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LOCAL EFFECT OF EQUILIBRIUM CURRENT ON TEARING MODE STABILITY

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The local effect of the equilibrium current on the linear stability of low poloidal number tearing modes in tokamaks is investigated analytically. The plasma response inside the tearing layer is derived from fluid theory and the local equilibrium current is shown to couple to the mode dynamics through its gradient, which is proportional to the local electron temperature gradient under the approximations used in the analysis.

The relevant eigenmode equations, expressing Ampére's law and the plasma quasineutrality condition, respectively, are suitably combined in a single integral equation, from which a variational principle is formulated to derive the mode dispersion relations for several cases of interest.

The local equilibrium current is treated as a small perturbation of the known results for the $m \geq 2$ and the $m = 1$ tearing modes in the collisional regime, and the $m \geq 2$ tearing mode in the semi-

collisional regime; its effect is found to enhance stabilization for the $m \geq 2$ drift-tearing mode in the collisional regime, whereas the $m = 1$ growth rate is very slightly increased and the stabilizing effect of the parallel thermal conduction on the $m \geq 2$ mode in the semi-collisional regime is slightly reduced. The possibility of new modes, which do not exist in the absence of local current, is also discussed, but no relevant result is found in this case.

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During this long journey of ours,

it is a joy and an honor

to share the ride with Nilla.

LOCAL EFFECT OF EQUILIBRIUM CURRENT
ON TEARING MODE STABILITY

by

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CHAPTER I

INTRODUCTION

Laboratory plasma devices are generally designed with ideal magnetodydrodynamic (MHD) stability in mind, so it is not surprising that the next most rapid and dangerous class of modes, the resistive instabilities,[1,2] figure prominently in the description of the behaviour of the confined plasma achieved.

Confinement in an axisymmetric, toroidal device such as the tokamak critically depends upon the existence of nested flux surfaces, on each of which the plasma pressure is constant. One of the features responsible for the good confinement properties of the tokamak is the rather strong shear in its magnetic field; fast (MHD-type) radial motions of the plasma toward the walls of the vessel (the radial direction is taken along the minor radius of the torus for concentric flux surfaces) are prevented from the field lines bending, which arises because the direction of the sheared $\underset{\sim}{B}$ field, which, according to the "frozen field" condition of MHD, is dragged along with the fluid, changes rapidly moving along the minor radius.

Rapid loss of confinement can instead take place because a radial field perturbation can locally break and reconnect the field lines, leading to the destruction of the flux surfaces in a finite volume, inside which the surfaces of constant pressure are chaotically scrambled and confinement no longer exists.

The local annihilation of field lines, and the formation of magnetic islands, which alter the topology of the magnetic field configuration, leading to a lower magnetic energy state, cannot however occur except in very localized regions, where the infinite conductivity, ideal MHD description is no longer valid. In the vicinity of a so-called mode rational surface, where the component of the magnetic field along the mode wave vector $\underset{\sim}{k}$, $\underset{\sim}{k} \cdot \underset{\sim}{B}/(kB)$, passes through a null, non ideal effects, such as finite plasma resistivity, become important, and a narrow tearing layer is formed around the rational surface, where an induced electric field is produced which accelerates the electrons along the local magnetic field. The perturbed current which results is also localized and, along with the current in the outer region, outside the tearing layer, self consistently generates the teared magnetic field. In the exterior region, the field lines remain "frozen in" with the fluid and the magnetic perturbation is essentially similar to the ideal kink mode.

This feature is characteristic of tearing-type modes; although the non ideal dynamics is restricted to a narrow inner layer, the mode itself is not localized; rather, it connects to a global disturbance, with a kink-like structure, far from the rational surface.

In the present dissertation, physically quite different modes will be collectively defined as (low poloidal number) tearing modes when the radial magnetic perturbation $\tilde{B}_r$ is not localized, in the above sense, and when, most importantly, $\tilde{B}_r \neq 0$ at the particular mode rational surface of interest, so that tearing and reconnection do actually occur. This definition includes in particular, but it is not restricted to, the classical tearing mode discussed in the famous 1963 article by Furth, Killeen and Rosenbluth (FKR).[1]

A considerable part of the history of a typical tokamak discharge is known, or at least conjectured, to be related to tearing mode activity. In the initial stages of a discharge, the double tearing mode is thought[3] to play a role in producing rapid current penetration into the plasma. During the discharge, there occur sawtooth oscillations[4] and Mirnov oscillations,[5] the former being identified with internal disruptions,[6,7] and the observed anomalous electron transport is possibly due to small magnetic islands and resulting large radial transport along the perturbed magnetic field.[8] Finally, abrupt termination of a discharge, through what is referred to as a major disruption, is usually associated[9] with the sudden overlap of islands growing around initially distinct rational flux surfaces.

The relevance of tearing instabilities, however, is not limited to magnetic fusion research; the tearing mode, which efficiently transfers the energy stored in a sheared magnetic field into the kinetic and thermal energy of the plasma, is considered a likely trigger to the onset of violent, high energy phenomena such as the solar flares.[10,11]

The relevance of tearing to fusion physics can be traced to the already mentioned paper by Furth, Killeen and Rosenbluth, although the notion of magnetic field line reconnection had already been discussed in the astrophysical literature.[12] After FKR treated a number of resistive instabilities, in the limit of vanishing gyro-radius, Coppi[13,14] included finite gyroradius effects, and other corrections to Ohm's law, which become significant as the plasma temperature and electrical conductivity increases.

The success of the tokamak experimental program in controlling fast, MHD instabilities brought a strong revival of interest toward the resistive instabilities in the last decade, as a number of observed tokamak phenomana were associated, as remarked, with the predicted properties of tearing instabilities.

Among the many significant contributions to the understanding of the linear stability properties of tearing modes, we can single out the extension of the analysis to the long mean free path regime of some tokamak experiments,[15,16] the development of powerful variational techniques to treat quite complex models of plasma behavior,[17-19] the use of fluid equations, rather than kinetic theory, to derive the electron response inside the tearing layer in the collisional and semicollisional regime, respectively.[20-22] Reviews of the significance of tearing instabilities in general, and also in the unified contest of the low frequency electromagnetic disturbances pertinent to a toroidally confined plasma, have also appeared in the literature.[23,24]

The noted long interest in the linear stability properties of tearing modes is significant, since most of phenomena of relevance in experiments[7,9] are patently non linear. The physics of tearing-type instabilities critically depends upon the coupling of the perturbed parallel current $\tilde{J}_{\|}$ to the perturbed parallel electric field $\tilde{E}_{\|}$ inside the tearing layer, and such a coupling between $\tilde{J}_{\|}$ and $\tilde{E}_{\|}$ becomes particularly delicate at low frequency; when the mode frequency is comparable to the electron drift frequency, a number of higher order terms, in both the plasma equation of motion and the generalized Ohm's law describing the electron response inside the layer, become potentially important.

It is precisely because of this characteristic of the mode, sensitivity to small details, that we felt the need of the present investigation. We recall that it is customary[17] to neglect terms proportional to the local equilibrium current in standard derivations of the relevant eigenmode equations in the inner region; because of this assumption, the resulting form of the equations allows the formulation of several elegant analytical techniques.[17-19] The eigenmode problem becomes much more difficult whenever such terms are kept and most of the methods of solution are no longer applicable.

The purpose of the present dissertation is to show how to include the local equilibrium current in the analysis and to assess its effect on the stability properties of a number of modes. The modification of the basic set[17,18] of eigenmode equations appears in an additional term in Ampére's law, due to resistivity perturbation in the generalized Ohm's law, and in the presence of the local current gradient, i.e., the so-called "kink term," in the equation expressing the quasi-neutrality condition. Although the equilibrium current term in the generalized Ohm's law does not explicitly have the form of a current gradient, it turns out that, assuming the local equilibrium parallel field $E_{\parallel}^{(0)}$ to be spatially constant, the two additional terms are exactly identical for two different forms of the generalized conductivity. Under the approximations used in the analysis, the effect of the local equilibrium current manifests itself only through its local gradient, and therefore we shall consider the local equilibrium current and its local gradient on the same footing.

We do not depart significantly from previous, "conventional" theories in treating the global effect of the equilibrium current. The current profile determines Δ', the well known[1,2,25] discontinuity in the derivative of the external solution; the problem in the ideal, MHD region is presumed to have been solved and the Δ' is treated as a known parameter in the theory.

In the most significant cases, we find that the inclusion of local current terms results in a small correction to the dispersion relations describing several known modes. The predictions of current theories on the linear stability of tearing instabilities are therefore confirmed, to leading order, by the present study, whose relevance is to have assessed the effect of the local equilibrium current on a number of modes by a detailed calculation, where previous analyses had argued a priori that such an effect was to be expected to be negligible.

The remainder of this dissertation is organized in the following manner. In Chapter II, we derive the eigenmode equations; since the derivation is quite standard, apart from the inclusion of equilibrium current terms, only the final result is presented. In Chapter III, we begin with a general discussion on the use of variational principles in drift-tearing mode stability theory; subsequently, we formulate in detail the method of constructing the variational functional for the problem under present consideration. In Chapter IV, appropriate limiting forms of the variational functional are shown to recover a number of known results in the collisional and semicollisional regime of the tearing mode, respectively. The effect of the local equilibrium current is then treated as a perturbation of these results, and the modified dispersion relations are obtained. Finally, in Chapter V, we summarize our conclusions. In Appendix A, we outline the derivation of the eigenmode equations using fluid theory. In Appendix B, we give a derivation of the equation expressing the relation between the perturbed parallel current and perturbed electric field in the collisional regime, using the drift kinetic equation. In Appendix C, we present a discussion on a choice of boundary conditions for the appropriate solution to the eigenmode equations, which is different from the one presented in Chapter III. In Appendix D, we give an alternate derivation of the semicollisional drift tearing result, in the absence of local current, using the variational principle formulation of Ref. 18.

C.G.S. units with $c = 1$ are used throughout.

CHAPTER II

EIGENMODE EQUATIONS

A. Tokamak Slab Geometry

Let $\hat{b}_0 \equiv \underset{\sim}{B}_0/B_0$ be a unit vector in the direction of the equilibrium magnetic field $\underset{\sim}{B}_0$; $\hat{r}$ be a unit vector in the radial direction, normal to a flux surface; and $\hat{e} \equiv \hat{b}_0 \times \hat{r}$. In the slab model of the tokamak, these unit vectors, as well as the equilibrium quantities, depend only on the radius.

A conventional Fourier representation of any linearly perturbed quantity, $\tilde{g}$, is introduced, such that

$$\frac{\partial}{\partial t} \tilde{g} = - i\omega\tilde{g} \quad , \tag{2.1a}$$

$$\hat{b}_0 \cdot \nabla\tilde{g} = ik_{\|}(x)\tilde{g} \quad , \tag{2.1b}$$

$$\hat{e} \cdot \nabla\tilde{g} = ik_{\perp}(x)\tilde{g} \quad , \tag{2.1c}$$

$$\hat{r} \cdot \nabla\tilde{g} = \tilde{g}' \quad , \tag{2.1d}$$

where the radial variable $x = r - r_s$ denotes the distance from a mode rational surface, located at $r = r_s$. In the vicinity of such surface, where $k_{\|}(0) = 0$, we have

$$k_{\parallel}(x) \simeq k_{\parallel}' x \quad . \tag{2.2}$$

It is convenient to formulate the eigenmode problem in terms of the perturbed electrostatic potential $\tilde{\phi}$ and the parallel component of the magnetic vector potential $\tilde{A}_{\parallel}$. For sufficiently small β (ratio between plasma pressure and magnetic pressure) it is consistent to assume[26]

$$\tilde{\underset{\sim}{A}} = \hat{b}_o \tilde{A}_{\parallel} \quad , \tag{2.3}$$

implying that the radial magnetic perturbation becomes

$$\tilde{B}_r = i k_{\perp} \tilde{A}_{\parallel} \quad , \tag{2.4}$$

and the components of the perturbed electric field are

$$\tilde{E}_{\parallel} \equiv \hat{b}_o \cdot \tilde{\underset{\sim}{E}} = - i k_{\parallel} \tilde{\phi} + i\omega \tilde{A}_{\parallel} \quad , \tag{2.5a}$$

$$\tilde{E}_{\perp} \equiv \hat{e} \cdot \tilde{\underset{\sim}{E}} = - i k_{\perp} \tilde{\phi} \quad , \tag{2.5b}$$

$$\tilde{E}_r \equiv \hat{r} \cdot \tilde{\underset{\sim}{E}} = - \tilde{\phi}' \quad . \tag{2.5c}$$

The parallel component of Ampére's law takes the form

$$\tilde{J}_{\parallel} = \frac{1}{4\pi} (-\tilde{A}_{\parallel}'' + k_{\perp}^2 \tilde{A}_{\parallel}) \simeq$$

$$\simeq - \frac{\tilde{A}_{\parallel}''}{4\pi} \quad , \tag{2.6}$$

where $\tilde{J}_{\parallel}$ is the perturbed parallel current. In Eq. (2.6) we introduced the approximation, used throughout the present analysis, of assuming the mode radial wavelength to be much shorter than the azimuthal wavelength.

B. Eigenmode Equations

The first eigenmode equation to be derived, describing essentially the electron dynamics inside the tearing layer, expresses the constitutive relationship between the perturbed parallel current, $\tilde{J}_\parallel$, given by Eq. (2.6), and the perturbed parallel electric field, $\tilde{E}_\parallel$. The electron dynamics of relevance in the present work is correctly represented by the following moment equations,[20,21] expressing particle conservation, momentum balance along the total magnetic field and energy balance, respectively:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\underset{\sim}{u}) = 0 \quad , \tag{2.7}$$

$$\begin{aligned}\varepsilon'' \frac{\partial}{\partial t} (nm_e\hat{b} \cdot \underset{\sim}{u}) = - ne\hat{b} \cdot \underset{\sim}{E} - \hat{b} \cdot \nabla p_e \quad + \\ - (ne)^2\eta\hat{b} \cdot \underset{\sim}{u}\left[1 + \frac{3}{2}\varepsilon''\left(\frac{m_e}{\eta ne^2}\right)\frac{\partial}{\partial t}\ln T_e\right] - \varepsilon n\hat{b} \cdot \\ \cdot \nabla T_e\left[1 - \frac{3\varepsilon'}{\nu}\frac{\partial}{\partial t}\ln T_e - \frac{\varepsilon'}{\nu}\frac{\partial}{\partial t}\ln(\hat{b} \cdot \nabla T_e)\right] \quad ,\end{aligned} \tag{2.8}$$

$$\begin{aligned}\frac{3}{2} n\left(\frac{\partial T_e}{\partial t} + \underset{\sim}{u} \cdot \nabla T_e\right) + nT_e\nabla \cdot \underset{\sim}{u} \quad + \\ + \nabla \cdot \left[\varepsilon nT_e\hat{b}\hat{b} \cdot \underset{\sim}{u} - \chi\hat{b}\hat{b} \cdot \nabla T_e\right] - \nabla \cdot \left[\frac{5}{2}\frac{nT_e}{eB}\hat{b} \times \nabla T_e\right] \quad + \\ - \hat{b} \cdot \underset{\sim}{u}[\varepsilon n\hat{b} \cdot \nabla T_e + (en)^2\eta\hat{b} \cdot \underset{\sim}{u}] = 0 \quad ,\end{aligned} \tag{2.9}$$

where

$$\underset{\sim}{u} = \hat{b}\hat{b} \cdot \underset{\sim}{u} + \frac{\underset{\sim}{E} \times \hat{b}}{B} - \frac{\hat{b} \times \nabla p_e}{enB} \quad . \tag{2.10}$$

These equations differ from the well known transport equations derived by Braginskii,[27] to the required order,[20] by the inclusion of a time dependent thermal force on the electrons and by the fact that the coefficient of the electron inertial term in the momentum equation, Eq. (2.8), is greater than unity. The physical origin of these terms, basically connected with the velocity dependence of the Coulomb cross section (for a plasma, $\nu \sim v_e^{-3}$, where ν is the electron collision frequency and v_e is the electron thermal velocity), has been discussed in detail in Ref. 20. Here, we only remark that the inclusion of the time dependent thermal force results in the thermoelectric growth rate[15,21] in the final dispersion relation and that this term does not couple with the local parallel equilibrium current J_0, and therefore does not play a crucial role in the present analysis. Also, the electron inertial term will be neglected in the subsequent derivation.

In Eqs. (2.7), (2.10), n is the electron density, T_e is the electron temperature, $p_e = nT_e$ is the electron pressure, m_e is the electron mass, e is the absolute value of the electron charge; $\underset{\sim}{E}$ and $\underset{\sim}{B}$ are the total (unperturbed) electric and magnetic fields, respectively, $\hat{b}$ is a unit vector in the direction of the total magnetic field $\underset{\sim}{B}$; η is the Spitzer-Braginskii resistivity; χ is the thermal conductivity along the ambient magnetic field; ν is the electron collision frequency; ε, ε', ε'' are numerical transport coefficients tabulated in

Ref. 20; in particular, ε is the familiar[27] coefficient of the thermal force and is equal to 0.71 for a singly charged ion plasma.

The linearization of Eqs. (2.7), (2.8), (2.9) is quite straightforward and an outline of the calculation is presented in Appendix A; the most important result, the expression for $\tilde{J}_{\parallel}$, is given by Eq. (A.5). We obtain for Ampére's Law, with $\tilde{J}_{\parallel}$ evaluated in terms of the perturbed electrostatic potential $\tilde{\phi}$ and the normalized perturbed vector potential $\psi \equiv \omega\tilde{A}_{\parallel}/k_{\parallel}'$,

$$\psi'' = \sigma_*(x^2)\left\{\psi - [x - \Lambda(x^2)]\phi\right\} \quad , \tag{2.11}$$

where the generalized conductivity $\sigma_*(x^2)$ is given by

$$\sigma_*(x^2) = \frac{-4\pi i}{\eta}\left\{\frac{(\omega - \omega_n^* - \hat{\varepsilon}\omega_T^*) + (\omega - \omega_n^*)i\varepsilon''' s\,\frac{k_{\parallel}^2 D}{\omega}}{\left(1 + i\varepsilon''' s\,\frac{k_{\parallel}^2 D}{\omega}\right)\left(1 + is\,\frac{k_{\parallel}^2 D}{\omega}\right) + \frac{2}{3}\,\hat{\varepsilon}(1 + \varepsilon)is\,\frac{k_{\parallel}^2 D}{\omega}}\right\} . \tag{2.12}$$

and the equilibrium current appears through the factor

$$\Lambda(x^2) = \left\{\frac{R}{-i(\omega - \omega_n^* - \hat{\varepsilon}\omega_T^*) + \varepsilon''' s(\omega - \omega_n^*)\,\frac{k_{\parallel}^2 D}{\omega}}\right\} \quad , \tag{2.13}$$

with the definitions

$$\omega_n^* \equiv - \frac{k_\perp}{eB}\,\frac{n'}{n}\,T_e \tag{2.14a}$$

$$\omega_T^* \equiv - \frac{k_\perp}{eB} T_e' \quad , \tag{2.14b}$$

$$\hat{\varepsilon} \equiv 1 + \varepsilon + i\varepsilon' \frac{\omega}{\nu} \quad , \tag{2.14c}$$

$$D \equiv \frac{T_e}{m_e \nu} \quad , \tag{2.14d}$$

$$s \equiv \eta(Z = 1)/0.51\eta(Z_{eff}) \quad , \tag{2.14e}$$

$$R \equiv - \frac{3}{2} \frac{\eta}{k_\parallel'} \frac{e\omega_T^*}{T_e} J_0 \quad , \tag{2.14f}$$

$$\frac{\chi}{n} \equiv \frac{3}{2} \varepsilon''' sD \quad ; \tag{2.14g}$$

ω_n^*, ω_T^* are the electron drift frequencies related to density gradients and electron temperature gradients, respectively; $\hat{\varepsilon}$ is defined in terms of the numerical transport coefficients ε, ε'; the quantity D is basically the diffusion coefficient of the electrons along the magnetic field B_o, when collisions inhibit free streaming at the thermal velocity; s is the numerical coefficient of the Spitzer-Braginskii conductivity and is equal to 1.96 for singly charged ions; the factor R is proportional to the local equilibrium parallel current J_o; the numerical transport coefficient ε''' has been defined[21] so that $(3/2)\varepsilon''' s$ is equal to the numerical coefficient of the parallel thermal conductivity, which is, $(3/2)\varepsilon''' s = 3.16$ for singly charged ions.

Equation (2.11), as remarked, is essentially a description of the electron dynamics; the other piece of needed information describes the ion dynamics, and it is usually[17] obtained considering the plasma equation of motion in the form

$$m_i n_i \frac{\partial \underset{\sim}{V}}{\partial t} = - \nabla P + \underset{\sim}{J} \times \underset{\sim}{B} \quad , \tag{2.15}$$

where $\underset{\sim}{V}$ is the ion (or bulk plasma) flow velocity, m_i is the ion mass, n_i is the ion density, $P = p_i + p_e$ is the total plasma pressure.

Solving Eq. (2.15) for the perpendicular component of the perturbed current, $\tilde{\underset{\sim}{J}}_\perp$, and imposing quasineutrality in the form

$$\nabla \cdot \tilde{\underset{\sim}{J}}_\parallel = - \nabla \cdot \tilde{\underset{\sim}{J}}_\perp \quad , \tag{2.16}$$

one finds[24] (see also Appendix A)

$$X_A^2 \phi'' = x\psi'' - \frac{4\pi k_\perp}{B k_\parallel'} J_o' \, \psi \quad , \tag{2.17}$$

where the Alfvén layer X_A, defined as the distance from the rational surface at which $\omega \simeq k_\parallel v_A$ ($v_A = B_o/(4\pi n_i m_i)^{\frac{1}{2}}$ is the Alfvén speed), is given by

$$X_A^2 \equiv \frac{\omega(\omega + \overset{*}{\omega}_i)}{(k_\parallel' v_A)^2} \quad , \tag{2.18a}$$

and the ion drift frequency is

$$\omega_i^* \equiv - \frac{k_\perp}{eB} \frac{p_i'}{n_i} \quad . \tag{2.18b}$$

In the derivation of Eqs. (2.11), (2.17) the following approximations have been used:

i) terms involving J_0 which are of order β have been neglected;

ii) terms involving J_0 which appear as Doppler shifts to the real part of the mode frequency have been neglected;

iii) contributions to the mode growth rate due to Ohmic heating terms are also neglected, since the corresponding thermal instability does not strongly couple to the tearing mode.

We remark that the J_0 term in Eq. (2.11), i.e. the Λ factor, comes from perturbations of the Spitzer-Braginskii resistivity in the generalized Ohm's law, Eq. (2.8).[(i)] The J_0' term in Eq. (2.17) is the so-called kink term.[24] By neglecting the J_0 terms, the derived eigenmode equations reduce to the well known set[17,18] studied in drift eigenmode theory, with the generalized conductivity in the form of Eq. (2.12). The structure of the eigenmode equations is significantly altered by the inclusion of these terms, to the extent that several elegant solution techniques are no longer applicable. In most tearing mode analyses, it has been customary to neglect the local equilibrium current inside the tearing layer and to consider the eigenmode equations in the resulting, considerably more symmetric form.

[(i)] In Appendix B we give an alternative derivation of the collisional form of Eq. (2.11), i.e., when terms proprotional to $k_\parallel^2 D/\omega$ are neglected, using kinetic theory.

In the following chapter, we present a method of solution which is suited to deal with Eqs. (2.11), (2.17) and which, at the same time, provides continuous contact with previous formulations in the appropriate limit.

CHAPTER III

VARIATIONAL PRINCIPLE

A. General Formulation

A particularly elegant and powerful method of solving the eigenmode equations derived in the previous chapter is provided by the use of a variational formulation. In this section, we briefly review the general ideas behind the variational approach; the calculations relevant to the present analysis are presented in the following sections.

The two eigenmode equations can be generally combined in a single equation of the form

$$\mathcal{L}\chi = M\chi \quad , \tag{3.1}$$

where the variable χ, which is a function of the potentials ψ and ϕ, must be chosen in such a way that it is localized, as well as continuous with its derivatives. The linear operators $\mathcal{L}$ and M have to be "self adjoint" with respect to an appropriately defined scalar product. Explicitly, let us define

$$\langle f,g \rangle \equiv \int_{-\infty}^{\infty} fg \, dx < \infty \quad , \tag{3.2}$$

for suitably behaved functions f and g. The operators $\mathcal{L}$ and M must then satisfy

$$\langle f,\mathcal{L}g\rangle = \langle \mathcal{L}f,g\rangle \quad , \tag{3.3a}$$

$$\langle f,Mg\rangle = \langle Mf,g\rangle \quad , \tag{3.3b}$$

for the variational method to be applicable; provided Eqs. (3.3a), (3.3b) hold, the functional S, defined by

$$S(f,f) \equiv \langle f,\mathcal{L}f\rangle - \langle f,Mf\rangle \quad , \tag{3.4}$$

can be easily shown to be the variational functional corresponding to the Euler-Lagrange equation, Eq. (3.1). Indeed, if f is a chosen trial function, containing variational parameters, the arbitrary variation of S with respect to these parameters is

$$\begin{aligned}\delta S &= S(\delta f,f) + S(f,\delta f) = \\ &= 2\langle \delta f,(\mathcal{L} - M)f\rangle \quad ,\end{aligned} \tag{3.5}$$

assuming the self adjointness of $\mathcal{L}$ and M. Requiring that $\delta S = 0$, the arbitrary character of δf implies that $(\mathcal{L} - M)f = 0$. Also, evaluating S at $f = \chi$, we find that

$$\begin{aligned}S(\chi,\chi) &= \langle \chi,\mathcal{L}\chi\rangle - \langle \chi,M\chi\rangle = \\ &= 0 \quad .\end{aligned} \tag{3.6}$$

The variational principle, Eqs. (3.5), (3.6), is used in practice in the following way. We substitute into Eq. (3.4) a test function, f, which depends upon one or more variational parameters α_j, $j = 1,2,\cdots,m$. The resulting function $S(\alpha_j)$ is required to satisfy the system of equations (extremum conditions)

$$\frac{\partial S}{\partial \alpha_j} = 0 \quad , \qquad j = 1,2,\cdots,m \quad , \tag{3.7}$$

which can be solved, in principle, for the α_j. We denote the solution to Eq. (3.7), which depends in general upon the complex mode frequency ω, by $\alpha_{j*}(\omega)$, and we denote the resulting, extremal value of S by S_*:

$$S_*(\omega) \equiv S[\alpha_{1*}(\omega),\alpha_{2*}(\omega),\cdots] \quad . \tag{3.8}$$

Now, if the chosen trial function is sufficiently flexible to roughly approximate the true solution,

$$f = \chi + \delta f \quad , \tag{3.9}$$

then $\delta S = 0$ implies that

$$S_* = S(\chi) + 0(|\delta f|^2) \quad , \tag{3.10}$$

(where the error estimate is best understood in terms of an L_2 norm). Then, Eq. (3.6) provides the mode dispersion relation

$$S_*(\omega) = 0 \quad , \tag{3.11}$$

which will be accurate to order δ^2, whereas the trial function was accurate only to order δ.

The advantages of variational principles are widely recognized and their use in drift eigenmode theory quite common.[17-19,27,28] Rather complex models of plasma behaviour, i.e., complicated forms of the conductivity $\sigma_*(x^2)$, become amenable to analytic treatment, and different modes, pertaining to a particular conductivity model, tend to be described by a single unified picture, which enhances the understanding of the underlying physics. Whenever approximations are required, an advantage of the variational procedure is that one needs only examine limiting forms of explicitly given functions, rather than having to determine the solutions to limiting forms of the differential equations. Whenever "exact", and usually much more complex, analytical methods are possible, the agreement with the variationally derived dispersion relation is very good; the qualitative form of the dispersion relation is reproduced perfectly, whereas the quantitative error does not typically exceed a few percent. It must be noted, however, that the application of variational techniques is possible if and only if the mode equation can be combined to yield the desired self-adjoint form Eq. (3.1). In the presence of local equilibrium current, the path to a variational formulation is far from obvious, although several distinct formulations are possible, when these terms are absent. In the past years, formulations in terms of the perturbed radial electric[17] field, perturbed parallel electric field,[18] Fourier transform of the perturbed parallel current,[28] and perturbed parallel current,[19] have been successfully employed in the stability analysis of electromagnetic disturbances. As noted at the end of Chapter II, the presence of the J_0 terms alters the symmetry of the eigenmode equations to the extent that some of the aforementioned formulations are no longer possible. However, the formulation in terms of the perturbed parallel current can be suitably extended to include the effect of the local equilibrium current. The next sections are devoted to the setting up of the appropriate formulation; the mode dispersion relations are derived in the following chapter.

B. Collisional and Semicollisional Regime of the Tearing Mode

In the previous section, we outlined the general procedure to carry out the eigenmode stability analysis using a variational principle formulation. The results to be obtained cannot be however derived by a direct, mathematically rigorous, manipulation of Eqs. (2.11), (2.17); rather, a good deal of "physical reasoning" is needed at almost each step of the derivation, and several necessary approximations are justified on the basis of some a priori knowledge of the form of the solution. To this purpose, it seems appropriate to introduce some concepts, as well as some terminology, through the following, somewhat heuristic, discussion.

As remarked in the Introduction, the interesting physics of tearing instabilities is confined to a very narrow region centered around a mode rational surface where local decoupling of fluid and magnetic field takes place, and breaking and reconnection of field lines occurs. The radial magnetic perturbation, $\tilde{B}_r$, which is responsible for the tearing process, is related to the perturbed parallel current $\tilde{J}_\parallel$, through the Maxwell equations, Eqs. (2.4) and (2.6). The perturbed current is itself expressed in terms of the perturbed parallel electric field $\tilde{E}_\parallel$ by an "Ohm's law" of the form

$$\tilde{J}_\parallel = \sigma_* \tilde{E}_\parallel \quad , \tag{3.12}$$

which is precisely Eq. (2.11), if one neglects the current-dependent term, Λ (for simplicity, we neglect the local equilibrium current in the present discussion, since we shall treat its effect as a small correction to existing theories in our analysis). From the form of Eq. (3.12), it appears that a large electron current $\tilde{J}_\parallel$ will exist unless one of the following effects occurs[16]:

i) the Doppler frequency ω_D resulting from electron thermal motion along $\underset{\sim}{B}_0$ is greater than the mode frequency ω, $\omega_D >> \omega$, causing the electrons to be subjected to an ac rather than a dc acceleration and thus reducing $\tilde{J}_\parallel$. When $\omega << \nu$ (which is the appropriate scaling for the present analysis, where we used fluid equations to derive the electron response), collisions inhibit electron motion along $\underset{\sim}{B}_0$ and the electrons diffuse along the field lines with a diffusion coefficient $D \equiv T_e/m_e\nu$. The resulting effective Doppler frequency of the electrons is $\omega_D = k_\|^2 D$, and from Eq. (2.12) we see that, when $k_\|^2 D \gg \omega$, $\sigma_* \to 0$, therefore quenching $\tilde{J}_\|$.

ii) The electric field $\tilde{E}_\|$ decays spatially before the conductivity does. This is so because the electrostatic part of $\tilde{E}_\|$, $-ik_\|\tilde{\phi}$, increases away from the rational surface until it cancels the inductive part of $\tilde{E}_\|$, $i\omega\tilde{A}_\|$.

Which one of the two effects actually causes the quenching of $\tilde{J}_\|$, depends on the radial width of the mode ℓ_W, which is convenient to identify with the radial extent of $\tilde{E}_\|$. The spatial scale over which the conductivity σ_* varies is evidently of the order of $\ell_\sigma \equiv (\omega/k_\|'^2 D)^{\frac{1}{2}}$. When the mode is very localized, $\ell_W \ll \ell_\sigma$, σ_* can be approximated by its value at $x = 0$, and we refer to the corresponding limit as the collisional regime of the tearing mode. When $\ell_W \gtrsim \ell_\sigma$, the full x^2 dependence of σ_* must be retained and the resulting scaling, $k_\|^2 D/\omega \sim 1$, is usually referred to as the semicollisional regime.

In the present dissertation, however, we shall restrict ourselves to a different definition of the semicollisional regime, corresponding to a mode width somewhat in between the two previously discussed cases. From the analysis of the dispersion relation of the collisional drift tearing mode, as shown in the next chapter, it turns out that the term $(\omega - \omega_n^* - \hat{\varepsilon}\omega_T^*)$ appearing in the numerator of σ_* tends to be much smaller than the factor $(\omega - \omega_n^*)$. Therefore, a scaling which is intermediate between the collisional, $k_\|^2 D/\omega \to 0$, and the semicollisional, $k_{\parallel}^2 D/\omega \sim 1$, is possible because of the presence of temperature gradients and it is characterized by

$$\frac{k_{\parallel}^2 D}{\omega} < 1 \quad , \tag{3.13}$$

but, at the same time,

$$(\omega - \omega_n^* - \hat{\varepsilon}\omega_T^*) \sim (\omega - \omega_n^*) \frac{k_{\parallel}^2 D}{\omega} \quad . \tag{3.14}$$

The existence of such a scaling, which includes in particular the effect of the thermal conduction, has been noted for example in Ref. 22, and is very interesting in the present context, since it is made possible by the temperature gradient, which also allows the coupling of the local equilibrium current to the eigenmode equations. We remark that, when $(\omega - \omega_n^* - \hat{\varepsilon}\omega_T^*) << (\omega - \omega_n^*)k_{\parallel}^2 D/\omega$, $\Lambda \to 0$ in Eq. (2.11), making the effect of the local current negligible. For this reason, since this study is intended to treat J_0 effects, we shall limit our discussion of the semicollisional case to the scaling expressed by Eqs. (3.13) and (3.14).

C. Collisional Form

As discussed in the previous section, the collisional form of the eigenmode equations is obtained by letting $k_\parallel^2 D/\omega \to 0$ in the conductivity σ_*, which can be approximated by

$$\sigma_* \simeq \mu_1 \equiv$$

$$\equiv -\frac{4\pi i}{\eta}\,(\omega - \omega_n^* - \hat{\varepsilon}\omega_T^*) \quad , \tag{3.15}$$

and similarly, by letting $k_\parallel^2 D/\omega \to 0$ in the current term Λ, which can be approximated by

$$\Lambda \simeq \lambda \equiv$$

$$\equiv \frac{iR}{(\omega - \omega_n^* - \hat{\varepsilon}\omega_T^*)} \quad . \tag{3.16}$$

We now show that, in the collisional regime, the terms involving J_0 and J_0' in Eqs. (2.11), (2.17) are in reality identical if we assume the equilibrium parallel field $E_\parallel^{(0)}$ to be spatially constant (from Maxwell equations, this is true if the equilibrium magnetic field does not appreciably change in time on the typical time scale for the perturbation, a property which is assumed for all the quantities describing the equilibrium). The equilibrium Ohm's law is

$$J_0 = \sigma_{SP}\, E_\parallel^{(0)} \quad , \tag{3.17}$$

where σ_{SP} is the Spitzer-Braginskii conductivity; from (3.17), since $\sigma_{SP} \sim T_e^{3/2}$ and we take $dE_\parallel^{(0)}/dx = 0$, we find

$$J_o' = \frac{3}{2}\frac{T_e'}{T_e} J_o \quad , \tag{3.18}$$

which yields, making use of Eqs. (2.14), (3.15), (3.16),

$$\frac{4\pi k_\perp}{Bk_\parallel'} J_o' = \lambda\mu_1 \quad . \tag{3.19}$$

Within the context of the approximations used in this work, the effect of the equilibrium current manifests itself through its local gradient, which is, in turn, directly proportional to the local electron temperature gradient.

The collisional form of the eigenmode equations can therefore be written as

$$X_A^2\phi'' = x\psi'' - \lambda\mu_1\psi \quad , \tag{3.20}$$

$$\psi'' = \mu_1[\psi - (x - \lambda)\phi] \quad ; \tag{3.21}$$

eliminating the factor $\mu_1\psi$ in Eq. (3.20) by using Eq. (3.21), we obtain

$$\phi'' + \frac{\lambda\mu_1}{X_A^2}(x - \lambda)\phi = \frac{(x - \lambda)\mu_1}{X_A^2}[\psi - (x - \lambda)\phi] \quad , \tag{3.22}$$

which along with Eq. (3.21) form our basic set. Let us now define a new radial variable

$$p \equiv x - \lambda \quad , \tag{3.23}$$

a new field variable

$$Q(p) \equiv \mu_1(\psi - p\phi) \quad , \tag{3.24}$$

and

$$\rho \equiv \frac{\lambda\mu_1}{X_A^2} \quad , \tag{3.25}$$

so that we can write the eigenmode equations, in the form

$$\psi'' = Q(p) \quad , \tag{3.26}$$

$$\phi'' + \rho p\phi = \frac{pQ(p)}{X_A^2} \quad . \tag{3.27}$$

By neglecting J_o terms, i.e., letting $\rho \to 0$, $\lambda \to 0$, $p \to x$, Eqs. (3.26), (3.27) are identical with the corresponding set of equations describing low poloidal mode number tearing modes in Ref. 19; in particular, the variable Q(p) becomes the perturbed parallel current $\tilde{J}_\parallel$. We remark that, by defining the shifted radial variable p, the local equilibrium current effect appears explicitly in the eigenmode equations only through the term $\rho p\phi$ of Eq. (3.27). This term is responsible for the mixing of the even and the odd part of the fields due to J_o, in the p variable formulation.

We can formulate a variational principle for the variable Q following the procedure of Ref. 19, which we briefly outline. Treating the variable Q as a given inhomogeneous term at the right-hand-side (r.h.s.) of Eqs. (3.26), (3.27), one formally solves these equations for ψ and ϕ in terms of Q; using the definition of Q as given by Eq. (3.24), one then constructs an appropriate combination of ψ and ϕ to obtain an integral equation for Q, which has the necessary structure to allow for a variational treatment. In the present analysis, the method of Ref. 19 is suitably extended to account for the presence of the extra J_0 term in Eq. (3.27). Equation (3.26) is formally identical with Eq. (4) of Ref. 19, and its solution is

$$\psi(p) = \frac{1}{2} \int_{-\infty}^{\infty} |p - p'| Q(p') dp' + a_1 + b_1 p \quad , \tag{3.28}$$

where a_1 and b_1 are arbitrary constants. The first term on r.h.s. of Eq. (3.28) is a particular integral, to which we have added a linear combination of the solutions of the corresponding homogeneous equation. To make contact with classical tearing mode theory, let us require the asymptotic behaviour[17,19]

$$\psi \to \psi_0 \left(1 + \frac{\Delta' |x|}{2} \right) \quad , \tag{3.29}$$

for large x, where, we recall, ψ_0 is the value toward which the exterior ψ solution tends and Δ' is the logarithmic jump in the derivative of the external solution, defined by

$$\Delta' \equiv \frac{\psi'_+ - \psi'_-}{\psi_0} \quad , \tag{3.30}$$

where ψ_+, ψ_- denote the external ψ solution on the right and on the left of the tearing layer, respectively. In the present context, no distinction is made between the radial variables x and p, because for $x \to \infty$, $p \to x$. It then appears[(i)] that we must choose

$$a_1 = \frac{1}{\Delta'} \int_{-\infty}^{\infty} Q(p')dp' \quad , \tag{3.31}$$

$$b_1 = 0 \quad , \tag{3.32}$$

so that the appropriate solution of Eq. (3.26) is

$$\psi = \frac{1}{2} \int_{-\infty}^{\infty} |p - p'|Q(p')dp' + \frac{1}{\Delta'} \int_{-\infty}^{\infty} Q(p')dp' \quad . \tag{3.33}$$

The solution of Eq. (3.27) is determined as follows: the associated homogeneous equation is

$$\phi'' + \rho p \phi = 0 \quad , \tag{3.34}$$

[(i)] A more detailed discussion on the boundary conditions in the presence of local J_0 terms is presented in Appendix C. It turns out that the change in the boundary conditions does not affect significantly the final answer.

whose solution can be expressed as a linear combination of the linearly independent set

$$\phi_1 = \sqrt{p}\, J_{1/3}(\zeta) \quad , \tag{3.35a}$$

$$\phi_2 = \sqrt{p}\, J_{-1/3}(\zeta) \quad , \tag{3.35b}$$

where $J_{\pm 1/3}(\zeta)$ are Bessel functions, and

$$\zeta \equiv \frac{2}{3}\, \rho^{1/2}\, p^{3/2} \quad . \tag{3.36}$$

The Wronskian of the set ϕ_1, ϕ_2 is (with standard normalization[29])

$$W_p(\phi_1,\phi_2) = \rho^{1/2} p^{3/2} \left[\frac{-2\,\sin\left(\frac{\pi}{3}\right)}{\frac{2}{3}\,\pi\rho^{1/2} p^{3/2}} \right] = \frac{-3\sqrt{3}}{2\pi} \quad . \tag{3.37}$$

Using the homogeneous solutions, the most general solution of Eq. (3.27) is

$$\phi(p) = \frac{2\pi}{3\sqrt{3}\, X_A^2} \times$$

$$\times \left\{ \sqrt{p}\, J_{1/3}(\zeta) \int^{p} p' Q(p') \sqrt{p'}\, J_{-1/3}(\zeta')\, dp' \; + \right.$$

$$\left. - \sqrt{p}\, J_{-1/3}(\zeta) \int^{p} p' Q(p') \sqrt{p'}\, J_{1/3}(\zeta')\, dp' \right\} \quad , \tag{3.38}$$

where the undefined limits allow any arbitrary addition of the homogeneous solution. It is equally convenient to write the general solution of Eq. (3.27) as

$$\phi(p) = a_2\sqrt{p}\, J_{1/3}(\zeta) + b_2\sqrt{p}\, J_{-1/3}(\zeta) + \frac{2\pi}{3\sqrt{3}\, X_A^2} \times$$

$$\times \left\{\frac{1}{2} \int_{-\infty}^{p} \sqrt{pp'}\, [J_{1/3}(\zeta)J_{-1/3}(\zeta') - J_{-1/3}(\zeta)J_{1/3}(\zeta')]p'Q(p')dp' + \right.$$

$$\left. - \frac{1}{2} \int_{p}^{\infty} \sqrt{pp'}\, [J_{1/3}(\zeta)J_{-1/3}(\zeta') - J_{-1/3}(\zeta)J_{1/3}(\zeta)]p'Q(p')dp' \right\} \quad , \tag{3.39}$$

where a_2, b_2 are arbitrary constants. When p is close to zero, the integrands in Eq. (3.39) contribute only for p' close to zero, since Q(p') is assumed to be localized about p' = 0. In the present study, we are interested in modes which decay spatially when ζ, defined by Eq. (3.36), is still quite small. Expanding the Bessel functions to significant order in ζ, so to include terms up to ρ^2, we find, after some straightforward algebra

$$\sqrt{pp'}\, [J_{1/3}(\zeta)J_{-1/3}(\zeta') - J_{-1/3}(\zeta)J_{1/3}(\zeta')] \simeq$$

$$\simeq \frac{1}{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{2}{3}\right)} (p - p')\mathbb{K}(p,p') \quad , \tag{3.40}$$

where

$$K(p,p') = K(p',p) \equiv$$

$$\equiv 1 - \frac{\rho}{12}(p + p')(p - p')^2 +$$

$$+ \frac{\rho^2}{504}[p^6 + p'^6 - \frac{9}{5}pp'(p^4 + p'^4) +$$

$$- \frac{9}{5}p^2p'^2(p^2 + p'^2) + \frac{26}{5}p^3p'^3] \quad . \qquad (3.41)$$

We can then write Eq. (3.39) as

$$\phi(p) = a_2\sqrt{p}\, J_{1/3}(\zeta) + b_2\sqrt{p}\, J_{-1/3}(\zeta) +$$

$$+ \frac{1}{2X_A^2}\int_{-\infty}^{\infty}|p - p'|K(p,p')p'Q(p')dp' \quad . \qquad (3.42)$$

If the arbitrary constants appearing in the above equation are chosen as

$$a_2 = 0 \quad , \qquad (3.43)$$

$$b_2 = 0 \quad , \qquad (3.44)$$

we obtain, for the appropriate solution of Eq. (3.27),

$$\phi(p) = \frac{1}{2X_A^2}\int_{-\infty}^{\infty}|p - p'|K(p,p')p'Q(p')dp' \quad , \qquad (3.45)$$

which together with Eq. (3.33), ensures the localization of the quantity Q, as given by Eq. (3.24), to lowest order in ρ.

Let us comment on this point. According to the procedure of Ref. 19, the choice of boundary conditions for ψ and ϕ is dictated by the requirement that Q, as given by Eq. (3.24), is a localized variable, and that Eq. (3.29) is satisfied. In the presence of J_0, the localization condition has to be understood in the following way.

In the present work, we treat the effect of the local current as a small modification to the mode dynamics inside the tearing layer; when $x >> \lambda$, $p \to x$, and the variable Q becomes the ordinary $\tilde{J}_\parallel$ ($\tilde{E}_\parallel$) which must be indeed localized, since in the exterior (MHD) region, $\tilde{E}_\parallel \to 0$. In the context of tearing mode theory, $|x| \to \infty$ has to be intended as $|x| >> \ell_w$. We assume the existence of a matching region where, even if x is sufficiently large (i.e., $x >> \ell_w$) the current-dependent dimensional parameter ζ, given by Eq. (3.36), is still very small, as already noted. We remark that assumptions of this kind are peculiar to tearing mode theories, even when local J_0 terms are neglected.[19] For the present discussion, it is appropriate to approximate K(p,p') in Eq. (3.45) by 1 and to neglect the odd part of Q in the particular integrals of Eqs. (3.33), (3.45) since, as it can be seen from the form of the eigenmode equations (see also the discussion at the beginning of Chapter IV), it appears that the odd part of Q must be linear in ρ, i.e., of order ζ^2. Notice, however, that all these corrections will have to be kept to derive the dispersion relation, which critically depends upon small contributions due to J_0.

The limit for large x of the particular integrals in Eqs. (3.33) and (3.45) becomes then analogous to $J_0 = 0$ case, for which the choice of boundary conditions expressed by Eqs. (3.31), (3.32), (3.43), (3.44) satisfies the mentioned requirements.[19]

Combining Eqs. (3.24), (3.33), (3.45) we obtain the integral equation

$$\frac{Q(p)}{\mu_1} = \frac{1}{\Delta'} \int_{-\infty}^{\infty} Q(p')dp' + \frac{1}{2} \int_{-\infty}^{\infty} g(p,p')Q(p')dp' \quad , \tag{3.46}$$

where

$$g(p,p') = g(p',p) \equiv |p - p'| \left[1 - \frac{pp'}{x_A^2} K(p,p') \right] \tag{3.47}$$

is a symmetric kernel; the symmetry of g is essential for an integral equation formulation, which is, in the language of Sec. III.A, for the integral operators in Eq. (3.46) to be self adjoint.

By letting $\rho \to 0$ in Eq. (3.46), i.e., $p \to x$, $K(p,p') \to 1$, we recover exactly Eq. (22) of Ref. 19. The variational principle follows then from Eq. (3.46) as

$$S = \int_{-\infty}^{\infty} \frac{Q^2(p)}{\mu_1} dp - \frac{1}{\Delta'} \left[\int_{-\infty}^{\infty} Q(p)dp \right]^2 + \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(p,p')Q(p)Q(p')dpdp' \quad . \tag{3.48}$$

D. Semicollisional Form

Let us examine the form of the eigenmode equations in the semicollisional form, defined by Eqs. (3.13), (3.14); σ_* can now be approximated by

$$\sigma_* \simeq \mu_1 + \mu_2 x^2 \quad , \tag{3.49}$$

where μ_1 is given by Eq. (3.15) and μ_2 is defined as

$$\mu_2 \equiv \frac{4\pi\varepsilon''' \breve{s}}{\eta} (\omega - \omega_n^*) \frac{k_\parallel'^2 D}{\omega} \quad . \tag{3.50}$$

We note that, when σ_* is approximated as in Eq. (3.49) (unlike when one retains its full x^2 dependence as in Eq. (2.12)), a similar relation to Eq. (3.19) still holds; namely,

$$(\mu_1 + \mu_2 x^2)\Lambda = \mu_1 \lambda = \frac{4\pi k_\perp J_0'}{k_\parallel' B} \quad . \tag{3.51}$$

Making use of the same manipulation used in Section C to cast the eigenmode equations in the form given by Eqs. (3.21), (3.22), we can then write the basic set for the semicollisional case in the following form

$$\psi'' = \sigma_*(x^2)\left\{\psi - [x - \Lambda(x^2)]\phi\right\} \quad , \tag{3.52}$$

$$\phi'' + \frac{\lambda\mu_1}{X_A^2}[x - \Lambda(x^2)]\phi =$$
$$= \frac{\sigma_*(x^2)}{X_A^2}[x - \Lambda(x^2)]\left\{\psi - [x - \Lambda(x^2)]\phi\right\} \quad , \qquad (3.53)$$

where, we repeat, $\sigma_*(x^2)$ is given by its approximated form, Eq. (3.49).

Even though Eqs. (3.52), (3.53) closely resemble Eqs. (3.21), (3.22) the x^2 dependence of Λ makes the procedure of Section C no longer applicable. The following approximation, however, will enable us to reduce the problem to the one solved before. Since the effect of the local equilibrium current shall be treated as a small perturbation of the known semicollisional result, a negligible error will be involved if we substitute x^2 by a suitable, average value over the radial extent of the mode; for the moment, let us denote this average simply as $\langle x^2\rangle$; in Chapter IV, an explicit form for $\langle x^2\rangle$ will be given. $\Lambda(x^2)$ can therefore be approximated by

$$\Lambda \simeq \frac{\rho X_A^2}{(\mu_1 + \mu_2\langle x^2\rangle)} \quad , \qquad (3.54)$$

making use of Eq. (3.25), and the eigenmode equations for the semicollisional case are now given by Eqs. (3.26), (3.27), with

$$Q(p) \equiv \sigma_*(p)(\psi - p\phi) \quad , \qquad (3.55)$$

$$\sigma_*(p) \equiv \mu_1 + \mu_2(p^2 + 2\Lambda p + \Lambda^2) \quad . \qquad (3.56)$$

The mathematical structure of the problem is now identical with the collisional case, the variational functional is therefore given by Eq. (3.48), substituting μ_1 with $\sigma_*(p)$. An alternative form of S is actually most useful for the subsequent calculations; defining

$$E(p) \equiv \frac{Q(p)}{\sigma_*(p)} \quad , \tag{3.57}$$

we obtain

$$S = \int_{-\infty}^{\infty} \sigma_*(p)E^2(p)dp - \frac{1}{\Delta'}\left[\int_{-\infty}^{\infty} \sigma_*(p)E(p)dp\right]^2 + \\ - \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(p,p')\sigma_*(p)\sigma_*(p')E(p)E(p')dpdp' \quad , \tag{3.58}$$

where g(p,p') is given by Eq. (3.48). The corresponding form of Eq. (3.58), Eqs. (13) and (22) of Ref. 19, is obtained as usual by letting $\rho \to 0$; in this case, E(p) becomes the perturbed parallel electric field. Also, letting $\mu_2 \to 0$ in Eq. (3.58), cfr. Eqs. (3.15) and (3.49), one recovers precisely the collisional form of S, Eq. (3.48).

CHAPTER IV

DISPERSION RELATIONS

A. Trial Function

In this chapter, we use the variational principle formulated in Sections III.C and III.D to solve the eigenmode problem by deriving the dispersion relations for a number of different modes, in the collisional and semicollisional regime, respectively. We first show that our formalism easily recovers known results in teh appropriate limit, and then we proceed to study the modifications on the mode dynamics brought by the inclusion of the equilibrium current; we also investigate the possibility of entirely new modes, which exist only in the presence of J_0 and discuss their relevance.

In order to evaluate explicitly the variational functional S, given by Eqs. (3.48), (3.58), respectively, it is necessary to choose a suitable trial function. As it has already been noticed, some a priori knowledge on the form of the solution is needed, if the results of the variational method are to be trusted. In addition to satisfying the symmetry properties of the eigenmode equations, such as parity invariance, an appropriate trial function should take into account as much physical information on the solution as possible; also, mathematical tractability is an important parameter of choice, in the form of a minimum number of variational parameters as well as ease in carrying out the relevant integrals.

In Section III.B, we remarked that it is natural to regard the perturbed parallel electric field as basic field variable. Of particular importance in the variational calculation is the fact that, whenever the conductivity exhibits a spatial scale, the $\tilde{J}_\parallel$ profile is constrained (cfr. Eq.(3.12)) to follow either the σ_* profile or the $\tilde{E}_\parallel$ profile, whichever is narrower, whereas the $\tilde{E}_\parallel$ profile is unconstrained by a particular model of the conductivity. That is why it is convenient to formulate the semicollisional form of S, Eq. (3.58), in terms of E(p) rather than Q(p); in the collisional case, of course, where σ_* is a constant, both variables are perfectly equivalent.

In conventional theories, where J_0 terms are neglected, the eigenmode equations are invariant under space reflection and one can choose a trial function of definite parity; solutions with tearing symmetry (cfr. the discussion in the introduction) have even parity (i.e., $\tilde{J}_\parallel$, $\tilde{E}_\parallel$, ψ even and ϕ odd) and the usual choice[17-19] is

$$E(x)_{trial} = \exp(-\alpha x^2/2) \quad , \tag{4.1}$$

where the variational parameter α measures the radial width of the mode, i.e. $\ell_\omega \sim \alpha^{-\frac{1}{2}}$.

The mode localization condition (cfr. also Eq. (3.2)) is ensured by demanding

$$\mathrm{Re}\alpha > 0 \quad ; \tag{4.2}$$

which we regard as a mathematical consistency condition; such a necessary condition, however, turns out not to be sufficient in some cases of interest. In evaluating S, the relevant integrals are calculated assuming $\mathrm{Re}\alpha^{\frac{1}{2}} > 0$; if the sign of the various terms in the explicit form of S does not depend on $\alpha^{\frac{1}{2}}$ (such as, when S reduces to a polynomial containing only integer powers of α), Eq. (4.2) is the only condition needed; when, however, the balancing of the various terms in S does depend on $\alpha^{\frac{1}{2}}$, we must require also

$$\mathrm{Re}\alpha^{\frac{1}{2}} > 0 \quad . \tag{4.3}$$

Once the dispersion relation has been obtained, any root must satisfy Eqs. (4.2) and (4.3) independently, in order to describe a mathematically consistent solution.

When J_0 terms are included, the space reflection symmetry of the eigenmode equations is broken, and one can no longer choose a trial function with definite parity; since we shall principally be treating the effect of the current as a correction to known results, we demand that the trial function reduces to Eq. (4.1) whenever the J_0 effect is neglected, to ensure continuous contact with previous analyses. The simplest choice for a mixed parity trial function with the required properties is

$$E(p) = (1 + \delta p)\exp(-\alpha p^2/2) \quad , \tag{4.4}$$

with variational parameters α and δ; α is still a measure of the radial mode width and δ, which is evidently proportional to ρ (cfr. Eq. (3.27)), measures the mixing of the even and odd part of the solution, caused by the inclusion of J_0. With the above choice for the trial function, S, as given by Eqs. (3.48) or (3.58), becomes a polynomial in α and δ, and Eqs. (3.7), (3.11) are in this case

$$\frac{\partial S}{\partial \delta} = 0 \quad , \tag{4.5}$$

$$\frac{\partial S}{\partial \alpha} = 0 \quad , \tag{4.6}$$

$$S = 0 \quad . \tag{4.7}$$

As noted, once the above system has been solved, the roots of the dispersion relation must be checked against Eqs. (4.2), (4.3), in order to describe a mathematically consistent solution; no similar restrictions have to be imposed on δ.

B. Collisional Form

The collisional form of S is obtained by substituting Eq. (4.4) into Eq. (3.48); we find, after a somewhat lengthy but straightforward calculation:

$$S = \frac{\alpha^{-1/2}}{\mu_1} + \frac{\delta^2}{2}\frac{\alpha^{-3/2}}{\mu_1} - \frac{2\pi^{1/2}}{\Delta'}\alpha^{-1} +$$
$$- \frac{\alpha^{-5/2}}{X_A^2}(1 + 2\alpha X_A^2) + \frac{2}{3}\delta\rho\frac{\alpha^{-9/2}}{X_A^2} +$$
$$+ \frac{7}{2}\delta^2\frac{\alpha^{-7/2}}{X_A^2}(1 + \frac{2}{7}\alpha X_A^2) - \frac{2}{15}\rho^2\frac{\alpha^{-11/2}}{X_A^2} \quad . \tag{4.8}$$

Notice that it is important to keep all the terms quadratic in δ, ρ, because we know that δ must scale as ρ. Effects due to the local equilibrium current appear then in S as terms of order ρ^2, as it should be expected by the form of Eqs. (3.26), (3.27), (4.4). Indeed, because of the term $\rho p\phi$ in Eq. (3.27), one must choose, as already noted, a trial function with mixed parity, which is used to evaluate the integrals in S, given by Eq. (3.48). In the first two integrals, terms linear in δ are odd in p and in the third integral terms linear in δ (coming from the trial function) or linear in ρ (coming from the expansion of the kernel K(p,p')) similarly vanish since, after the integration variables p and p' are changed to (p + p') and (p - p') to carry out the double integral, they are odd in (p + p'). Only terms quadratic in J_0, i.e., terms in δ^2, $\delta\rho$, ρ^2, contribute to S (we are neglecting higher order corrections due to J_0, such as terms of order ρ^4). That is why one has to expand the kernel K(p,p') (cfr. Eq. (3.41)) to order ρ^2 to include correctly all terms up to order ρ^2 in S.

Before proceeding to study the effects of the local J_0 terms, we show that Eq. (4.8) easily recovers a number of previous known results derived in the absence of local current. By letting $\rho \to 0$, Eq. (4.5) implies $\delta \to 0$ (as, of course, it must be), and we obtain the "classical" form of S:

$$S_{cl.} = \frac{\alpha^{-1/2}}{\mu_1} - \frac{2\pi^{1/2}}{\Delta'}\alpha^{-1} - \frac{\alpha^{-5/2}}{X_A^2}(1 + 2\alpha X_A^2) \quad . \tag{4.9}$$

We recall[23] that low poloidal member tearing modes can be basically grouped in two categories: the $m \geq 2$ modes, such as the $m = 2$ tearing mode, which are characterized by a finite discontinuity of ψ' across the tearing layer, whereas ψ is roughly constant inside it, and the $m = 1$ tearing mode, for which $\psi \to 0$ inside the layer and is therefore characterized by a very large value of Δ'. Although the nature of the magnetic perturbation ψ is very different for the two varieties of modes, the nature of $\tilde{E}_\parallel$ is essentially the same, enabling us to derive the dispersion relation for the two cases by taking appropriate limits of the same expression for S.

The $m \geq 2$ modes are characterized by a radial extent wider than the Alfvén layer X_A and by a finite value of Δ'; the condition $\ell_w > |X_A|$ is equivalent to $|\alpha^{1/2}X_A| < 1$ and therefore the appropriate limit of Eq. (4.9) to describe $m \geq 2$ modes is

$$S_{cl.} \simeq \frac{\alpha^{-1/2}}{\mu_1} - \frac{2\pi^{1/2}}{\Delta'}\alpha^{-1} - \frac{\alpha^{-5/2}}{X_A^2} \quad . \tag{4.10}$$

Solving Eqs. (4.6), (4.7) for S given by Eq. (4.10), we obtain for $\alpha^{1/2}$ and the dispersion relation, respectively,

$$\alpha^{1/2} = \frac{3}{2}\frac{\pi^{1/2}}{\Delta'}\mu_1 \quad ; \tag{4.11}$$

$$\mu_1^3 \chi_A^2 = -\left(\frac{2\Delta'}{3^{3/4}\pi^{1/2}}\right)^4 \quad , \tag{4.12}$$

which can be written in more conventional notation as[17,18,21]

$$\omega(\omega + \omega_i^*)(\omega - \omega_n^* - \hat{\varepsilon}\omega_T^*)^3 = i\left(\frac{2\Delta'}{3^{3/4}\pi^{1/2}}\right)^4 \left(\frac{\eta}{4\pi}\right)^3 (k_\parallel v_A)^2 \quad . \tag{4.13}$$

Although the result expressed by Eq. (4.13) is well known, we analyze it in some detail as an example of the use of Eqs. (4.2), (4.3).

The classical FKR result[1] is obtained by letting $\omega^* \to 0$ and setting $\omega = i\gamma$ in Eq. (4.13); we find

$$\gamma_{FKR} = \left(\frac{2\Delta'}{3^{3/4}\pi^{1/2}}\right)^{4/5} \left(\frac{\eta}{4\pi}\right)^{3/5} (k_\parallel v_A)^{2/5} \quad . \tag{4.14}$$

The above chosen root, $(1)^{1/5} = 1$, is consistent from Eqs. (4.2), (4.3), (4.11) only for $\Delta' > 0$, confirming the well known FKR instability criterion.

The $m \geq 2$ drift tearing result follows from Eq. (4.13) by assuming $\omega^* \gg \gamma$; we find that the real mode frequency is approximately given by

$$\omega_0 \simeq \omega_n^* + (1 + \varepsilon)\omega_T^* \quad , \tag{4.15}$$

whereas the total growth rate is given by

$$\gamma \simeq \varepsilon\varepsilon' \frac{\omega_0}{\nu} \omega_T^* - \frac{\gamma_{FKR}^{5/3}}{\omega_0^{1/3}(\omega_0 + \omega_i^*)^{1/3}} \quad . \tag{4.16}$$

The first term on the r.h.s. of Eq. (4.16) is the thermoelectric growth rate,[15] which in the present fluid theory derivation is due to the inclusion of the time dependent thermal force in the electron momentum balance equation.[21] The source of free energy for this instability, which was originally found using kinetic theory,[15] is in the electron temperature gradient. The second term provides a small damping contribution to the overall growth rate[30]; this choice for the root $(-1)^{1/3} = -1$ is the only possible, since the growing μ_1 roots $e^{\pm i\pi/3}$ have $Re\alpha < 0$, as can be seen from Eq. (4.11), and are therefore not consistent. Also, we note that the validity of Eq. (4.3) requires $\Delta' < 0$ in this case.

In the drift tearing regime, therefore, the tearing mode can be linearly unstable only because of the thermoelectric part of the growth rate and the mode can no longer tap the magnetic free energy made available by the magnetic activity in the external, kinked region. It is perhaps worthwhile noting that this feature of the collisional drift tearing mode does not seem to be properly appreciated in some of the relevant literature.

The dispersion relation for the m = 1 tearing mode, which is characterized by a very large value of Δ', can be obtained from Eq. (4.9) by formally letting $\Delta' \to \infty$; S becomes

$$S_{cl.} \simeq \frac{\alpha^{-1/2}}{\mu_1} - \frac{\alpha^{-5/2}}{X_A^2} - 2\alpha^{-3/2} \quad , \tag{4.17}$$

from which we find α and the dispersion relation to be given respectively by

$$\alpha X_A^2 = -1 \quad , \tag{4.18}$$

and

$$\mu_1 X_A^2 = -1 \quad , \tag{4.19}$$

which can be written as[17,18]

$$\omega(\omega + \omega_i^*)(\omega - \omega_n^* - \hat{\varepsilon}\omega_T^*) = -i\left(\frac{\eta}{4\pi}\right)(k_\parallel' v_A)^2 \quad . \tag{4.20}$$

Since S does not depend in this case explicitly on $\alpha^{1/2}$, we simply must require the validity of Eq. (4.2). The classical m = 1 tearing mode growth rate is obtained by letting $\omega^* \to 0$ and setting $\omega = i\gamma$ in Eq. (4.20); we obtain the well known result[31]

$$\gamma = \left(\frac{\eta}{4\pi}\right)^{1/3}(k_\parallel' v_A)^{2/3} \quad . \tag{4.21}$$

Since $X_A^2 = -\gamma^2/(k_\parallel' v_A)^2$ in this regime, the chosen root, $(1)^{1/3} = 1$, is consistent having $\mathrm{Re}\alpha > 0$. We note that Eq. (4.18) implies that Eq. (4.2) cannot be satisfied for a drift tearing-type solution with $\omega^* \gg \gamma$. This property of the classical m = 1 tearing mode has been noticed.[32]

We now show that the scaling $|\alpha^{1/2} X_A| > 1$, corresponding to a mode radial width narrower than the Alfvén layer X_A, does not lead to a significant result. In this limit, S reduces to

$$S_{cl.} \simeq \frac{\alpha^{-1/2}}{\mu_1} - \frac{2\pi^{1/2}}{\Delta'} \alpha^{-1} - 2\alpha^{-3/2} \quad . \tag{4.22}$$

The corresponding $\alpha^{1/2}$ and dispersion relation are found, as usual, by solving Eqs. (4.6), (4.7); $\alpha^{1/2}$ is given by

$$\alpha^{1/2} = \frac{\pi^{1/2}}{\Delta'} \mu_1 \quad , \tag{4.23}$$

and the dispersion relation is

$$\mu_1 = -2\left(\frac{\Delta'}{\pi^{1/2}}\right)^2 \quad , \tag{4.24}$$

or, equivalently,

$$\omega - \omega_n^* - \hat{\varepsilon}\omega_T^* = -i \frac{\eta}{2\pi^2} (\Delta')^2 \quad . \tag{4.25}$$

The mode described by Eq. (4.25) has not been discussed in the literature, to the best of our knowledge. From Eq. (4.23), the mode is consistent for $\Delta' < 0$ and can therefore be driven unstable when the thermoelectric part of the growth rate is larger than the r.h.s. of Eq. (4.25). We remark, however, that this result is at best of academic interest, since the condition $|\alpha^{1/2}X_A| > 1$ corresponds to $|\Delta' X_A| > 1$, as can be seen from Eqs. (4.23) and (4.24); such a condition is manifestly not satisfied for typical tokamak parameters, where low poloidal mode numbers have $|\Delta' X_A| << 1$.

Equation (4.25) can therefore be said to describe a physically inconsistent mode, since although the result is mathematically consistent in the sense of Eqs. (4.2), (4.3), low poloidal numbers tearing modes simply do not satisfy the scaling used in deriving the above result. Such a distinction between physical and mathematical consistency is more semantic than practical and, in fact, the basic conclusion is that the result expressed by Eq. (4.25) has to be rejected as a real mode.

We now begin to investigate the effect of J_o on the modes discussed; to describe the m ≥ 2 tearing mode in the presence of current, let us take the limit of S, Eq. (4.8), for $|\alpha^{1/2}X_A| < 1$. This is given by

$$S \approx \frac{\alpha^{-1/2}}{\mu_1} - \frac{\alpha^{-5/2}}{X_A^2} - \frac{2\pi^{1/2}}{\Delta'}\alpha^{-1} + + \delta^2\left(\frac{\alpha^{-3/2}}{2\mu_1} + \frac{7}{2}\frac{\alpha^{-7/2}}{X_A^2}\right) + \frac{2}{3}\delta\rho\frac{\alpha^{-9/2}}{X_A^2} + - \frac{2}{15}\rho^2\frac{\alpha^{-11/2}}{X_A^2} \quad . \tag{4.26}$$

The variational parameter δ is determined by solving Eq. (4.5), which gives

$$\delta = -\frac{2}{3}\frac{\rho\mu_1}{7\alpha\mu_1 + \alpha^3 X_A^2} \quad ; \tag{4.27}$$

in principle, one should substitute this expression for δ in Eqs. (4.6), (4.7) to obtain the dispersion relation, but we can derive the correction to the mode dynamics due to J_0 in a very simple way, if we treat the ρ^2 terms as a perturbation of the result derived in absence of current, i.e., Eqs. (4.11), (4.12). This approach is consistent with the previous calculations, where we have already assumed the effect of the current to be small, expanding the Bessel functions in Eq. (3.39) for small argument.

Since S is a variational quantity, the perturbative calculation is of the outmost simplicity; indeed, if we write Eq. (4.26) as

$$S = S_0 + \varepsilon^2 S_1 \quad , \tag{4.28}$$

where S_0 is the corresponding "classical" S, given by Eq. (4.10), and $\varepsilon^2 << 1$ is proportional to ρ^2, and we assume

$$\alpha = \alpha_0 + \varepsilon^2 \alpha_1 \quad , \tag{4.29}$$

with α_0 given by Eq. (4.11), we have

$$S \simeq S_0(\alpha_0) + \left.\frac{\partial S_0}{\partial \alpha}\right|_{\alpha_0} \varepsilon^2 \alpha_1 +$$

$$+ \varepsilon^2 S_1(\alpha_0) =$$

$$= S_0(\alpha_0) + \varepsilon^2 S_1(\alpha_0) \quad , \qquad (4.30)$$

where only terms up to order ε^2 have been kept, and we made use of the extremum condition on S_0. Therefore, there is no need of determining α from Eq. (4.4), and the dispersion relation, correct to order ρ^2, is obtained by substituting Eq. (4.27) into Eq. (4.26), using Eqs. (4.11) and (4.12) in terms of order ρ^2, and setting the resulting expression, which is of the form of Eq. (4.30), equal to zero. We then obtain the dispersion relation showing the leading order effect of the equilibrium current on the m ≥ 2 tearing mode,

$$\mu_1^3 \chi_A^2 = -\left(\frac{2\Delta'}{3^{3/4}\,\pi^{1/2}}\right)^4 \left(1 + \frac{17}{90}\,\rho^2 \alpha^{-3}\right) \quad . \qquad (4.31)$$

By recalling that the variational parameter α measures the radial mode width, $\ell_w \sim \alpha^{-1/2}$, we see that the J_0 term in Eq. (4.31) is of order ζ^4, where ζ, given by eq. (3.36), has been treated as a small quantity in the derivation of S. Thus, the validity of our derivation in Chapter III ensures the validity of the present perturbative calculation and, using Eqs. (3.25), (4.11), (4.12), we can write Eq. (4.31) as

$$\omega(\omega + \omega_i^*)(\omega - \omega_n^* - \hat{\varepsilon}\omega_T^*)^3 =$$

$$= i\left(\frac{2\Delta'}{3^{3/4}\,\pi^{1/2}}\right)^4\left(\frac{\eta}{4\pi}\right)^3 (k_\parallel' v_A)^2\left[1 + \frac{17\pi}{360}\left(\frac{4\pi k_\perp J_0'}{k_\parallel' B\Delta'}\right)^2\right] \quad . \tag{4.32}$$

If we neglect the J_0' term in Eq. (4.32) we, of course, recover the known result, Eq. (4.13); as remarked, the validity of our derivation require the J_0' term to be a small correction, i.e.,

$$\left|\frac{4\pi k_\perp J_0'}{k_\parallel' B\,\Delta'}\right| < 1 \quad . \tag{4.33}$$

For the Text tokamak, estimating the relevant plasma parameters[33] at approximately two-thirds of the minor radius, where for example the q = 2 mode rational surface is located (q is the safety factor), the quantity $|4\pi k_\perp J_0' a/k_\parallel' B|$, where a is the plasma radius, is of the order of 1/3; since $|\Delta' a|$ is typically greater than one,[25] the above condition is usually well satisfied.

In the drift tearing limit of Eq. (4.32), $\gamma << \omega^*$, we see that the local current gradient enhances stabilization of the $m \geq 2$ drift tearing mode, since, as discussed after Eq. (4.16), the r.h.s. of Eq. (4.13) provide a small damping to the overall growth rate. The "pure" FKR result, obtained by letting $\omega_* \to 0$ in Eq. (4.13), is not altered by the local current term which, as remarked, is directly proportional to ω_T^*. If, however, we simply consider the limit $\gamma >> \omega^*$ of Eq. (4.32), we see that the leading order effect of the J_0' term is to (slightly) increase the growth rate given by Eq. (4.14).

To examine the effect of the local current gradient on the $m = 1$ tearing mode, let us take the limit of S for $\Delta' \to \infty$. We obtain

$$S \simeq \frac{\alpha^{-1/2}}{\mu_1} - \frac{\alpha^{-5/2}}{X_A^2}(1 + 2\alpha X_A^2) + \frac{\delta^2}{2}\frac{\alpha^{-3/2}}{\mu_1} +$$

$$+ \frac{7}{2}\delta^2 \frac{\alpha^{-7/2}}{X_A^2}(1 + \frac{2}{7}\alpha X_A^2) + \frac{2}{3}\delta\rho\frac{\alpha^{-9/2}}{X_A^2} +$$

$$- \frac{2}{15}\rho^2\frac{\alpha^{-11/2}}{X_A^2} \quad . \tag{4.34}$$

Solving Eq. (4.5), we find that δ is given by

$$\delta = - \frac{\frac{2}{3}\frac{\rho}{\alpha X_A^2}}{\left(\frac{\alpha^2}{\mu_1} + \frac{7}{X_A^2} + 2\alpha\right)} \quad ; \tag{4.35}$$

by proceeding as in the previous case, namely, treating the ρ^2 terms in Eq. (4.34) as a perturbation to the result expressed by Eqs. (4.18), (4.19), we find

$$\mu_1 X_A^2 = - \frac{1}{\left(1 - \frac{17}{90}\rho^2\alpha^{-2}\right)} \quad , \tag{4.36}$$

where, analogously to the $m \geq 2$ case, the local current effect turns to be of order ζ^4, confirming the validity of the perturbative calculation. In this case, α is given by Eq. (4.18) and is, of course, different from the α appearing in the $m \geq 2$ result given by Eq. (4.31).

By substituting for α we find

$$\omega(\omega + \overset{*}{\omega}_i)(\omega - \overset{*}{\omega}_n - \hat{\varepsilon}\overset{*}{\omega}_T) =$$

$$= \frac{-i\left(\frac{n}{4\pi}\right)(k_\parallel' v_A)^2}{\left[1 + \frac{17}{90}\left(\frac{4\pi k_\perp J_0'}{k_\parallel' B}\right)^2 \frac{(\omega + \overset{*}{\omega}_i)\omega}{(k_\parallel' v_A)^2}\right]} \quad ; \qquad (4.37)$$

The condition for the validity of the derivation in this case can be written as

$$\left|\frac{4\pi k_\perp J_0' X_A}{k_\parallel' B}\right| < 1 \quad , \qquad (4.38)$$

where X_A must be evaluated with γ given by Eq. (4.21) for the classical $m = 1$ tearing mode. For the Text tokamak, estimating the relevant plasma parameters near the location of the $q = 1$ rational surface,[33] the small parameter in Eq. (4.38) is of the order of $5 \cdot 10^{-3}$, making the effect of J_0 on the $m = 1$ mode very small.

Equation (4.20) is obtained by neglecting the J_0' term in Eq. (4.37); the modification to the classical $m = 1$ result due to J_0' is found by substituting $\omega \simeq i\gamma$ in Eq. (4.37), which then shows that the growth rate given by Eq. (4.21) is (very slightly) increased. Since, as remarked in the discussion of Eqs. (4.20), (4.21), the m = 1 tearing mode is inconsistent in the drift-tearing limit, we cannot discuss the limit of Eq. (4.37) for $\omega^* \gg \gamma$.

Having examined the effect of J_o on the standard $m \geq 2$ and the m = 1 tearing modes, it will be interesting to determine whether the presence of J_o makes the scaling $|\alpha^{1/2} X_A| > 1$, i.e. a mode with a radial width narrower than the Alfvén layer, possible. In this limit, S becomes:

$$S \simeq \frac{\alpha^{-1/2}}{\mu_1} - 2\alpha^{-3/2} - \frac{2\pi^{1/2}}{\Delta'} \alpha^{-1} + \frac{\delta^2}{2} \frac{\alpha^{-3/2}}{\mu_1} + \delta^2 \alpha^{-5/2} + \frac{2}{3} \delta\rho \frac{\alpha^{-9/2}}{X_A^2} - \frac{2}{15} \rho^2 \frac{\alpha^{-11/2}}{X_A^2} \quad . \tag{4.39}$$

From Eq. (4.5), we find that δ is given by

$$\delta = -\frac{2}{3} \frac{\rho}{\alpha^2 X_A^2} \left(\frac{\mu_1}{\alpha + 2\mu_1} \right) \quad ; \tag{4.40}$$

we remark that the mode described by Eqs. (4.23), (4.24) is characterized by $\alpha = -2\mu_1$. From Eq. (4.40), it appears that the result given by Eq. (4.25) cannot simply be extended to the case in which equilibrium current terms are retained; Eq. (4.25) is valid only for $\rho = 0$. Furthermore, we already concluded that the result expressed by Eq. (4.25) was not relevant since it required $|\Delta' X_A| > 1$ in order to be physically consistent. It is therefore appropriate to proceed, in the present case, without treating terms in ρ^2 as a perturbation. To avoid misunderstandings, we do emphasize that the current terms are always "small" in our work, since all the forms of S have been derived by the method of Section IIIC, assuming the small ζ expansion; we note however that the dispersion relation in many cases of interest follows by balancing terms in S which are indeed small (e.g., the $m \geq 2$ result follows from balancing $\Delta' X_A$ against $\alpha^{3/2} X_A^3$). In the present case, therefore, we are looking for a new dominant balance in S in the limit $|\alpha^{1/2} X_A| > 1$, when J_o is present, and find out if it is possible to obtain a consistent result.

Since we are not treating ρ^2 terms as a small perturbation, we must now solve Eqs. (4.5), (4.6), (4.7) independently; substituting δ from Eq. (4.40) into Eqs. (4.6), (4.7) we obtain

$$\frac{\alpha^{-1/2}}{\mu_1} - 2\alpha^{-3/2} - \frac{2\pi^{1/2}}{\Delta'}\,\alpha^{-1} - \frac{2}{15}\,\rho^2\,\frac{\alpha^{-11/2}}{X_A^2} +$$

$$- \frac{2}{9}\,\rho^2\,\frac{\alpha^{-13/2}}{X_A^4}\left(\frac{\mu_1}{2\mu_1 + \alpha}\right) = 0 \quad ; \qquad (4.41)$$

$$- \frac{1}{2} \frac{\alpha^{-1/2}}{\mu_1} + 3\alpha^{-3/2} + \frac{2\pi^{1/2}}{\Delta'} \alpha^{-1} +$$

$$+ \frac{11}{15} \rho^2 \frac{\alpha^{-11/2}}{X_A^2} + \frac{4}{3} \rho^2 \frac{\alpha^{-13/2}}{X_A^4} \left(\frac{\mu_1}{2\mu_1 + \alpha}\right) +$$

$$- \frac{4}{9} \rho^2 \frac{\alpha^{-13/2}}{X_A^4} \left(\frac{\mu_1}{2\mu_1 + \alpha}\right)^2 = 0 \quad . \qquad (4.42)$$

Equations (4.41), (4.42) are, however, still too complicated for a simple analytical solution; this is obtained by assuming

$$\left| \frac{\alpha}{\mu_1} \right| > 1 \quad , \qquad (4.43)$$

which is equivalent to looking for a solution describing a radially narrow mode, i.e., large α, in agreement with the condition $|\alpha^{1/2} X_A| > 1$, or a mode with a small value for μ_1, or both. We remark that we have not been able to find a sensible answer for a scaling opposite to the one given by Eq. (4.43). Eqs. (4.41), (4.42) therefore reduce to

$$\frac{\alpha^{-1/2}}{\mu_1} - \frac{2\pi^{1/2}}{\Delta'} \alpha^{-1} - \frac{2}{15} \rho^2 \frac{\alpha^{-11/2}}{X_A^2} = 0 \quad ; \qquad (4.44)$$

$$- \frac{1}{2} \frac{\alpha^{-1/2}}{\mu_1} + \frac{2\pi^{1/2}}{\Delta'} \alpha^{-1} + \frac{11}{15} \rho^2 \frac{\alpha^{-11/2}}{X_A^2} = 0 \quad , \qquad (4.45)$$

which allows a simple analytical solution for the eigenmode problem;

let us also note that the ρ^2 term is now critical for the balancing in S and that the result depends crucially on the inclusion of J_o terms. From the above equations we obtain

$$\alpha^{1/2} = \frac{9}{5} \frac{\pi^{1/2}}{\Delta'} \mu_1 \quad , \tag{4.46}$$

and

$$\mu_1^9 X_A^6 = - \frac{2}{3} \left(\frac{5}{9}\right)^9 \left(\frac{4\pi k_\perp}{k_\parallel' B} J_o'\right)^2 \left(\frac{\Delta'}{\pi^{1/2}}\right)^{10} \quad , \tag{4.47}$$

or

$$\omega^3(\omega + \omega_i^*)^3(\omega - \omega_n^* - \hat{\varepsilon}\omega_T^*)^9 =$$

$$= -i \frac{2}{3} \left(\frac{5}{9}\right)^9 \left(\frac{4\pi k_\perp}{k_\parallel' B} J_o'\right)^2 \left(\frac{\Delta'}{\pi^{1/2}}\right)^{10} \left(\frac{\eta}{4\pi}\right)^9 (k_\parallel' v_A)^6 \quad . \tag{4.48}$$

We note that, as expected, this new result is made possible by the presence of J_o'; also, a mathematically consistent solution, corresponding to a growing root, is given by Eqs. (4.46), (4.48) by trying a drift-tearing type solution, $\gamma << \omega^*$, and choosing the $(-1)^{1/9} = e^{\pm i\pi/9}$ root for μ_1. In this case, both the thermoelectric part, the $i\varepsilon\varepsilon'\omega/\nu$ factor in $\hat{\varepsilon}$, as well as the r.h.s. of Eq. (4.48) provide growth. However, we must reject the above result as physically inconsistent, on the base of the following argument. In deriving Eq. (4.48) we demanded $|\alpha^{1/2} X_A| > 1$, $|\alpha/\mu_1| > 1$ and the validity for the small ζ expansion in the derivation of S can be expressed, as already noted, by the condition $|\rho^{1/2}\alpha^{-3/4}| < 1$. Substituting the appropriate expressions for α and μ_1, the above conditions are expressed as

$$\left| \left(\frac{2}{3\pi^{1/2}}\right)^{1/9} \left(\frac{4\pi k_\perp J_0'}{k_\parallel' B\Delta'}\right)^{2/9} (\Delta' X_A)^{1/3} \right| \gg 1 \quad , \tag{4.49}$$

$$\left| \frac{9}{5} \left(\frac{2\pi^4}{3}\right)^{1/9} \left(\frac{4\pi k_\perp J_0'}{k_\parallel' B\Delta'}\right)^{2/9} (\Delta' X_A)^{-2/3} \right| > 1 \quad , \tag{4.50}$$

$$\left| \left(\frac{3\pi^{1/2}}{2}\right)^{1/6} \left(\frac{4\pi k_\perp J_0'}{k_\parallel' B\Delta'}\right)^{1/6} \right| < 1 \quad . \tag{4.51}$$

Equations (4.49), (4.50), (4.51) form a consistent set of conditions only for $2.61 < |\Delta' X_A| < 3.50$, which makes the result given by Eq. (4.48) quite irrelevant, as remarked. It is perhaps worth noting that Eq. (4.51), demanding the validity of the small ζ argument expansion, and Eq. (4.50), expressing the simplifying assumption given by Eq. (4.43), are usually well satisfied. It is Eq. (4.49), expressing the condition $|\alpha^{1/2} X_A| > 1$, which cannot possibly be satisfied.

We do stress that not only the condition $|\Delta' X_A| > 1$ is manifestly violated for low poloidal number modes, but also the corresponding region of parameter space is so limited, that the mode described by Eq. (4.48) must be rejected as physically inconsistent.

We therefore conclude that the scaling $|\alpha^{1/2} x_A| > 1$, already shown not to describe modes of relevance in the absence of local current, is similarly not important whenever J_0 terms are included in the analysis.

C. Semicollisional Form

Let us now consider the semicollisional scaling given by Eqs. (3.13), (3.14), which has been discussed in detail in Section III.B. The main additional effect on the mode dynamics is, as noted, the inclusion of the parallel thermal conduction.

The appropriate form of the variational for the present case is provided by Eq. (3.58), evaluated with the trial function given by Eq. (4.4), which yields, after a somewhat lengthy calculation,

$$S = \mu_1 \left(\alpha^{-1/2} + \frac{\delta^2 \alpha^{-3/2}}{2} \right) +$$

$$+ \mu_2 \left(\frac{\alpha^{-3/2}}{2} + \frac{3}{4} \delta^2 \alpha^{-5/2} + 2\Lambda\delta\alpha^{-3/2} + \Lambda^2 \alpha^{-1/2} \right) +$$

$$- \frac{2\pi^{1/2}}{\Delta'} \alpha^{-1} [\mu_1 + \mu_2 (\alpha^{-1} + 2\Lambda\delta\alpha^{-1} + \Lambda^2)]^2 +$$

$$- \mu_1^2 \left(2\alpha^{-3/2} + \frac{\alpha^{-5/2}}{x_A^2} \right) +$$

$$- \mu_1 \mu_2 \left(6\alpha^{-5/2} + \frac{5\alpha^{-7/2}}{x_A^2} \right) +$$

$$
\begin{aligned}
&- \mu_2^2 \left(\frac{7}{2}\, \alpha^{-7/2} + \frac{27}{4} \frac{\alpha^{-9/2}}{X_A^2} \right) + \\
&+ \delta^2 \left[\mu_1^2 \left(\alpha^{-5/2} + \frac{7}{2} \frac{{}^{-7/2}}{X_A^2} \right) + \right. \\
&\qquad + \mu_1 \mu_2 \left(5\alpha^{-7/2} + \frac{49}{2} \frac{\alpha^{-9/2}}{X_A^2} \right) + \\
&\qquad \left. + \mu_2^2 \left(\frac{27}{4}\, \alpha^{-9/2} + \frac{321}{8} \frac{\alpha^{-11/2}}{X_A^2} \right) \right] + \\
&+ 4\delta\Lambda \left[\mu_1 \mu_2 \left(-2\alpha^{-5/2} + \frac{\alpha^{-7/2}}{X_A^2} \right) + \right. \\
&\qquad \left. + \mu_2^2 \left(-\alpha^{-7/2} + \frac{11}{2} \frac{\alpha^{-9/2}}{X_A^2} \right) \right] + \\
&+ \Lambda^2 \left[-2\mu_1 \mu_2 \left(2\alpha^{-3/2} + \frac{\alpha^{-5/2}}{X_A^2} \right) + \right. \\
&\qquad \left. + \mu_2^2 \left(-2\alpha^{-5/2} + \frac{9\alpha^{-7/2}}{X_A^2} \right) \right] + \\
&+ \frac{2}{3} \frac{\delta\rho}{X_A^2} \left(\mu_1^2 \alpha^{-9/2} + 9\mu_1 \mu_2 \alpha^{-11/2} + \frac{51}{4}\, \mu_2^2 \alpha^{-13/2} \right) + \\
&+ \frac{4}{3} \frac{\Lambda\rho}{X_A^2} \left(\mu_1 \mu_2 \alpha^{-9/2} - \frac{15}{2}\, \mu_2^2 \alpha^{-11/2} \right) + \\
&+ \frac{\rho^2}{X_A^2} \left(- \frac{2}{15}\, \mu_1^2 \alpha^{-11/2} - \frac{22}{15}\, \mu_1 \mu_2 \alpha^{-13/2} + \frac{3}{10}\, \mu_2^2 \alpha^{-15/2} \right) \quad . \qquad (4.52)
\end{aligned}
$$

In Eq. (4.52), we recall, α and δ are variational parameters, μ_1 is given by Eq. (3.15), μ_2 is given by Eq. (3.50), the local equilibrium current appears linearly in ρ and Λ, given by Eqs. (3.25) and (3.54), respectively. We notice that, as in the collisional case, the corrections due to the local current appear in S as terms quadratic in ρ since δ must be of order ρ itself. Terms of higher order in ρ (e.g., terms of fourth order in J_o such as $\delta^2\rho^2$ or $\delta^2\Lambda^2$) have been systematically ignored in the calculation.

We notice that, by letting $\mu_2 \to 0$ in Eq. (4.52), one recovers the collisional limit of S, Eq. (4.8), as it should be expected from the form of Eqs. (3.15), (3.56). Also, from the previous derivation, it appears that we can choose for the average value of x^2, $\langle x^2 \rangle$, appearing in Λ its average over a gaussian trial function, which is

$$\Lambda \simeq \frac{\rho X_A^2}{\mu_1 + (\mu_2/2\alpha)} \quad . \tag{4.53}$$

In analogy to the collisional case, we first proceed to show that S, given by Eq. (4.52), recovers the known result in the absence of the equilibrium current, in the appropriate limit. The "classical" form of S is obtained by neglecting terms proportional to J_o in Eq. (4.52); also, since in the present section we shall only be interested in modes which are broader than the Alfvén layer, we consider the limit of S for $|\alpha^{1/2}X_A| < 1$:

$$S_{cl.} \simeq \mu_1\alpha^{-1/2} + \mu_2 \frac{\alpha^{-3/2}}{2} +$$

$$- \frac{2\pi^{1/2}}{\Delta'} \alpha^{-1}(\mu_1 + \mu_2\alpha^{-1})^2 +$$

$$- \mu_1^2 \frac{\alpha^{-5/2}}{X_A^2} - 5\mu_1\mu_2 \frac{\alpha^{-7/2}}{X_A^2} +$$

$$- \frac{27}{4} \mu_2^2 \frac{\alpha^{-9/2}}{X_A^2} = 0 \quad . \tag{4.54}$$

The above form of S is still too complicated for a simple analytical solution, but it is possible to obtain a quite accurate answer by means of the following argument. In Appendix D, we present an exact variational calculation based on the method of Ref. 18, which allows a very simple treatment of the semicollisional conductivity given by Eq. (3.49) but it is not extendable to the case in which J_o terms are included. We remark that the dispersion relations obtained by the two methods differ only by a few percent in the numerical factor in front of the damping rate.

Since the mode which we are going to discuss is characterized by a negligible value of Δ',[22] the corresponding term in S containing Δ' will be negligible, provided

$$\alpha = - \frac{\mu_2}{\mu_1} \quad , \tag{4.55}$$

which, substituted in Eq. (4.54) and setting the resulting expression for S equal to zero, yields the dispersion relation

$$\mu_1^3 = \frac{2}{11} \mu_2^2 \chi_A^2 \quad , \tag{4.56}$$

which can be written as

$$(\omega - \omega_n^* - \hat{\varepsilon}\omega_T^*)^3 = -i \frac{2}{11} \left(\frac{\eta}{4\pi}\right) [\varepsilon''' s(\omega - \omega_n^*) k_\parallel' D]^2 \frac{(\omega + \omega_i^*)}{\omega v_A^2} \quad , \tag{4.57}$$

and which, for a drift tearing-type solution, $\omega^* >> \gamma$, gives[22]

$$\omega \simeq \omega_0 + i\varepsilon\varepsilon' \frac{\omega_0}{\nu} \omega_T^* + $$
$$+ e^{7\pi i/6} \left(\frac{2}{11}\right)^{1/3} \left(\frac{\eta}{4\pi}\right)^{1/3} (\varepsilon''' s\hat{\varepsilon}\omega_T^* k_\parallel' D)^{2/3} \frac{\left(1 + \frac{\omega_i^*}{\omega_0}\right)^{1/3}}{v_A^{2/3}} \quad , \tag{4.58}$$

where the real mode frequency ω_0 is given by Eq. (4.15).

The first imaginary term on the r.h.s. of Eq. (4.58) is the usual thermoelectric growth rate (as in the collisional drift tearing result), while the last term provides a damping of the mode due to the stabilizing effect of the parallel thermal conduction. This dispersion relation corresponds to the root $(1)^{1/3} = e^{2\pi i/3}$ for μ_1, which is mathematically consistent, as it can be seen from Eq. (4.55).

We now investigate the effect of J_0 on the result expressed by Eqs. (4.55), (4.56), by treating the effect of the local current as a perturbation, in analogy to the discussion of the collisional regime in Section IV.B. The appropriate form of S is obtained by taking the

$|\alpha^{1/2} X_A| < 1$ limit of Eq. (4.52), which gives

$$S \simeq \mu_1 \left(\alpha^{-1/2} + \delta^2 \frac{\alpha^{-3/2}}{2}\right) +$$

$$+ \mu_2 \left(\frac{\alpha^{-3/2}}{2} + \frac{3}{4} \delta^2 \alpha^{-5/2} + 2\Lambda\delta\alpha^{-3/2} + \Lambda^2 \alpha^{-1/2}\right) +$$

$$- \mu_1^2 \frac{\alpha^{-5/2}}{X_A^2} - 5\mu_1\mu_2 \frac{\alpha^{-7/2}}{X_A^2} - \frac{27}{4} \mu_2^2 \frac{\alpha^{-9/2}}{X_A^2} +$$

$$+ \frac{\delta^2}{X_A^2} \left(\frac{7}{2} \mu_1^2 \alpha^{-7/2} + \frac{49}{2} \mu_1\mu_2 \alpha^{-9/2} + \frac{321}{8} \mu_2^2 \alpha^{-11/2}\right) +$$

$$+ \frac{4\delta\Lambda}{X_A^2} \left(\mu_1\mu_2\alpha^{-7/2} + \frac{11}{2} \mu_2^2 \alpha^{-9/2}\right) +$$

$$+ \frac{\Lambda^2}{X_A^2} (-2\mu_1\mu_2\alpha^{-5/2} + 9\mu_2^2\alpha^{-7/2}) +$$

$$+ \frac{2}{3} \frac{\delta\rho}{X_A^2} \left(\mu_1^2 \alpha^{-9/2} + 9\mu_1\mu_2\alpha^{-11/2} + \frac{51}{4} \mu_2^2 \alpha^{-13/2}\right) +$$

$$+ \frac{4}{3} \frac{\Lambda\rho}{X_A^2} \left(\mu_1\mu_2\alpha^{-9/2} - \frac{15}{2} \mu_2^2 \alpha^{-11/2}\right) +$$

$$+ \frac{\rho^2}{X_A^2} \left(- \frac{2}{15} \mu_1^2 \alpha^{-11/2} - \frac{22}{15} \mu_1\mu_2\alpha^{-13/2} + \frac{3}{10} \mu_2^2 \alpha^{-15/2}\right) \quad ; \qquad (4.59)$$

the term involving Δ' has been neglected since, with α given by Eq. (4.55), it provides a correction of order ρ^4 (i.e., of fourth order in J_o) to S.

The perturbative calculation for Eq. (4.59) is carried out in as much the same way as in the previous collisional cases; after δ has been obtained by solving Eq. (4.5), we set S equal to zero, substituting α from Eq. (4.55) and making use of Eq. (4.56) in evaluating terms of order ρ^2. We then obtain, after some algebra,

$$\mu_1^3 = \frac{2}{11} \frac{\mu_2^2 \chi_A^2}{(1 - 164\, \rho^2 \alpha^{-3})} , \qquad (4.60)$$

or, making use again of Eqs. (4.55), (4.56),

$$(\omega - \omega_n^* - \hat{\varepsilon}\omega_T^*)^3 = \frac{-i\, \frac{2}{11} \left(\frac{n}{4\pi}\right) [\varepsilon''' s(\omega - \omega_n^*) k_\parallel' D]^2 \, \frac{(\omega + \omega_i^*)}{\omega v_A^2}}{\left\{ 1 + \frac{30 \left(\frac{n}{4\pi}\right) \left(\frac{4\pi k_\perp J_0'}{k_\parallel' B}\right)^2}{\left[\varepsilon''' s(\omega - \omega_n^*) D \, \frac{(\omega + \omega_i^*)}{v_A^2}\right]} \right\}} \qquad (4.61)$$

Equation (4.61) provides the modification to the result of Eq. (4.57) due to J_0'; we note from the form of Eq. (4.60) that the effect of the equilibrium current turns out to be, as in the collisional case treated previously, of order ζ^4, ζ being the small, current dependent, expansion parameter given by Eq. (3.36).

In the present case, the validity of the perturbative calculation is ensured by the condition

$$\left| \frac{4\pi k_\perp J_o'}{k_\parallel' B \mu_2^{1/2} x_A} \right| < 1 \quad ; \qquad (4.62)$$

for typical Text parameters, evaluated near the location of the q = 2 rational surface,[32] the small quantity of Eq. (4.62) is of the order of $2 \cdot 10^{-2}$. Therefore, even though the numerical factors in front of the J_o corrections in Eqs. (4.60), (4.61) are rather large, the equilibrium current does modify very little the result expressed by Eq. (4.57).

For a drift type solution, $\omega^* \gg \gamma$, recalling that the mathematically consistent root of the dispersion relation is the one shown in Eq. (4.58), we conclude that the inclusion of J_o results in (slightly) decreasing the damping contribution given by the r.h.s. of Eq. (4.61).

CHAPTER V

CONCLUSIONS

In the present dissertation, the local effect of the equilibrium parallel current J_0 on the stability properties of several low poloidal number tearing modes has been investigated analytically.

The mode dispersion relations are obtained by solving the two relevant eigenmode equations, which are derived from fluid theory; the first equation is a statement of the plasma quasineutrality condition, the second is Ampére's law with the perturbed parallel current expressed in terms of the perturbed electrostatic potential ϕ and the perturbed parallel vector potential ψ.

The equilibrium current enters the first eigenmode equation through its gradient, in the so-called "kink term", whereas J_0 appears in Ampére's law through resistivity perturbations in the "Ohm's law" provided by the electron momentum balance equation. Assuming the equilibrium parallel electric field to be spatially constant, both J_0 terms are identical for two different forms of the generalized conductivity and have the form of local current gradients, directly related to the local electron temperature gradient, because of the $T_e^{3/2}$ dependence of the Spitzer-Braginskii conductivity in the equilibrium Ohm's law.

The inclusion of J_0 alters considerably the symmetry of the eigenmode equations, so that several method of solutions derived in the absence of equilibrium current are no longer applicable. Our method of solution, presented in Sections III.C, III.D, suitably combines the two eigenmode equations to obtain an integral equation with a symmetric kernel for the perturbed parallel current $\tilde{J}_{\parallel}$ (or, alternatively, the perturbed parallel electric field $\tilde{E}_{\parallel}$), from which a variational principle can be formulated; this formulation is a generalization of the method of Ref. 19, whose results are precisely reproduced when J_0 terms are correspondingly neglected in our equations.

The main approximation behind our approach is the expansion for small argument of the Bessel functions in the solution for ϕ, in Section III.C. We remark that the resulting variational principle formulation for $\tilde{J}_{\parallel}$ (or $\tilde{E}_{\parallel}$) is reasonably tractable, it provides continuous contact with previous calcualtions where local J_0 terms were ignored and, perhaps most importantly, physical situations of interest (e.g., tokamak physics) do correspond to the relative expansion parameter being indeed small. However, we must acknowledge that the possibility of an arbitrary strong current is ruled out a priori in ur formalism, even though, as noted, this is not restrictive for most cases of interest.

The dispersion relations for several modes, including the corrections due to J_0, have been derived in Chapter IV; the most important results of the present work are expressed by Eqs. (4.32), (4.37), (4.61), describing the $m \geq 2$ and $m = 1$ tearing mode in the collisional regime and the $m \geq 2$ tearing mode in the semicollisional regime (in the sense of Eqs. (3.13), (3.14)), respectively.

These results, derived by perturbation theory, show that the effect of the local current turns out to modify the final dispersion relations in basically the same way, as it is apparent from the above equations in the form of Eqs. (4.31), (4.36), (4.60); namely, the $(J_0')^2$ contributions are all proportional to the fourth power of the small expansion parameter previously mentioned, although the $(J_0')^2$ correction itself is different for the various modes.

We explicitly checked for the Text tokamak that the current contributions were indeed small, thereby confirming the goodness of the approximation used.

Depending on the particular root of the dispersion relation derived neglecting J_0 effects, which, we recall, has to be chosen so that the mode is mathematically consistent, the effect of the current was found to be stabilizing in one case and destabilizing in the others. Since the current-dependent term is a small correction to known results, the above statement must be understood in the sense that the inclusion of J_0 slightly modifies the term which provides growth or damping to the mode, according to the particular case, but it cannot overtake the leading term.

From our analysis, the inclusion of J_0 enhances stabilization of the $m \geq 2$ drift tearing mode in the collisional regime, whereas the clssical growth rates for the $m = 1$ and the $m \geq 2$ tearing modes are very slightly increased by J_0, in the non drift regime, $\gamma >> \omega^*$. In the collisional regime, we also considered the possibility that the local current is responsible for a new mode, which does not exist when

J_0 terms are neglected; although a mathematically consistent growing root can be found from the derived dispersion relation, the result does not appear to be relevant since it corresponds to an extremely narrow region of parameter space, which does not pertain to low poloidal number modes.

The analysis of the $m \geq 2$ drift tearing mode has been extended to the semicollisional regime (in the sense of Eqs. (3.13), (3.14)) in Section IV.C, where we found that the inclusion of J_0 results in slightly decreasing the stabilizing effect of the parallel thermal conduction.

Current theoretical prediction on the linear stability of tearing modes are therefore confirmed, to leading order, by the present work, since the modifications due to the inclusion of the local current has been shown to be typically very small.

As a concluding remark, we could tentatively predict that the discussed J_0 effects should be quite negligible in the weakly non-linear regime. Even though our calculation is restricted to the linear case, it shows that the equilibrium current enters the analysis through the local electron temperature gradient if the equilibrium parallel electric field is spatially constant. If one confirms itself to a time scale over which the equilibrium magnetic field does not change appreciably, such an approximation should be satisfied. As a consequence of the formation of magnetic islands, and of the resulting large radial electron thermal conduction, the local flattening of the temperature profile will prevent the equilibrium current to effectively couple to the mode dynamics.

APPENDIX A

DERIVATION OF THE EIGENMODE EQUATIONS

The linearization of Eq. (2.8) gives

$$\tilde{E}_{\parallel} = \tilde{\eta} J_o + \eta \tilde{J}_{\parallel} - \frac{T_e}{en} \widetilde{\hat{b} \cdot \nabla n} + $$

$$- \left(1 + \varepsilon + i\varepsilon' \frac{\omega}{\nu}\right) \frac{\widetilde{\hat{b} \cdot \nabla T_e}}{e} \quad , \qquad \text{(A.1)}$$

where corrections due to electron inertia, (i.e., terms involving the coefficient ε"), have been neglected.[21] Here, $\widetilde{\hat{b} \cdot \nabla T_e}$ represent a perturbation of both $\hat{b} \cdot \nabla$ and T_e, i.e., $\widetilde{\hat{b} \cdot \nabla T_e} \equiv ik_{\parallel}\tilde{T}_e + \tilde{B}_r T_e'/B$, and similarly for $\widetilde{\hat{b} \cdot \nabla n}$.

Using Eqs. (2.5), we can write Eq. (A.1) as

$$i(\omega - \omega_n^* - \hat{\varepsilon}\omega_T^*)\tilde{A}_{\parallel} - ik_{\parallel}\tilde{\phi} =$$

$$= -\frac{3}{2} \eta J_o \frac{\tilde{T}_e}{T_e} + \eta \tilde{J}_{\parallel} - \frac{ik_{\parallel} T_e}{e} \frac{\tilde{n}}{n} - i\hat{\varepsilon} \frac{k_{\parallel} T_e}{e} \frac{\tilde{T}_e}{T_e} \quad , \qquad \text{(A.2)}$$

where we defined, cfr. Eq. (2.14c), $\hat{\varepsilon} \equiv 1 + \varepsilon + i\varepsilon'\omega/\nu$.

Let us now make use of the approximations discussed in the text, after Eqs. (2.18); from the form of Eq. (A.2), it follows that the perturbation in density and temperature can be obtained by the linearized forms of Eqs. (2.7) and (2.9), respectively, where J_o terms are systematically neglected. Indeed, by keeping the equilibrium current contributions in Eqs. (2.7) and (2.9), all J_0 terms appearing in the final answer for the generalized conductivity can be neglected, as being of the nature described after Eqs. (2.18). Making use of Eq. (2.10) in Eqs. (2.7), (2.9), we obtain[21]

$$\omega \frac{\tilde{n}}{n} = \omega_n^* \frac{e\tilde{\phi}}{T_e} - \frac{k_\parallel \tilde{J}_\parallel}{en} , \tag{A.3}$$

$$\left(\omega + \frac{2}{3} ik_\parallel^2 \frac{\chi}{n}\right) \frac{\tilde{T}_e}{T_e} =$$

$$= \omega_T^* \frac{e\tilde{\phi}}{T_e} - \frac{2}{3} (1 + \varepsilon) \frac{k_\parallel \tilde{J}_\parallel}{en} - \frac{2}{3} ik_\parallel \frac{\chi}{n} \frac{k_\perp T_e'}{B T_e} \tilde{A}_\parallel . \tag{A.4}$$

Substituting Eqs. (A.3), (A.4) in Eq. (A.2), and neglecting terms in J_0 which merely give rise to Doppler shifts to the real part of ω, we find

$$\eta \tilde{J}_\parallel \left[1 + \frac{isk_\parallel^2 D}{\omega} + \frac{isk_\parallel^2 D}{\omega} \frac{\frac{2}{3}(1 + \varepsilon)\hat{\varepsilon}}{\left(1 + i\varepsilon''' s \frac{k_\parallel^2 D}{\omega}\right)}\right] =$$

$$= i(\omega\tilde{A}_\parallel - k_\parallel \tilde{\phi})\left[1 - \frac{\omega_n^*}{\omega} - \frac{\omega_T^*}{\omega} \frac{\hat{\varepsilon}}{\left(1 + i\varepsilon''' s \frac{k_\parallel^2 D}{\omega}\right)}\right] +$$

$$+ \frac{3}{2} \eta J_\parallel^{(0)} \frac{e\omega_T^*}{T_e} \tilde{\phi}(\omega + i\varepsilon''' sk_\parallel^2 D)^{-1} , \tag{A.5}$$

where the definitions in Eqs. (2.14) have been used.

Eq. (A.5), which is the first main result of this appendix, expresses the perturbed parallel current $\tilde{J}_\parallel$ in terms of the potentials ϕ and ψ, and has been previously derived neglecting the J_0 contribution in Ref. 21. By now relating $\tilde{J}_\parallel$, given by Eq. (A.5), to ψ'' through Ampére's law, Eq. (2.6), and after a straightforward manipulation which makes use of Eq. (2.2), one finally obtains the first eigenmode equation in the form of Eq. (2.11).

The second eigenmode equation involving ψ and ϕ, Eq. (2.17), is obtained in the following way: by linearizing Eq. (2.15) and taking its cross product with $\hat{b}_0/B$, we obtain

$$\tilde{\underset{\sim}{J}}_\perp = -\frac{i\omega m_i n\hat{b}_0 \times \underset{\sim}{\tilde{V}}}{B} + \frac{\hat{b}_0 \times \nabla\tilde{p}}{B} - \frac{\hat{b}_0}{B} \times (\underset{\sim}{J}^{(0)} \times \underset{\sim}{\tilde{B}}) \quad , \tag{A.6}$$

where $\tilde{\underset{\sim}{J}}_\perp$ denotes the perpendicular component of the perturbed current.

The perpendicular components of the perturbed velocity are given, for low frequency perturbations (e.g., $\omega \sim \omega_i^*$), by the familiar formula

$$\hat{b}_0 \times (\underset{\sim}{\tilde{V}} \times \hat{b}_0) = \frac{\hat{b}_0}{eB} \times \left(\frac{\nabla\tilde{p}_i}{n_i} + e\nabla\tilde{\phi} \right) \quad . \tag{A.7}$$

To calculate $\tilde{p}_i$, we assume that the ions are "hydrodynamic", in the sense that

$$\omega >> k_\parallel v_i \quad ; \tag{A.8}$$

where $v_i = (2T_i/m_i)^{1/2}$ is the ion thermal speed. The crucial consequence of Eq. (A.7) is that the ion energy flow along $\underset{\sim}{B}_0$ is small, so that temporal changes in the ion pressure are balanced by convection:

$$\frac{\partial p_i}{\partial t} + \underset{\approx}{V} \cdot \nabla p_i \simeq 0 \quad . \tag{A.9}$$

Neglecting equilibrium ion flow along $\underset{\sim}{B}_0$ and small corrections of order β, the linearization of Eq. (A.9) gives

$$\frac{\tilde{p}_i}{p_i} = \frac{\omega_i^*}{\omega} \frac{e\tilde{\phi}}{T_i} \quad , \tag{A.10}$$

with ω_i^* given by Eq. (2.18b). Equation (A.7) reduces therefore to

$$\hat{b}_0 \times (\tilde{\underset{\sim}{V}} \times \hat{b}_0) = \left(1 + \frac{\omega_i^*}{\omega}\right) \frac{\hat{b}_0 \times \nabla\tilde{\phi}}{B} \quad . \tag{A.11}$$

We now substitute Eq. (A.11) in Eq. (A.6) and take its divergence; assuming that the perturbed quantities vary on a much shorter spatial scale than quantities describing the equilibrium,[17] and neglecting the small contribution from the $(\hat{b}_0 \times \nabla\tilde{p})/B_0$ term in Eq. (A.6), we obtain

$$\nabla \cdot \underset{\sim}{J}_\perp = \frac{i m_i n_i}{B^2} (\omega + \omega_i^*) \nabla_\perp^2 \tilde{\phi} + \frac{i k_\perp J_0'}{B} \tilde{A}_\parallel \quad . \tag{A.12}$$

Let us now impose the quasineutrality condition in the form

$$\nabla \cdot \tilde{\underset{\sim}{J}}_{\parallel} = - \nabla \cdot \tilde{\underset{\sim}{J}}_{\perp} \quad ; \tag{A.13}$$

expressing $\tilde{J}_{\parallel}$ in the terms of ψ'' through Ampére's law, Eq. (2.6), and substituting $\nabla \cdot \tilde{\underset{\sim}{J}}_{\perp}$ from Eq. (A.12), we obtain

$$\frac{k_{\parallel}}{4\pi} \tilde{A}''_{\parallel} = \frac{m_i n_i}{B^2} (\omega + \omega_i^*)\tilde{\phi}'' + \frac{k_{\perp} J'_0}{B} \tilde{A}_{\parallel} \quad , \tag{A.14}$$

where we assumed the mode radial wavelength to be much shorter than the azimuthal wavelength.

Eq. (A.14), which, together with Eq. (A.5), is the main result of this appendix, is the second eigenmode equation and can evidently be cast in the form of Eq. (2.17) by using Eq. (2.2).

APPENDIX B

KINETIC THEORY DERIVATION OF EQ. (2.11) IN THE COLLISIONAL REGIME

The most significant departure from conventional tearing mode theories in the present work is the inclusion of the parallel equilibrium electric field $E_\parallel^{(0)}$ in the derivation of the eigenmode equations, which alters, in particular, the character of Eq. (3.12) into the non-diagonal form given by Eq. (A.5); we remark that the constitutive relationship between $\tilde{J}_\parallel$ and $\tilde{E}_\parallel$ expressed by Eq. (3.12) is not a law of nature; rather, it only pertains when $E_\parallel^{(0)}$ is neglected.

In this appendix, we show that Eq. (2.11) can be also derived from kinetic theory, and is therefore quite general; for simplicity, we shall restrict ourselves only to the collisional regime in the present derivation.

The drift kinetic equation is[34]

$$\frac{\partial f_1}{\partial t} + v_\parallel \hat{b}_o \cdot \nabla f_1 - \frac{(\nabla\tilde{\phi} \times \underset{\sim}{B})}{B^2} \cdot \nabla f_o + \\ + i v_\parallel \frac{k_\perp \tilde{A}_\parallel}{B} \hat{r} \cdot \nabla f_o + \\ + \frac{e}{m}\left[E_\parallel^{(0)} \frac{\partial}{\partial v_\parallel} f_1 + \tilde{E}_\parallel \frac{\partial f_o}{\partial v_\parallel}\right] = C_\ell(f_o, f_1) \quad , \qquad \text{(B.1)}$$

where f_o and f_1 are the equilibrium and perturbed electron distribution functions, respectively, $C_\ell(f_o,f_1)$ is the linearized collision operator,[34] e is the electron charge, and m is the electron mass.

In the collisional regime, it is appropriate to choose

$$f_1 = \delta n \frac{\partial f_o}{\partial n} + \delta T \frac{\partial f_o}{\partial T} + \delta f \quad , \tag{B.2}$$

where δn, δT are density and temperature perturbations, respectively, and δf may be taken as a small (odd in $v_\parallel$) additional term.

The equilibrium distribution function satisfies

$$\frac{e}{m} E_\parallel^{(0)} \frac{\partial f_o}{\partial v_\parallel} = C(f_o,f_o) \quad , \tag{B.3}$$

for any density and temperature (we consider $E_\parallel^{(0)}$ spatially constant); f_o can then be written as

$$f_o = f_M[1 + v_\parallel h_{sp}] \quad , \tag{B.4}$$

where

$$f_M \equiv \frac{n}{\pi^{3/2} v_e^3} \exp(-v^2/v_e^2) \tag{B.5}$$

is the Maxwellian distribution function, $v_e = (2T/m)^{1/2}$ is the electron thermal speed, and the Spitzer function h_{sp} satisfies (cfr. Eq. (3.17))

$$J_o \equiv e \int d^3v \;\; v_\parallel f_o =$$

$$= e \int d^3v \;\; v_\parallel^2 f_M h_{sp} =$$

$$= \sigma_{sp} \; E_\parallel^{(0)} \quad , \tag{B.6}$$

with the Spitzer-Braginskii conductivity given by

$$\sigma_{sp} \equiv s \, \frac{ne^2}{m\nu} \quad ; \tag{B.7}$$

s is the numerical transport coefficient defined by Eq. (2.14e).

From Eq. (B.3), we have

$$\left[\delta T \frac{\partial}{\partial T} + \delta n \frac{\partial}{\partial n}\right] \frac{e}{m} \left[E_\parallel^{(0)} \frac{\partial f^{(0)}}{\partial v_\parallel}\right] =$$

$$= C_\ell\left(f_o, \delta n \frac{\partial f_o}{\partial n} + \delta T \frac{\partial f_o}{\partial T}\right) \quad , \tag{B.8}$$

so that the term $\frac{e}{m} E_\parallel^{(0)} \frac{\partial}{\partial v_\parallel} \left[\delta n \frac{\partial f_o}{\partial n} + \delta T \frac{\partial f_o}{\partial T}\right]$ in Eq. (B.1) evidently cancels with the corresponding term from $C_\ell(f_o, f_1)$.

We may next determine the perturbed δn and δT from the density and energy moments of Eq. (B.1), noting that the collision operator does not affect particle number and energy conservation. Actually, we shall use $\delta n_i = \delta n_e$ and $\nabla \cdot \tilde{\underset{\sim}{J}} = 0$, so that we do not need the density moment. The energy moment of Eq. (B.1), neglecting Ohmic heating and small $k_\parallel^2 D/\omega$ corrections from parallel thermal conduction, yields

$$-i\omega \int d^3v \; \frac{mv^2}{2} \left[\delta n \frac{\partial f_o}{\partial n} + \delta T \frac{\partial f_o}{\partial T}\right] \quad +$$

$$- ik_\perp \frac{\tilde{\phi}}{B} \int d^3v \; \frac{mv^2}{2} \left[\frac{dT}{dx} \frac{\partial f_o}{\partial T} + \frac{dn}{dx} \frac{\partial f_o}{\partial n}\right] = 0 \quad , \tag{B.9}$$

which reduces to

$$- i\omega\left[\delta n \frac{\partial}{\partial n} nT + \delta T \frac{\partial}{\partial T} nT\right] \quad +$$

$$- ik_\perp \frac{\tilde{\phi}}{B} \left[\frac{dT}{dx} \frac{\partial}{\partial T} nT + \frac{dn}{dx} \frac{\partial}{\partial n} nT\right] = 0 \quad . \tag{B.10}$$

Hence,

$$n\delta T + T\delta n = - \frac{k_\perp \tilde{\phi}}{\omega B} \frac{d}{dx} (nT) \quad , \tag{B.11}$$

which, if we take

$$\frac{\delta n}{n} = \frac{e\tilde{\phi}}{T} \frac{\omega_n^*}{\omega} \quad , \tag{B.12}$$

gives

$$\frac{\delta T}{T} = \frac{e\tilde{\phi}}{T} \frac{\omega_T^*}{\omega} \quad , \tag{B.13}$$

with ω_n^*, ω_T^* given by Eqs. (2.14a), (2.14b), respectively.

To obtain the current, it is useful to use the self adjoint properties of C_ℓ. We multiply Eq. (B.1) by $v_\parallel h_{sp}$ and integrate over

d^3v; taking note of the previous cancellation, we have

$$ik_\parallel \int d^3v \quad v_\parallel^2 h_{sp}\left[\delta n \frac{\partial f_o}{\partial n} + \delta T \frac{\partial f_o}{\partial T}\right] +$$

$$+ i \frac{k_\perp \tilde{A}_\parallel}{B} \int d^3v \quad v_\parallel^2 h_{sp}\left[\frac{dn}{dx}\frac{\partial f_o}{\partial n} + \frac{dT}{dx}\frac{\partial f_o}{\partial T}\right] +$$

$$- \frac{e}{T}\tilde{E}_\parallel \int d^3v \quad v_\parallel^2 h_{sp} f_o =$$

$$= \int d^3v \quad v_\parallel h_{sp} \, C_\ell(f_o, \delta f) \quad . \tag{B.14}$$

Making use of the self adjoint character of C_ℓ,

$$\int d^3v \quad v_\parallel h_{sp} \, C_\ell(f_o,\delta f) =$$

$$= \int d^3v \quad \frac{\delta f}{f_o} \, C_\ell(f_o, v_\parallel h_{sp} f_M) =$$

$$= \int d^3v \quad \frac{\delta f}{f_o}\frac{e}{m} E_\parallel^{(0)} \frac{\partial f_o}{\partial v_\parallel} =$$

$$= -\frac{e}{T} E_\parallel^{(0)} \int d^3v \quad v_\parallel \delta f \quad . \tag{B.15}$$

Making use of Eqs. (B.6), (B.12), (B.13), (B.15) in Eq. (B.14), and using Eq. (2.5a), we find

$$e \int d^3v \quad v_\parallel \delta f = \sigma_{sp}\tilde{E}_\parallel\left[1 - \frac{\omega_n^*}{\omega} - (1+\varepsilon)\frac{\omega_T^*}{\omega}\right] \quad , \tag{B.16}$$

where ε is the numerical transport coefficient defined in Chapter II.

With f_1 given by Eq. (B.2), the perturbed parallel current, defined by

$$\tilde{J}_{\parallel} \equiv e \int d^3v \quad v_{\parallel} f_1 \quad , \tag{B.17}$$

is equal to

$$\tilde{J}_{\parallel} = \delta n \frac{\partial J_o}{\partial n} + \delta T \frac{\partial J_o}{\partial T} + e \int d^3v \quad v_{\parallel} \delta f \quad ; \tag{B.18}$$

taking into account the $T^{3/2}$ dependence of J_o through σ_{sp} and making use of Eqs. (B.13) and (B.16), we obtain

$$\tilde{J}_{\parallel} = \sigma_{sp} \tilde{E}_{\parallel} \left[1 - \frac{\overset{*}{\omega}_n}{\omega} - (1 + \varepsilon) \frac{\overset{*}{\omega}_T}{\omega} \right] - \frac{3}{2} \frac{e\tilde{\phi}}{T} \frac{\overset{*}{\omega}_T}{\omega} J_o \quad , \tag{B.19}$$

which is precisely the collisional form of Eq. (A.5), and that, substituting $\tilde{J}_{\parallel}$ in Ampére's law, Eq. (2.6), yields the collisional form of the first eigenmode equation, Eq. (2.11).

APPENDIX C

EFFECTS OF LOCAL CURRENT ON BOUNDARY CONDITIONS

In the construction of the variational S, Eq. (3.48), the asymptotic behaviour for ψ has been assumed to be given by Eq. (3.29). This choice, which is standard in tearing mode theory,[17-19] explicitly assumes that ψ has even parity inside the layer, as it follows from the structure of the eigenmode equations when local J_0 terms are neglected. The question arises whether it is possible to formulate a self-consistent set of boundary conditions for Eqs. (3.26), (3.27), which allows for the mixed parity character of the inner solution, due to presence of the local equilibrium current, and provides at the same time contact with conventional theories when we let the current-dependent parameter $\rho \to 0$.

Let us recall that the solution of Eq. (3.20) in the matching region, where the term $X_A^2\phi''$, which represents plasma inertia, can be neglected, can be written as[24,25]

$$\psi = \psi_0 + A_{\pm}x + \rho X_A^2\, \psi_0 x\, \ell n|x| + O(x^2) \quad , \tag{C.1}$$

where the constants A_+, A_- are related to Δ' by

$$A_+ - A_- \equiv \psi_0 \Delta' \tag{C.2}$$

In conventional theories, where local effects of J_0 are neglected, one imposes the even parity character of ψ by choosing

$$A_+ = -A_- = \frac{\psi_0 \Delta'}{2} \quad , \tag{C.3}$$

so that the asymptotic behaviour for ψ can be chosen as in Eq. (3.29).

Since $x \ln|x| \sim x$ for any physically relevant range of x, let us choose, for the asymptotic behaviour of ψ in the matching region,

$$\psi \approx \psi_0 \left[1 + \frac{\Delta'|x|}{2} + \rho X_A^2 x\right] \quad . \tag{C.4}$$

If no distinction is made between the radial variables x and p for sufficiently large value of x, the above asymptotic form of ψ is in agreement with Eq. (3.28), provided one chooses

$$a_1 = \frac{1}{\Delta'} \int_{-\infty}^{\infty} Q(p')dp' \quad , \tag{C.5}$$

$$b_1 = \frac{\rho X_A^2}{\Delta'} \int_{-\infty}^{\infty} Q(p')dp' \quad , \tag{C.6}$$

for the arbitrary constants in Eq. (3.28). The appropriate solution of Eq. (3.26) is now

$$\psi(p) = \frac{1}{2} \int_{-\infty}^{\infty} |p - p'|Q(p')dp' + \frac{(1 + \rho X_A^2 p)}{\Delta'} \int_{-\infty}^{\infty} Q(p')dp' \quad , \tag{C.7}$$

where, as already noted, the adimensional quantity $\rho X_A^2 p$, which depends linearly on J_o, has to be regarded as a small correction << 1 in the present theory, even in the matching region, where x is "large".

Let us now impose appropriate boundary conditions for ϕ by choosing the arbitrary constants a_2, b_2, in Eq. (3.42), so that contributions of order $\rho X_A^2 x$ coming from terms outside the integrals, exactly cancel, leaving only terms of order ρ^2. By expanding the Bessel functions in Eq. (3.42) for small argument, we find that proper balancing follows if we choose

$$a_2 = 0 \quad , \tag{C.8}$$

$$b_2 = b_1 \frac{\rho^{1/6}\Gamma(2/3)}{3^{1/3}} \quad . \tag{C.9}$$

We then have the following integral equation for Q

$$\frac{Q(p)}{\mu_1} = \frac{1}{2}\int_{-\infty}^{\infty} g(p,p')Q(p')dp' +$$

$$+ \frac{1}{\Delta'}\left(1 + \frac{\rho^2 X_A^2 p^4}{6}\right)\int_{-\infty}^{\infty} Q(p')dp' \quad , \tag{C.10}$$

where g(p,p') has been defined in Eq. (3.47); the above equation is, of course, identical with Eq. (3.46), apart from the (small) additional term ρ^2, quadratic in J_o, which represents the change in the boundary conditions for ψ. Since this additional term in $\rho^2 \chi_A^2 p^4$ is a small correction, a negligible error is involved by replacing p^4 with $(p^4 + p'^4)/2$, which provides a symmetric kernel in the corresponding integral.

The variational S, following from Eq. (C.10), is therefore

$$
\begin{aligned}
S = & \int_{-\infty}^{\infty} \frac{Q^2(p)}{\mu_1} \, dp \; + \\
& - \frac{1}{\Delta'} \left[\int_{-\infty}^{\infty} Q(p) dp \right]^2 \; + \\
& - \frac{\rho^2 \chi_A^2}{12\Delta'} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (p^4 + p'^4) Q(p) Q(p') dp dp' \; + \\
& - \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(p,p') Q(p) Q(p') dp dp' \quad . \qquad \text{(C.11)}
\end{aligned}
$$

Let us now derive the mode dispersion relation, describing the effect of the change in the boundary conditions for ψ on the $m \geq 2$ drift tearing result in the collisional regime; proceeding as in Section IV.B, we evaluate S by substituting the trial function given by Eq. (4.4) in Eq. (C.11) and we obtain

$$S = \frac{\alpha^{-1/2}}{\mu_1} + \frac{\delta^2}{2}\frac{\alpha^{-3/2}}{\mu_1} - \frac{2\pi^{1/2}}{\Delta'}\alpha^{-1} +$$

$$- \frac{\alpha^{-5/2}}{X_A^2}(1 + 2\alpha X_A^2) + \frac{2}{3}\delta\rho\frac{\alpha^{-9/2}}{X_A^2} +$$

$$+ \frac{7}{2}\delta^2\frac{\alpha^{-7/2}}{X_A^2}\left(1 + 2\,\alpha X_A^2\right) +$$

$$- \rho^2\left(\frac{2}{15}\frac{\alpha^{-11/2}}{X_A^2} + \frac{5}{6}\pi^{1/2}\frac{X_A^2\alpha^{-3}}{\Delta'}\right) . \qquad (C.12)$$

Apart from the last term, representing the change in boundary conditions, the above form of S is identical with the one given by Eq. (4.8). Assuming $|\alpha^{1/2}X_A| < 1$ and treating all the terms in ρ^2 as a perturbation of the result expressed by Eqs. (4.11), (4.12) we obtain the following dispersion relation,

$$\omega(\omega + \omega_i^*)(\omega - \omega_n^* - \hat{\varepsilon}\omega_T^*)^3 =$$

$$= i\left(\frac{2\Delta'}{3^{3/4}\pi^{1/2}}\right)^4\left(\frac{n}{4\pi}\right)^3(k_\parallel' v_A)^2 \times$$

$$\times \left\{1 + \frac{17\pi}{360}\left(\frac{4\pi k_\perp J_0'}{k_\parallel' B\Delta'}\right)^2\left[1 - \frac{150}{17}\left(\frac{2\Delta' X_A}{\pi^{1/2}}\right)^{2/3}\right]\right\} , \qquad (C.13)$$

where the modification to the $m \geq 2$ tearing mode result due to local current include the change in the boundary conditions for ψ through the correction of order $\Delta' X_A$ to the J_0' term.

Recalling that $|\Delta' X_A| << 1$ for the $m \geq 2$ tearing mode and comparing Eq. (C.13) with Eq. (4.32), we conclude that the change in the boundary conditions for ψ induced by J_o' has a negligible effect on the final dispersion relation.

APPENDIX D

ALTERNATIVE DERIVATION OF THE SEMICOLLISIONAL TEARING MODE DISPERSION RELATION IN THE ABSENCE OF LOCAL CURRENT

Because of the fairly complex expression for the semicollisional form of S, Eq. (4.54), even in the absence of J_0 terms, a somewhat approximate procedure has been employed to derive the mode dispersion relation given by eq. (4.56). In this appendix, we present an alternative derivation of Eq. (4.56), based on the variational principle formulation of Ref. 18, which enables an exact and simple variational calculation in the present context, even though this method of solution cannot be extended to the case in which equilibrium current terms are included in the eigenmode equations.

If the J_0 terms are neglected, Eqs. (2.11), (2.17) can be combined into a single, integro-differential equation of the form[18]

$$\left(\frac{X_A^2 x^2 \mathcal{E}'}{x^2 - X_A^2}\right)' + x^2 \sigma_* \mathcal{E} = -\left(\Delta' + \frac{i\pi}{X_A}\right)^{-1} \frac{4x X_A^4}{(x^2 - X_A^2)^2} \int_{-\infty}^{+\infty} \frac{x \mathcal{E}\, dx}{(x^2 - X_A^2)^2} \quad , \qquad (D.1)$$

where the field variable $\mathcal{E}$ is defined by

$$\mathcal{E}(x) \equiv \psi - x\phi = \frac{\tilde{E}_\parallel}{ik_\parallel' x} \quad . \qquad (D.2)$$

The variational functional reproducing the Euler-Lagrange equation, Eq. (D.1), is evidently

$$S = (\Delta' + i\pi/X_A)(I_1 + I_2) + I_3^2 \quad , \tag{D.3}$$

where

$$I_1 = \int_{-\infty}^{+\infty} \mathcal{E} \left(\frac{\mathcal{E}' x^2 X_A^2}{x^2 - X_A^2} \right)' dx \quad , \tag{D.4}$$

$$I_2 = \int_{-\infty}^{+\infty} x^2 \sigma_* \mathcal{E}^2 \, dx \quad , \tag{D.5}$$

and

$$I_3 = 2X_A^2 \int_{-\infty}^{+\infty} \frac{x\mathcal{E}}{(x^2 - X_A^2)^2} dx \quad . \tag{D.6}$$

By comparison with Eq. (4.1), it appears that the trial function for tearing-symmetry solution is

$$\mathcal{E}(x)_{trial} = \frac{\exp(-\alpha x^2/2)}{x} \; ; \tag{D.7}$$

substituting the above trial function in Eqs. (D.4) - (D.6), using $\sigma_* = \mu_1 + \mu_2 x^2$, and simplifying the resulting expression for S making use of the fact that $|\alpha^{1/2} X_A| < 1$, $|\Delta' X_A| \ll 1$ for $m \geq 2$ tearing modes, we obtain

$$S = \mu_1 \alpha^{-1/2} + \frac{\mu_2}{2} \alpha^{-3/2} +$$

$$- \left(\frac{4\sqrt{2} - 5}{4}\right) \alpha^{3/2} X_A^2 - \frac{\Delta'}{\pi^{1/2}} \quad . \tag{D.8}$$

The semicollisional mode discussed in Section IV.D corresponds to the

$$|\Delta' X_A| < \alpha^{3/2} X_A^3 \tag{D.9}$$

limit of Eq. (D.8), which gives

$$S \simeq \mu_1 \alpha^{-1/2} + \frac{\mu_2}{2} \alpha^{-3/2} +$$

$$- \left(\frac{4\sqrt{2} - 5}{3}\right) \alpha^{3/2} X_A^2 \quad ; \tag{D.10}$$

the variational parameter α and the dispersion relation are obtained, as usual, solving Eqs. (4.6), (4.7); they are respectively given by

$$\alpha = - \frac{3}{4} \frac{\mu_2}{\mu_1} \quad , \tag{D.11}$$

and (cfr. with Eq. (4.56))

$$\mu_1^3 = \left(\frac{3}{4}\right)^2 (4\sqrt{2} - 5) \mu_2^2 X_A^2 \quad . \tag{D.12}$$

For a drift-type solution, Eqs. (D.11) and (D.12) yield

$$\omega \simeq \omega_o + i\varepsilon' \frac{\omega_o}{\nu} \omega_T^* + $$

$$+ e^{\frac{7\pi i}{6}} \left(\frac{3}{4}\right)^{2/3} (4\sqrt{2} - 5)^{1/3} \left(\frac{n}{4\pi}\right)^{1/3} (\varepsilon''' s\hat{\varepsilon}\omega_T^* k_\parallel' D)^{2/3} \frac{\left(1 + \frac{\omega_i^*}{\omega_o}\right)^{1/3}}{v_A^{2/3}} , \quad (D.13)$$

with ω_o given by Eq. (4.15). We remark that the above result differs from the one expressed by Eq. (4.58) only by a 21% difference in the numerical factor in front of the damping rate, and by a 15% difference in the numerical factor in front of the damping rate given by Eq. (39) of Ref. 22.

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