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Abstract

Numerical studies show that the $m = 1$ tearing mode continues to grow exponentially well into the nonlinear regime, in contrast with the slow, "Rutherford," growth of $m > 1$ modes. We present a single helicity calculation which generalizes that of Rutherford to the case when the constant-ψ approximation is invalid. As in that theory, the parallel current becomes an approximate flux function when the island size, W, exceeds the linear tearing layer width. However for the $m = 1$ mode, W becomes proportional to δB, rather than $(\delta B)^{1/2}$ above this critical amplitude. This implies that the convective nonlinearity in Ohm's law, which couples the $m = 0$ component to the $m = 1$ component, dominates the resistive diffusion term. The balance between the inductive electric field and this convective nonlinearity results in exponential growth. Assuming the form of the perturbed fields to be like that of the linear mode, we find that growth occurs at 71% of the linear rate.

I. Introduction

Tokamak tearing and kink instabilities[1] have a distinctive character when the poloidal mode-number, m, is unity.[1-3] For large aspect ratio, internal kink modes are damped for $m \neq 1$ and neutrally stable for $m = 1$. Furthermore the marginally stable $m = 1$ displacement has a peculiar spatial structure. It is concentrated inside the resonant flux surface (i.e., the surface on which the safety factor, q, is unity) where it is nearly constant[4]:

$$\begin{aligned} \xi &= \text{const.}, \quad r < r_s, \\ \xi &= 0, \quad r > r_s. \end{aligned} \tag{1}$$

Here ξ is the radial displacement and r_s is the radius of the $q = 1$ surface.

The $m = 1$ tearing mode is unstable but, since tearing growth rates are very slow on Alfvénic time scales, it is still described by Eq. (1) outside the narrow reconnection layer. Outside this layer the parallel electric field is nearly zero, and the linearized Faraday's law

$$\partial B_r / \partial t \approx \mathbf{B} \cdot \nabla \xi,$$

implies that B_r is proportional to the parallel wave vector, i.e., to the distance from the rational surface:

$$B_r \propto k_{\|} \xi \propto |r - r_s|, \quad \text{for} \quad r < r_s, \tag{2}$$

since ξ is constant. The relation (2) distinguishes the $m = 1$ tearing mode, since for $m > 1$, B_r is roughly constant inside the layer ("constant-ψ" approximation).

The $m = 1$ mode is also peculiar nonlinearly. We recall first the crucial feature of nonlinear magnetic island growth for $m > 1$. In that case Rutherford[5] showed that evolution becomes, for sufficiently large islands, linear in time:

$$\partial B_r / \partial t = \text{const.}, \quad \text{for} \quad W \geq w_L, \quad m > 1. \tag{3}$$

Here W denotes the island width and w_L is the width of the linear tearing layer. The constant in Eq. (3) is proportional to plasma resistivity (and Δ'), implying that saturated islands grow on the resistive time scale.

The derivation of Eq. (3) depends on the constant-ψ approximation, which is inapplicable for $m = 1$. Thus it's not very surprising that numerical simulation[6] reveals a very different evolution for the $m = 1$ island. In fact it continues to grow exponentially, at some fraction of the linear growth rate, well into the nonlinear regime:

$$\partial B_r/\partial t = \mu\gamma_L B_r, \quad \text{for} \quad W \leq w_L, \quad m = 1, \tag{4}$$

where γ_L is the linear growth rate of the $m = 1$ tearing mode and μ is a number of order unity.

Although the behavior indicated by Eq. (4) is universally observed in $m = 1$ simulations it has previously not been deduced analytically. The present work provides an analytical derivation of Eq. (4). The formulation is a generalization of Rutherford's treatment[5] that also reproduces Eq. (3) and therefore illuminates the underlying differences between the two cases of nonlinear island growth.

The paper is organized as follows. In Sec. II several features necessary for the formulation of the calculation are discussed. Firstly, we review our notation in the context of low-β single-helicity reduced MHD[7], the basic dynamical system that is our starting point. Secondly, we review linear tearing mode theory and describe the $m = 1$ eigenmode structure. We conclude Sec. II with a discussion of magnetic island structure, which has a distinctive character for $m = 1$. In particular we compare the scalings of the $m = 1$ and $m > 1$ island widths with the field perturbation. Section III contains the main part of the $m = 1$ island evolution calculation. Here we generalize Rutherford's calculation by including nonlinear coupling via the $m = 0$ component of Ohm's law, and the nonlinear island growth rate is evaluated. We conclude in Sec. IV with a summary of our main assumptions and an assessment of the experimental significance of our result.

II. Formulation

A. Notation

The magnetic field is conveniently expressed as

$$\mathbf{B}/B_T = \hat{\varsigma} - \epsilon\hat{\varsigma} \times \nabla\psi \tag{5}$$

where B_T is a constant measure of the vacuum toroidal field, $\hat{\varsigma}$ is a unit vector in the toroidal direction, ϵ is the small inverse aspect ratio and ψ is the normalized poloidal flux.

Equation (5) expresses tokamak orderings in the conventional manner of reduced MHD;[7] the reduced normalizations have become sufficiently standard (apart from some disparity in sign conventions) to require no detailed review here.[3]

The toroidal current density is denoted by $-J$ and given by the reduced Ampere's law,

$$J = \nabla_\perp^2 \psi, \tag{6}$$

where $\nabla_\perp = \nabla - \hat{\varsigma}\hat{\varsigma}$ is the transverse gradient.

The crucial operator in the analysis of magnetic islands is the parallel gradient,

$$\nabla_\| = (\mathbf{B}/B_T) \cdot \nabla.$$

We introduce the conventional bracket,

$$[f, g] = \hat{\varsigma} \cdot \nabla f \times \nabla g,$$

and use Eq. (5) to write

$$\nabla_\| f = \partial/\partial\varsigma - [\psi, f], \tag{7}$$

for any function f. The two terms in Eq. (7) are comparable because derivatives with respect to ς are considered first order in ϵ, reflecting the long toroidal scale length. Notice that $\nabla_\|$ is the gradient along the full magnetic field, including perturbed contributions to ψ; it is a nonlinear operator.

The normalized electrostatic potential is denoted by φ and defined such that the parallel electric field is proportional to $\partial\psi/\partial t + \nabla_\| \varphi$ (where "t" measures time in units of the poloidal Alfvén time). Thus Ohm's law is expressed as

$$\partial\psi/\partial t + \nabla_\| \varphi = \eta J. \tag{8}$$

Here η, the normalized resistivity, can be identified with the inverse of the poloidal magnetic Reynold's number: it is the ratio of the (short) poloidal Alfvén time to the (very long) resistive skin time.

We restrict our attention to a low beta plasma, in which the full dynamics are described by Eq. (8) and the shear-Alfvén law,

$$\partial U/\partial t + [\varphi, U] + \nabla_\| J = 0, \tag{9}$$

where

$$U = \nabla_\perp^2 \varphi$$

is the parallel vorticity. Thus we have two dynamical equations for the two potentials φ and ψ.

Equations (8) and (9) summarize (incompressible) resistive MHD in a large aspect-ratio tokamak. As a three-dimensional nonlinear system, the equations are amenable to analysis only in certain simplified cases. Here we are interested in the evolution of a single, coherent magnetic island and therefore reduce the dimensionality by imposing helical symmetry. For $m = 1$ the appropriate helical angle is

$$\alpha = \varsigma - \theta,$$

where θ is the ordinary poloidal angle. Helical symmetry is expressed by

$$f(r, \theta, \varsigma) = f(r, \alpha),$$

for any field variable f. Of course r is the usual cylindrical coordinate measuring minor radius.

Helical symmetry allows the parallel gradient to be expressed as

$$\nabla_{||} f = \left(1 + r^{-1} \frac{\partial \psi}{\partial r}\right) \frac{\partial f}{\partial \alpha} - r^{-1} \frac{\partial \psi}{\partial \alpha} \frac{\partial f}{\partial r},$$

or

$$\nabla_{||} f = -[\psi_h, f], \tag{10}$$

where

$$\psi_h = \psi + r^2/2 \tag{11}$$

is the helical flux. Evidently ψ_h is a flux label of the perturbed magnetic surfaces,

$$\nabla_{||} \psi_h = 0;$$

it therefore can serve as a convenient "radial" variable in the island geometry. Without symmetry no such ψ_h, nor even (exact) flux surfaces themselves, are likely to exist.

The flux-surface average is indicated by angle brackets and defined by the usual normalized volume integral

$$\langle f\rangle(\psi_h) = \frac{\int d\mathbf{x} f}{\int d\mathbf{x}}, \tag{12}$$

where f is arbitrary, $d\mathbf{x}$ is the volume element and both integrals extend over an infinitesimal volume enclosing the flux surface labelled by ψ_h. Dependence on the value of ψ_h is often left implicit. This operator is important because it annihilates $\nabla_{\|}$; that is,

$$\langle \nabla_{\|} f\rangle = 0 \tag{13}$$

for any single-valued function f. A more explicit expression for the flux surface average is

$$\langle f\rangle = \frac{\oint hrf d\alpha}{\oint hrd\alpha}, \tag{14}$$

where

$$h \equiv |\partial\psi_h/\partial r|^{-1} \tag{15}$$

and the integrals are performed at fixed ψ_h.

B. Linear mode structure

Here we review salient features of the linear $m = 1$ tearing mode. First Eqs. (8) and (9) are reduced to coupled ordinary differential equations by writing

$$\begin{aligned} \psi(r,\alpha,t) &= \bar{\psi}(r) + \mathrm{Re}\Big\{\psi_1(r)\exp(i\alpha - i\omega t)\Big\} \\ \varphi(r,\alpha,t) &= \mathrm{Re}\Big\{\varphi_1(r)\exp(i\alpha - i\omega t)\Big\} \end{aligned} \tag{16}$$

and keeping only first order terms in φ_1 and ψ_1. The overbar represents an α-average at fixed r,

$$\bar{f}(r) \equiv \oint (d\alpha/2\pi) f(r,\alpha). \tag{17}$$

In linear theory it is assumed that $\bar{\psi}$ coincides with the equilibrium poloidal flux. Of course this is not true in general. The parallel gradient operator linearizes according to $\nabla_{\|} \to ik_{\|}$, where, from Eq. (10),

$$k_{\|} = r^{-1}\partial\bar{\psi}_h/\partial r. \tag{18}$$

Alternatively, $k_{||}(r) = 1 - (1/q)$, where $1/q = -r^{-1}\partial\bar{\psi}/\partial r$ is the inverse of the rotational transform, in view of Eqs. (11) and (16).

The linearized equations,

$$\begin{aligned} -\omega\psi_1 + k_{||}\varphi_1 &= -i\eta\psi_1'' , \\ -\omega\varphi_1'' + k_{||}\psi_1'' &= 0, \end{aligned} \tag{19}$$

have the expected singularity at the rational surface, in the vicinity of which

$$k_{||} \approx k_{||}'(r_s)(r - r_s).$$

We denote radial distance from the rational surface by $x = r - r_s$ and derivatives with respect to x by primes. Equations (19) have been simplified by assuming that, because of the singularity, radial or x-derivatives dominate derivatives with respect to α.

After one imposes boundary conditions appropriate to the $m = 1$ case [recall Eq. (2)] one finds that the coupled linear equations have only one unstable solution, the $m = 1$ tearing mode.[1] It's growth rate is

$$-i\omega \equiv \gamma_L = (k_{||}')^{2/3}\eta^{1/3}, \tag{20}$$

and the linear eigenfunction is given by

$$\psi_1 = (\hat{b}/2)\left\{x\mathrm{erfc}(x/w_L) - (w_L/\sqrt{\pi})\exp[-(x/w_L)^2]\right\},$$

$$\varphi_1 = i(\hat{b}/2)(w_L/\sqrt{2})\mathrm{erfc}(x/w_L),$$

where $\hat{b}$ is a constant, erfc is the complementary error function, and

$$w_L = \sqrt{2}(\eta/k_{||}')^{1/3} \tag{21}$$

is the linear layer width.

The ψ_1 and φ_1 are of course complex amplitudes, rather than real field perturbations. Distinguishing the latter with a tilde, we have

$$\tilde{\psi}(x,\alpha,t) = \mathrm{Re}\left\{\psi_1\exp(i\alpha - i\omega t)\right\} = b_L g_L(x)\cos\alpha, \tag{22}$$

$$\tilde{\varphi}(x,\alpha,t) = -b_L(w_L/\sqrt{2})(dg_L/dx)\sin\alpha. \tag{23}$$

where

$$b_L \equiv \hat{b}\exp(\gamma_L t), \tag{24}$$

is the time-dependent amplitude and the linear magnetic eigenfunction is

$$g_L(x) \equiv (x/2)\mathrm{erfc}(x/w_L) + (w_L^2/4)(d/dx)\mathrm{erfc}(x/w_L), \tag{25}$$

which is plotted in Fig. 1. It has the limits

$$g_L(x) = \begin{cases} w_L^3 e^{-(x/w_L)^2}/(4\sqrt{\pi}x^2) & x \gg w_L \\ w_L/\sqrt{\pi}, & x = 0 \\ x, & x \ll -w_L. \end{cases} \tag{26}$$

The proportionality between $\tilde{\varphi}$ and $\partial^2\tilde{\psi}/\partial x\partial\alpha$, apparent in Eqs. (22) and (23), is not obvious from the eigenmode equations and probably fortuitous.

The asymptotic forms of the potentials, for large $|x|$, are especially important. First note from Eq. (26) that $g_L(x)$ is exponentially small for large positive x, localizing the $m = 1$ disturbance inside the $q = 1$ surface. For negative x, one finds that

$$\tilde{\psi} \to b_L x\cos\alpha, \qquad x \ll -w_L, \tag{27}$$

providing $|x|$ is still small compared to r_s. Thus, in particular, b_L measures the $m = 1$ field perturbation far from the tearing layer. In terms of the physical poloidal magnetic field perturbation $\tilde{B}_p$ we have

$$\tilde{B}_p/B_{p0} = \partial\tilde{\psi}/\partial x \to b_L\cos\alpha, \tag{28}$$

where B_{p0} is the equilibrium poloidal field. In the same limit, the electrostatic potential becomes independent of x:

$$\varphi \to b_L(w_L/\sqrt{2})\sin\alpha, \quad \text{for} \quad x \ll -w_L. \tag{29}$$

When x becomes of order $-r_s$, cylindrical geometry slightly modifies these formulae.

C. Magnetic island structure

The perturbed flux surfaces are determined at each time by

$$\psi_h(r,\alpha) = \text{constant}. \tag{30}$$

It is convenient to write

$$\psi_h(r,\alpha) = \bar{\psi}(r) + \tilde{\psi}(r,\alpha), \tag{31}$$

suppressing the subscript. In the linear regime, when the island width is small compared to w_L, the function $\tilde{\psi}$ is given by Eq. (22). Nonlinearly we suppose that

$$\tilde{\psi}(r,\alpha,t) \approx b(t)g(x)\cos\alpha, \tag{32}$$

for some functions $b(t)$ and $g(x)$. In other words we suppose that $\tilde{\psi}$ remains separable and that its α-dependence is dominated by a single harmonic. Analogous assumptions are made in the $m > 1$ case; here we note that the unique stability properties of $m = 1$, mentioned in the introduction, make Eq. (32) seem an especially realistic approximation. The point is that higher harmonics are subject to stabilizing line-bending forces.

Next, assuming that the island width, W, is small on the scale length of $\bar{\psi}$,

$$W \lessdot r_s, \tag{33}$$

we Taylor-expand

$$\bar{\psi}(r) = \bar{\psi}(r_s) + (1/2)\bar{\psi}''(r_s)x^2. \tag{34}$$

The absence of a linear term is a consequence of Eq. (18), which also implies

$$\bar{\psi}_h''(r_s) = r_s k_{\|}'(r_s). \tag{35}$$

We obtain an equation for the perturbed (island) flux surfaces by substituting Eqs. (32) and (34) into Eq. (31). The result is simplified by introducing normalized measures of the helical flux,

$$F \equiv 2[\psi_h - \bar{\psi}(r_s)]/(r_s k_{\|}'), \tag{36}$$

and of the perturbation amplitude,

$$\Delta(t) \equiv b(t)/(r_s k_{\|}'). \tag{37}$$

Then we have

$$F = x^2 + 2\Delta g(x) \cos \alpha. \tag{38}$$

We have not expanded $\tilde{\psi}$ because we expect its logarithmic derivative to be large near r_s. Indeed, in the linear case, $\tilde{\psi} \approx w_L \partial\tilde{\psi}/\partial x$, for $|x| < w_L$.

Since g remains unspecified, Eq. (38) applies equally to the $m = 1$ and $m > 1$ cases (provided we replace $\cos\alpha$ by $\cos m\alpha$). For $m > 1$, the constant-ψ approximation of linear theory is extended nonlinearly by assuming that g is constant in the radial domain of interest. The analogous extension for $m = 1$, $g \approx g_L$, is difficult to justify in detail. However the most important features of sufficiently large magnetic islands involve only the region $|x| > w_L$ where it is plausible to assume that the linear forms, Eqs. (26) and (27), remain valid. In other words we assume the gross features of the internal kink perturbation, Eq. (1), to have nonlinear asymptotic validity. This assumption is qualitatively supported by the numerical results, at least while the island width remains less than r_s. Thus g is assumed to be exponentially small for positive x and linear for negative x,

$$g(x) = x, \quad x < 0. \tag{39}$$

The resulting island structure has an o-point at $(x = -\Delta, \alpha = 0)$ and an x-point at $(x = 0, \alpha = \pi)$. Because of Eq. (26), the islands become very thin for $|\alpha| > \pi/2$; the inner $(x < 0)$ and outer $(x > 0)$ separatrices nearly coincide in this region. For $|\alpha| < \pi/2$, the outer separatrix is approximated by $x = 0$, i.e., $r = r_s$, while the inner one is given by $x = -\Delta \cos\alpha$.

The most important feature of the nonlinear $m = 1$ magnetic island is that its width, W, is proportional to the perturbation amplitude:

$$W = 2\Delta, \quad \text{for} \quad W \geq w_L, \quad m = 1. \tag{40}$$

This is a consequence of Eq. (39). In contrast the width of an $m > 1$ island, for which g is constant, varies as the square root of the amplitude. We remark that in the linear regime, $W < w_L$, the $m = 1$ island width obeys the more conventional $\Delta^{1/2}$ scaling; but the linear threshold is exceeded at very small amplitude.

III. Island Evolution

A. Generalized island dynamics

Our purpose in this section is to derive a zero-dimensional description of nonlinear magnetic island evolution: an ordinary differential equation for the function $b(t)$. This is accomplished by integrating over the spatial variables, much as one eliminates velocity dependence in a kinetic equation by taking moments. The particular moment of interest is chosen on the basis of Eqs. (32) and (39): b evidently measures the change, across the rational surface, of the radial slope of the $\cos\alpha$-Fourier component of ψ. Explicitly,

$$b(t) = -2\oint (d\alpha/2\pi)\cos\alpha \int dr\partial^2\psi/\partial x^2, \tag{41}$$

where the r-integral extends from a radius far inside the reconnection or island region ($r \ll r_s$) to one far outside; recall Eq. (27). Without serious approximation the last factor in the integrand of Eq. (41) can be replaced by J, the parallel current. Furthermore the two-dimensional integral can be transformed according to

$$\oint (d\alpha/2\pi)\int dr = \int d\psi_h \oint (d\alpha/2\pi)|\partial r/\partial\psi_h| = \int d\psi_h \oint (d\alpha/2\pi)h,$$

in view of Eq. (15). Thus Eq. (41) becomes

$$b(t) = -2\int d\psi_h \langle J\cos\alpha\rangle \oint (d\alpha/2\pi)h, \tag{42}$$

where the flux surface average is defined by Eq. (14).

To clarify further analysis we introduce the parameter,

$$\delta \equiv w_L/W. \tag{43}$$

By definition δ is small in the nonlinear regime. Under typical tokamak experimental conditions δ becomes small as soon as the perturbation amplitude of Eq. (28) exceeds $(R/L_s)^{2/3} \times 10^{-3}$, where R is the tokamak major radius and L_s is the shear length. (It must be acknowledged that estimates of L_s near the magnetic axis are very difficult; nonetheless R/L_s is unlikely to be more than an order of magnitude smaller or larger than unity.)

We next consider the flux surface average in Eq. (42), using Eq. (9) for the variation of J on a surface. Recall that for $m > 1$, Eq. (9) reduces to

$$\nabla_{\|} J = 0, \tag{44}$$

since the slow resistive evolution keeps ion inertia small.[5] Thus J is a flux function, equal to its average, and

$$\langle J \cos\alpha \rangle = \langle J \rangle \langle \cos\alpha \rangle.$$

The $m = 1$ case, in which nonlinear evolution remains rapid, is more complicated; in fact Eq. (44) is not globally valid for the nonlinear $m = 1$ mode. However, it is easily seen that $\nabla_{\|} J$ is localized to the tearing layer, just as in linear theory, even as the $m = 1$ magnetic island becomes wider than w_L. The point is that $U \propto \xi''$ is localized [recall Eq. (1)], while the nonlinear parallel gradient operator, like $k_{\|}$, grows in proportion to the distance from the rational surface.

In other words Eq. (44) is replaced in the $m = 1$ case by the weaker statement,

$$J = \langle J \rangle + O(\delta), \quad \text{for} \quad |x| > w_L. \tag{45}$$

Significantly, Eq. (45) allows the current sheet characterizing linear evolution to persist, with some deformation, into the nonlinear regime. The remnant sheet-current – a radial spike at r_s whose narrow width is amplitude independent – is apparent in numerical simulations. If the amplitude of the sheet current were larger by a factor of δ^{-1} than that of the bulk current described by Eq. (45), then both current components would contribute comparably to the integral in Eq. (42). Because such current concentration seems unlikely (especially since δ^{-1} grows exponentially) and because the simulations seem to show a less extreme degree of concentration, we assume that for sufficiently small δ the radially integrated current comes mostly from the broad island region, $|x| \approx W \gg w_L$, rather than from the remnant sheet. Then the integral in Eq. (42) can be simplified by substitution from Eq. (45) and we have

$$b(t) = -2 \int d\psi_h \langle J \rangle \langle \cos\alpha \rangle \oint (d\alpha/2\pi) h. \tag{46}$$

Next consider Ohm's law, Eq. (8). Because of Eq. (13) its flux surface average is $\langle \partial\psi/\partial t\rangle = \langle \partial\psi_h/\partial t\rangle = \eta\langle J\rangle$ or

$$\langle J\rangle = \eta^{-1}\langle \partial\bar{\psi}/\partial t + \partial\tilde{\psi}/\partial t\rangle. \tag{47}$$

After using Eq. (32) to express $\tilde{\psi}$ in terms of b, and substituting the result into Eq. (46) we have

$$\eta b(t) = -2\int d\psi_h \left\{ \left\langle \frac{\partial\bar{\psi}}{dt} \right\rangle + \frac{db}{dt}\langle g\cos\alpha\rangle \right\} \langle \cos\alpha\rangle \oint \frac{d\alpha}{2\pi} h, \tag{48}$$

an island dynamical equation in which only the $m = 0$ evolution, $\partial\bar{\psi}/\partial t$, remains to be specified.

Before evaluating $\partial\bar{\psi}/\partial t$ we briefly comment on Eq. (48) and its relation to previous work. First note that if $\partial\bar{\psi}/\partial t$ is neglected and g is taken to be constant, then Eq. (48) quickly reproduces the well known constant-ψ evolution of Eq. (3). The point is that $\langle\cos\alpha\rangle^2$ is localized to the island region, so that the width of the integration domain is W and Eq. (48) roughly implies $\eta b \approx (\partial b/\partial t)W$ or, since $b \propto W^2$, $\partial W/\partial t \approx \eta$. Moreover, the neglect of $\partial\bar{\psi}/\partial t$ is justified in the $m > 1$ case because the fastest rate in the $m > 1$ problem is the linear growth rate which satisfies

$$\gamma_L \ll \eta/w_L^2, \quad \text{for} \quad m > 1, \tag{49}$$

or $w_L \ll x_R$, where $x_R = (\eta/\gamma_L)^{1/2}$ is the resistive layer width (again, for $m > 1$). Equation (49) requires the $m = 0$ flux perturbation to be small, as shown by Rutherford.[5] Because the growth rate and layer width of the $m = 1$ mode are related by

$$\gamma_L \approx \eta/w_L^2 \tag{50}$$

instead of Eq. (49), it is not surprising that $m = 0$ couplings affect $m = 1$ nonlinear dynamics.

B. Coupling to $m = 0$

We write the $m = 0$ component of Ohm's law as

$$\partial\bar{\psi}/\partial t = -N + \eta\bar{J}, \tag{51}$$

where N represents the nonlinear term,

$$N \equiv \int (d\alpha/2\pi)[\tilde{\varphi}, \tilde{\psi}]. \tag{52}$$

The relative size of the two terms on the r.h.s of Eq. (51) can be estimated as

$$\eta \bar{J}/N \approx \delta^2 \tag{53}$$

using Eqs. (23) and (35). This implies that the $m = 0$ component has the same nonlinearity threshold, $W \approx w_L$, as the $m = 1$ equation. [Because of Eq. (49) this statement is not true for $m > 1$.] It follows that in the strongly nonlinear regime of Eq. (43) we can approximate

$$\partial\bar{\psi}/\partial t = -N.$$

We evaluate the nonlinearity N by using the linear eigenfunctions,[8] Eqs. (22) and (23), obtaining

$$\langle \partial\bar{\psi}/\partial t\rangle = \sqrt{2}(b/2)^2(w_L/r_s), \quad \text{for} \quad x \ll -w_L. \tag{54}$$

This procedure is not rigorous but, for reasons given at the end of Sec. II, it should be qualitatively reliable in the asymptotic region, $|x| \gg w_L$, where it is needed. Note in particular that our result for N is spatially constant and therefore unaffected by such key physical processes as island separatrix motion, fluid convection or magnetic diffusion. (The fact that the nonlinear perturbation in $\bar{\psi}$ is spatially constant for large $|x|$ also means that the shear $k'_{\|}$ and width w_L remain close to their equilibrium values.) Using Eq. (54) in (48) yields

$$\eta b(t) = -2\int d\psi_h \langle\cos\alpha\rangle \oint (d\alpha/2\pi) h \left\{ \sqrt{2}\frac{w_L}{r_s}\Theta(-x)\left(\frac{b}{2}\right)^2 + \frac{db}{dt}\langle g\cos\alpha\rangle \right\}, \tag{55}$$

where the step-function, Θ, results from the smallness of N for $r > r_s$.

Observe next that, in contrast to the $m > 1$ case, the left-hand side of Eq. (55) is negligible in the nonlinear regime. The first term on the right-hand side, due to the convective nonlinearity, is estimated by

$$\int d\psi_h \langle\cos\alpha\rangle \oint (d\alpha/2\pi) h \left[b^2(w_L/r_s)\right] \approx b^2(w_L/r_s)W,$$

where the W-factor reflects the fact that $\langle\cos\alpha\rangle$ is small outside the magnetic island. Using Eqs. (21) and (37) the ratio of the resistive term to the convective term is

$$\eta b/\left[b^2(w_L/r_s)W\right] \approx (\eta/k'_{\|})W^{-2}w_L^{-1} \approx \delta^2.$$

Thus in lowest order Eq. (55) can be expressed as

$$\int_{-} d\psi_h\langle\cos\alpha\rangle \oint (d\alpha/2\pi)h\left\{\sqrt{2}\frac{w_L}{r_s}\left(\frac{b}{2}\right)^2 + \frac{db}{dt}\langle g\cos\alpha\rangle\right\} = 0. \tag{56}$$

Here we have introduced the abbreviation,

$$\int_{-} \equiv \int_{x<0}.$$

That is, we are to integrate over those ψ_h and α such that $x < 0$. This integration domain results ultimately from Eq. (1); recall that g is exponentially small for positive x.

Exponential growth of the nonlinear $m = 1$ island is apparent in Eq. (56). The point is that $\langle g\cos\alpha\rangle$ yields, upon integration, a factor of $-W \approx -(k'_{\|}r_s)b$, as can be seen from Eqs. (38) and (39). Therefore Eq. (56) reduces roughly to

$$db/dt \approx (k'_{\|}w_L)b, \tag{57}$$

in agreement with Eq. (3), since $\gamma_L \approx k'_{\|}w_L$.

C. Nonlinear growth rate

It is possible to evaluate the integrals in Eq. (56) exactly in the limit $\delta \to 0$. First we make the flux surface average explicit, by solving Eq. (38) for $x(\psi_h, \alpha)$. Since the integration domain has been restricted to $x < 0$, the result need not be accurate for positive x, and we can write, from Eq. (38),

$$F = x^2 + 2\Delta x\cos\alpha.$$

The solution is conveniently expressed as

$$x(\psi_h, \alpha) = -\Delta z_\sigma(F, \alpha),$$

where

$$z_\sigma(F,\alpha) = \cos\alpha - \sigma[\cos^2\alpha + F/\Delta^2]^{1/2}, \tag{58}$$

and $\sigma = \pm 1$. Notice that F can have either sign, but the domains of F, α, and the discrete variable σ are limited by

$$\cos^2\alpha + F/\Delta^2 > 0, \tag{59}$$

$$z_\sigma(F,\alpha) > 0. \tag{60}$$

Furthermore, Eq. (58) implies that the natural radial integration variable is $z \equiv F/\Delta^2$, rather than ψ_h or F.

Now we can express Eq. (14) for $\langle f\rangle$ in terms of z. By Eq. (15) $h^{-1} \propto dF/dx = 2(x + \delta\cos\alpha) \propto [\cos^2\alpha + F/\Delta^2]^{1/2}$, so we have

$$\langle f\rangle(z) = V^{-1}\oint (d\alpha/2\pi)[z + \cos^2\alpha]^{-1/2} f,$$

where

$$V(z) = \oint (d\alpha/2\pi)[z + \cos^2\alpha]^{-1/2}. \tag{61}$$

Next we return to Eq. (56). Notice that after using Eq. (39) to write g in terms of z_σ, we must include contributions from both choices of σ, constrained only by Eqs. (59) and (60). Hence we have

$$\sum_\sigma \int_- dz\langle\cos\alpha\rangle V\left\{\sqrt{2}\frac{w_L}{r_s}\left(\frac{b}{2}\right)^2 - \Delta\frac{db}{dt}\langle z_\sigma\cos\alpha\rangle\right\} = 0.$$

Straightforward simplification using Eq. (37) reduces this form to

$$\sum_\sigma \int_- dz\langle\cos\alpha\rangle V\left[\gamma_L b - 2\langle z_\sigma\cos\alpha\rangle\frac{db}{dt}\right] = 0. \tag{62}$$

The normalizations make z and its integration domain of order unity, so Eq. (62) is just a more precise version of Eq. (4); the coefficient μ can be identified as a rather complicated-looking ratio of integrals.

Actually μ is relatively simple, essentially because of the identity

$$\langle\cos\alpha\rangle V = (1/2)\Theta(-z), \tag{63}$$

stating that only the island interior contributes. Equation (63), the right-hand side of which would clearly be smoothed by corrections of higher order in δ, can be verified from the definition,

$$\langle \cos\alpha \rangle V = \oint (d\alpha/2\pi) \cos\alpha [z + \cos^2\alpha]^{-1/2}.$$

For positive z this integral includes all α and vanishes by symmetry; for negative z, the turning point integral is exact and yields Eq. (63). The result is to reduce Eq. (62) to

$$\mu^{-1} = \sum_\sigma \int_{-1}^{0} dz \langle z_\sigma \cos\alpha \rangle, \tag{64}$$

where

$$\gamma_{NL} \equiv \mu \gamma_L, \tag{65}$$

as in Eq. (4). Equation (64) further simplifies because for negative z, Eq. (60) allows both branches of the square root in z_σ to contribute:

$$\mu^{-1} = \int_{-1}^{0} dz \big\langle (z_+ + z_-) \cos\alpha \big\rangle = 2 \int_{-1}^{0} dz \langle \cos^2\alpha \rangle.$$

We finally use elementary identities and Eq. (61) to evaluate the flux surface average, noting that only the negative-z domain matters. The result is

$$\mu^{-1} = 2 \int_0^1 dk^2 E(k^2)/K(k^2) \approx (0.71)^{-1}, \tag{66}$$

where E and K are complete elliptic integrals of the first and second kind respectively. In other words, Eq. (56) predicts the nonlinear growth rate of the $m = 1$ island to be about 30% less than the linear one.

IV. Discussion

A. Main assumptions

The result that the $m = 1$ tearing mode continues to grow at an exponential rate in the nonlinear regime depends on the validity of three essential assumptions:

1. Single helicity: $\psi(r, \theta, \varsigma, t) = \psi(r, \alpha = \varsigma - \theta, t)$. This assumption rules out the nonlinear interaction of modes on various rational surfaces which could cause stochastic field lines and enhanced transport, thus accelerating the growth of the mode.

2. Separability: $\psi(r, \alpha, t) = b(t)g(r, \alpha)$. This approximation is a Galerkin truncation, where all the other radial modes and α harmonics have been neglected. The linear problem can be expanded in a complete set of eigenfunctions, which are Laguerre functions radially ($|x| \ll r_s$) and sinusoidal in α. All of the modes are strongly damped except for the tearing mode we retain. We expect if the other modes are retained, their amplitudes will be slaved to that of the tearing mode and thus provide an additional source of dissipation.
3. The neglect of sheet current at the rational surface: numerical experiments show the formation of a sheet current at the rational surface with a width given by the linear layer width, w_L, and an amplitude increasing in time.[6] Our result, Eq. (46), which depends on the current integrated over the island, will have contributions from this sheet current as well. Notice that the contribution of the sheet current to Eq. (42) will scale as $w_L b$, since the width of the sheet, w_L, is constant in time. Because the non-sheet contribution that we retain scales as $Wb \approx b^2$, it should dominate at sufficiently large island width. Of course the omission of sheet-current effects as well as other w_L/W corrections make our quantitative result inexact.

Other simplifications, such as the evaluation of the convective nonlinear term using the linear mode functions, and the neglect of cylindrical effects, will also affect the exact numerical value of the coefficient μ, but should not change the basic result of exponential growth. This result is already apparent from Eq.(48) when it is realized that $\bar{\psi}$ evolves proportional to the square of the $m = 1$ disturbance and dominates resistive diffusion in the nonlinear regime.

B. Experimental significance

There is considerable experimental evidence that an $m = 1$ disturbance is present just before and during the temperature crash of the sawtooth oscillation. Conventional theory assumes that ohmic heating of the plasma drives the q on axis below one and triggers an $m = 1$ instability.[3] In order for the $m = 1$ tearing mode to be a possible cause of the crash, the nonlinear growth rate must be larger than the inverse crash time.

At this point it is convenient to revert to dimensional variables. Thus the growth rate of Eq. (20) is multiplied by ϵ/τ_A to obtain the dimensional growth rate

$$\gamma_{NL} = \frac{\mu}{\tau_A S^{1/3}} \left(\frac{a^2}{r_s L_s} \right)^{2/3}, \tag{67}$$

where τ_A is the Alfvén time, S is the magnetic Reynolds number, and L_s is the shear length at the $q = 1$ surface, defined by $R/(r_s q')$. Interestingly, the last factor in Eq. (66), which is the most uncertain because the value of L_s is difficult to measure, is probably very close to unity. The values of these parameters as well as the value of γ_{NL} and the measured time of the electron temperature crash, τ_{cr}, are given in Table 1 for TEXT[11] and TFTR[12] (Fall of '84 parameters) tokamaks. The growth rate for the $m = 1$ tearing mode appears sufficient to explain the sawtooth crash.

In low field tokamaks, such as ISXB,[13] the diamagnetic drift frequency, ω_*, can be larger than the linear $m = 1$ growth rate, and its modification to the growth rate should be included. However in the machines listed in Table 1 as well as in reactor regimes, ω_* is smaller than the computed growth rate. In this case one presumes that it can be neglected nonlinearly as well.

TABLE I

	TEXT	TFTR
τ_A (s)	$3.4(10)^{-8}$	$7.0(10)^{-8}$
S	$6.8(10)^7$	$3.2(10)^8$
a (cm)	27	83
r_s (cm)	10	20
L_s (cm)	200	500
$\omega_*(s^{-1})$	$4.5(10)^3$	$9.1(10)^2$
$\gamma_{NL}(s^{-1})$	$2.6(10)^4$	$1.1(10)^4$
$\tau_{cr}^{-1}(s^{-1})$	$5(10)^3$	$7(10)^3$

The collisionality regime of present day tokamaks near the $q = 1$ surface is determined by two criteria. Firstly, the ratio of the electron collision rate ν_e, to γ_{NL} is about 5 in both TEXT and TFTR and even larger in reactor regimes. Secondly, the assumption of short mean-free-path is equivalent to $k'_{||} v_e w_L \ll (\gamma \nu_e)^{1/2}$, where v_e is the electron thermal speed. Since $k'_{||} v_A w_L = \gamma_L$, short mean-free-path theory is appropriate for $m = 1$ tearing only if $\beta m_i / m_e \ll \nu_e / \gamma$. This criterion is hardly satisfied in present machines and unlikely to be satisfied for fusion reactor parameters. The implication is that the semi-collisional regime,[10] with a current channel due to long mean-free-path effects in the neighborhood of the rational surface, is the relevant regime. While the linear growth rate of the mode in this regime is probably close to that given by the resistive result, the nonlinear aspects of $m = 1$ tearing at long mean-free-path have not yet been treated.

In conclusion, the $m = 1$ tearing mode, in our opinion, remains a prime candidate for the explanation of the sawtooth crash in tokamaks.

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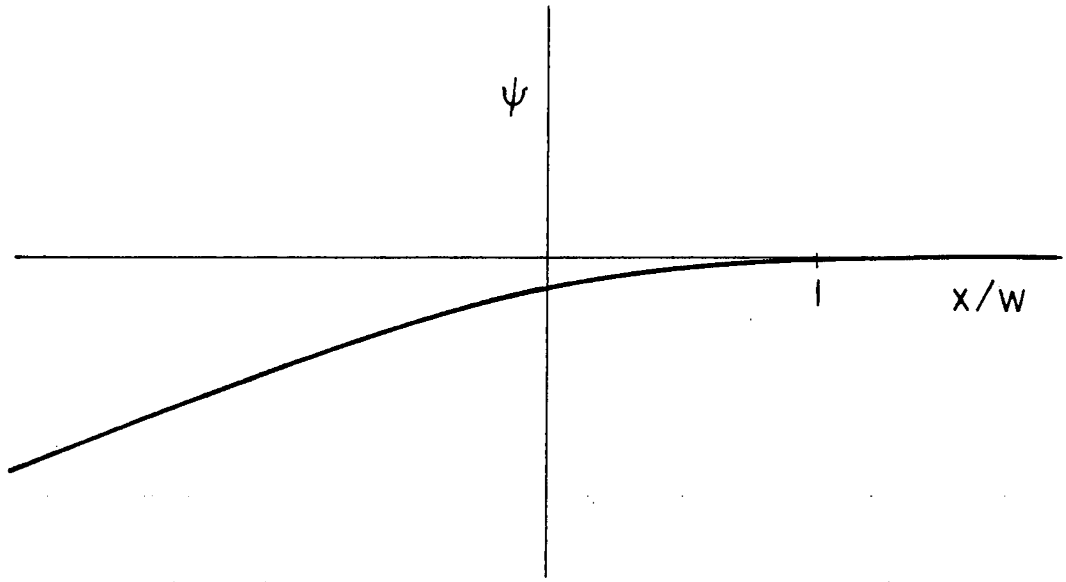
References

1. B. Coppi, R. Galvão, M.N. Rosenbluth, and P.H. Rutherford, Sov. J. Plas. Phys. **2**, 533 (1976); G. Aru, B. Basu, B. Coppi, G. Laval, M.N. Rosenbluth, and B.V. Waddel, Ann. Phys. **112**, 443 (1978).
2. R.B. White, "Resistive Instabilities and Field Line Reconnection," Handbook of Plasma Physics, M.N. Rosenbluth and R.Z. Sagdeev (eds.), (North Holland, 1983).
3. R.D. Hazeltine and J.D. Meiss, Phys. Reports **121**, 1 (1985).
4. W.A. Newcomb, Ann. Phys. **10**, 232 (1960).
5. P.H. Rutherford, Phys. Fluids **16**, 1903 (1973).
6. B.V. Waddell, M.N. Rosenbluth, D.A. Monticello, and R.B. White, Nuc. Fusion **16**, 528 (1976); A. Aydemir, private communication, 1985.
7. H.R. Strauss, Phys. Fluids, **19**, 134 (1976).
8. R. Pellat, M. Frey, and M. Tagger, J. Physique **45**, 1615 (1984).
9. G. I. Jahns, M. Soler, B. V. Waddell, J. D. Callen, and H. R. Hicks, Nucl. Fusion **18**, 609 (1978).
10. J.F. Drake and Y.C. Lee, Phys. Fluids **20**, 1341 (1977).
11. R.V. Bravenec, K.W. Gentle, P.E. Phillips, T.R. Price, W.L. Rowan, K. Empson, W.L. Hodge, C. Klepper, T.P. Kochanski, D.M. Patterson, J. Porter, and B. Richards, Plasma Phys. and Cont. Fusion **27**, 1335 (1985).
12. K.M. Young, M. Bell, W.R. Blanchard, N. Bretz, J. Cecchi, J. Coonrod, S. Davis, H.F. Dylla, P.C. Efthimion, R. Fonck, R. Goldston, D.J. Grove, R.J. Hawryluk, H. Hendel, K.W. Hill, J. Isaccson, L.C. Johnson, R. Kaita, R.B. Krawchuk, R. Little, M. McCarthy, D. McCune, K. McGuire, D. Meade, S.S. Medley, D. Mikkelson, D. Mueller, E. Nieschmidt, D.K. Owens, A. Ramsey, A.L. Roquimore, L. Samuelson, N. Sauthoff, J. Schievell, J.A. Schmidt, S. Sesnic, J. Sinnis, J. Strachan, G.D. Tait, G. Taylor, F. Tenney, and M. Ulrickson, Plasma Phys. and Cont. Fusion **26**, 11 (1984).
13. M. Murakami, P.H. Edmonds, G.A. Hallock, R.C. Isler, E.A. Lazarus, G.H. Neilson, A.J. Wootton, J.D. Bell, C.E. Bush, A. Carnevali, B.A. Carreras, J.L. Dunlap, A.C. England, W.L. Gardner, R.L. Hickok, H.C. Howe, D.P. Hutchinson,

W.C. Jennings, R.R. Kindsfather, R.A. Langley, C.H. Ma, J. Mathew, P.K. Mioduszewski, J.K. Nunro, V.K. Pare, M.J. Saltmarsh, S.D. Scott, D.J. Sigmar, M.L. Simpson, C.E. Thomas, R.M. Wieland, W.R. Wing, and K.E. Yokoyama, *Plasma Physics and Controlled Nuclear Fusion Research, 10th International Conference, London*, 1984 (IAEA, Vienna, 1985), Vol. 1, p. 87.

Figure Caption

Magnetic perturbation for the linear $m = 1$ tearing mode.


ψ
1
x/w