the anchor due to strong favorable pressure. Hence Eq. (22) is still valid. To lowest order Ω is pure imaginary and taking Re$\omega > 0$ for definiteness, we get

$$C_D(\beta-\beta_c) = \Omega\,{\rm trgh}\,\Omega = \begin{cases} -W\tan W & \text{even modes} \\ W\cot W & \text{odd modes} \end{cases} \tag{27}$$

where $W=\sqrt{-\Omega^2}$. Consider even modes for definiteness. Since $C_DL_c(\beta_c-\beta)/L_k \gg 1$, to lowest order $W=(n+1/2)\pi$ for some positive integer n. The next order correction gives

$$W_0 = \pi(n+1/2)\left[1+\frac{L_k}{C_DL_c(\beta_c-\beta)}\right]$$
$$\omega_0 = \frac{\omega_{*i}}{2}+\frac{1}{2}\left[\omega_{*i}^2+\frac{4(n+1/2)^2}{L_c^2}\pi^2v_A^2\left(1+\frac{L_k}{C_DL_c(\beta_c-\beta)}\right)^2\right]^{1/2}+\mathcal{O}(\beta) \tag{28}$$

where $n=0,1,2,\ldots$. The $\mathcal{L}_{\rm kin}$ term gives an additional frequency correction

$$\delta\omega = \frac{ig\beta_i^{1/2}\frac{L_c}{v_A}\left[\omega_{*i}\left(1+\frac{\eta_i}{2(1+\eta_i)}\right)-\omega_0\right]}{\pi(n+1/2)\left(\frac{\partial W}{\partial\omega}\right)_0\left[\frac{L_c}{r_0}C_D(\beta-\beta_c)\right]^2} \tag{29}$$

where

$$\left(\frac{\partial W}{\partial \omega}\right)_0 = \frac{L_c^2}{v_A^2}\left[\omega_{*i}^2 + \frac{4(n+1/2)}{L^2}^2 \pi^2 v_A^2\right]^{1/2} \tag{30}$$

Substituting ω_0 in Eq. (28) into Eq. (29), gives instability if

$$(n+1/2)\frac{\pi v_A}{\omega_{*i} L_c} < \frac{\eta_i^{1/2}}{1+\eta_i}(1+2\eta_i)^{1/2} \tag{31}$$

. Now to achieve FLR stability $\ell \gtrsim 2$ at beta values substantially above marginal stability, we need $\omega_{*i} \gtrsim 2C_D v_A \beta_c / L_k \approx 2\gamma_{\mathrm{MHD}}$. With this criterion we observe that the η_i mode gives instability even for $\beta < \beta_c$ if,

$$\frac{L_k}{L_c}(2n+1) \lesssim \frac{\ell \eta_i^{1/2}}{8(1+\eta_i)}\beta_c(2+3\eta_i)^{1/2} \tag{32}$$

where we have used Eq. (13) for C_D.

If Eq. (32) is strongly satisfied, the growth rate for the η_i mode is,

$$\delta\omega = \frac{i g v_{thi} \eta_i r^2}{\sqrt{2}\pi(n+1/2)L_c^3(1+\eta_i)C_D^2(\beta-\beta_c)^2} \tag{33}$$

where $v_{thi} = (T_i/m_i)^{1/2}$. If $C_D(\beta_c - \beta) \lesssim L_k/L_c$, Eq. (33) and (26) fail. In this regime, both Eq. (32) (for $n = 0$) and Eq. (26) predict comparable growth rates, given by

$$\delta\omega \approx \frac{i g v_{thi} \eta_i r^2}{(1+\eta_i)L_k^2 L_c}. \tag{34}$$

Thus for β near β_c, we have instability for all ℓ if $\eta_i > 0$, including low ℓ.

If $\eta_i < 0$, Eq. (33) is stable. However, Eq. (33) was obtained by choosing the plus sign in the square root of Eq. (28). If the minus sign is chosen instead, instability can arise if $\eta_i < -2/3$.

IV. Discussion

First, let us consider the implications of the lowest order dispersion relation given in Eq. (22), when there is a choke coil. The lowest order FLR stabilization condition with $T_e = T_i$ needs to satisfy,

$$\frac{4\gamma_{\mathrm{MHD}}^2}{\omega_{*i}^2} = \frac{16(\beta-\beta_c)^2 r^4 C_D^2}{\ell^2 \beta L_k^2 a_H^2} < 1$$

where a_H is the mean ion Larmor radius of a *D-T* plasma, and $C_D \approx .2$. This condition appears to be satisfied in the reactor designs found in Ref. (3) where typical $\dfrac{L_k}{r} \simeq 8$ parameters are, $B_0 = 47\mathrm{kg}$, $r = 49\mathrm{cm}$, $T_i = T_c = 28$ keV, $\beta = .3$,and $L_c/r \approx 260$ (in the choke coil). For these parameters we find $a_H = .57\mathrm{cm}$, $n_c \equiv$ electron density $\equiv$ 3.3×10^{14} cm^{-3}. The critical beta for stability β_c, is reported in Ref. 3 as $\beta_c = .3$ for the ℓ=1 "rigid" mode, but not for $\ell \to \infty$ modes. The ratio of the line bending term to the curvature term determines β_c. For m=1 the line bending term has roughly equal contributions from radial and transverse gradients, whereas for a high ℓ mode the former is absent. For an ℓ=2 mode the β_c is slightly larger, so we take β_c=.2. The stability parameter for the lowest order FLR theory is then,

$$\frac{4\gamma_{\mathrm{MHD}}^2}{\omega_{*i}^2} = \frac{21(1-\beta_c/\beta)^2}{\ell^2}.$$

Thus, this "typical" design does satisfy the lowest order FLR stability condition for ℓ=2, though not strongly. The reason is that the FLR stabilization mechanism is not as effective for a ballooning mode that localizes near the choke coil, as compared to a flute mode where the finite Larmor radius stabilization term can be weighted over the entire machine. Residual instability from $\mathcal{L}_{\mathrm{kin}}$ will occur if $\eta_i \geq 0$. However, if the plasma containment time fulfills the Lawson criterion, one can tolerate residual instability. We use mixing arguments to estimate the transport lifetime, $\tau_{p\ell}$, as

$$\tau_{p\ell} = \frac{A\ell^2}{\gamma}$$

with the growth rate γ given by Eq. (26), and we choose $\ell = 2$ and A an empirical constant that follows from the nonlinear behavior of the mode (A is $\mathcal{O}(1)$). The growth rate from Eq. (26) is found to be,

$$\gamma = \max\left[7\times 10^{-2}\frac{v_{thi}}{r_0}\left(\frac{r_0}{L_k}\right)^5 \frac{r_0^2}{a_H^2}\frac{(\beta-\beta_c)^3}{\ell^2\beta} \quad , \quad \frac{2v_{thi}}{r}\frac{\eta_i(\beta-\beta_c)}{1+\eta_i}\left(\frac{r^3}{L_k^3}\right)\right] .$$

For a machine that will satisfy $n_e\tau_{p\ell} > 10^{14}\,\mathrm{sec}/\mathrm{cm}^3$, we require for the nominal beta, magnetic field and temperatures selected above, that the choke coil be sufficiently tapered to satisfy

$$\frac{L_k}{r} > \max\left[\frac{10}{A^{1/5}} \quad , \quad \frac{30\eta_i^{1/3}}{(1+\eta_i)^{1/3}A^{1/3}}\right] .$$

Note that A enters with a small fractional power, so that these estimates are not sensitive to departures from the admittedly very crude estimate of the turbulent diffusion. With η_i=0, these instabilities might be significant in the present Mars design, but modest design changes could render them tolerable. However, with $\eta_i \sim$1, they require a much longer mirror throat.

We have also observed that even at or below marginal stability, one may obtain substantial growth rates if $\eta_i > 0$ and Eq. (31) is satisfied. If we use Eq. (33) as an estimate of the growth rate near ideal MHD marginal stability, we observe that the criterion to fulfill $n\tau_{p\ell} > 10^{14}$ at $\beta = \beta_{\rm cr} = .2$ with the other parameters the same as above, is

$$\frac{L_k}{r} > 22\left[\frac{A\eta_i}{1+\eta_i}\right]^{1/2} .$$

Even this condition appears restrictive.

In principle, mirror machines can operate with η_i negative by allowing the axial loss rate to be faster than the radial diffusion rate, and by heating the edge moderately to control the cooling from incoming gas. In such a case it appears from Eq. (26) that stable operation in accordance to the lowest order finite Larmor radius theory is possible. However, one should maintain $-2/3 < \eta_i < 0$ to prevent additional temperature gradient modes.

The above estimates are based on rough models on the shape of the magnetic field in a tandem mirror. The actual curvature has a more complicated structure which will lead to quantitative departures from the analytical model. These can be readily evaluated numerically using Eq. (22). It is also easy to numerically consider the spatial structure of the lowest order eigenmode in the curvature region, since this can be found with a simple MHD code.

It is possible that with proper design, the above estimates can be considerably improved. Nonetheless, it is clear that higher order terms in the curvature expansion should be considered in optimizing a tandem mirror design. The restrictions seem particularly severe with positive ion temperature gradients and systems with choke coils that provide strong and spatially localized MHD drives. The stability result of lowest order FLR theory is correct if negative ion temperature gradients can be produced. However, one should maintain $-2/3 < \eta_i < 0$ to prevent additional temperature gradient modes. If a tandem mirror can operate with predominantly axial loss, such a temperature profile may be possible to establish, and should lead to the best configuration as far as stability consideration is concerned.

References

1. T.B. Kaiser, W.M. Nevins and L.D. Pearlstein, Phys. Fluids **26**, 351 (1983).
2. W.M. Nevins, L.D. Pearlstein, in "Physics Basis for an Axial Design for the Eng. Phys. of MFTFB," edited by D.E. Baldwin and B.G. Logan, LLNL Report# UCID-19359 (1982), Appendix B.
3. "Mirror Advanced Reactor Study", UCRL-53480, Lawrence Livermore National Laboratory, July 1984.
4. W.A. Newcomb, J. Plasma Phys. **26**, 529 (1981).
5. C.E. Seylor and J.P. Freidberg, Phys. Fluids **23**, 33 (1980).
6. M. Kotschenreuther, Institute for Fusion Studies Report, The University of Texas at Austin (1985), submitted to Phys. Fluids.
7. H.L. Berk and B.G. Lane, Phys. Fluids **29**, 1076 (1986).
8. H.L. Berk and B.G. Lane, (submitted to Phys. Fluids) Plasma Fusion Center Report, PFC/JA-869 Fes. 1986.
9. H.L. Berk and L.D. Pearlstein, Phys. Fluids **14**, 1910 (1971).

Figure Caption

Fig. 1 —

Model for the axial magnetic field in a tandem mirror configuration L_c is the length of the uniform central field region. L_k is the axial length of the unfavorable curvature region. In configurations with choke coils $B_M/B_0 \gg 1$. It is further assumed that $L_k \ll L_{tr} \ll L_c$.