DOI/ET-53088-202 IFSR #202

Effect of Toroidicity during Lower Hybrid Mode Conversion

S. Riyopoulos and *S. Mahajan*
Institute for Fusion Studies
The University of Texas at Austin
Austin, Texas 78712

November 1985

Effect of Toroidicity during Lower Hybrid Mode Conversion

*S. Riyopoulos** and *S. Mahajan*

Institute for Fusion Studies

The University of Texas at Austin

Austin, Texas 78712

Abstract

The effect of toroidicity during lower hybrid mode conversion is examined by treating the wave propagation in an inhomogeneous medium as an eigenvalue problem for $\omega^2(m,n)$, m,n poloidal and toroidal wave numbers. Since the frequency regime near $\omega^2 = \omega_{\mathrm{LH}}^2$ is an accumulation point for the eigenvalue spectrum, the degenerate perturbation technique must be applied. The toroidal eigenmodes are constructed by a *zeroth* order superposition of monochromatic solutions with different poloidal dependence m, thus they generically exhibit a wide spectrum in $k_{\|}$ for given fixed ω^2 even for small inverse aspect ratio ϵ. In case that the average $\langle k_{\|}\rangle$ is in the neighborhood of k_{min}, the minimum wave number for accessibility of the mode conversion regime, it is possible that excitation of toroidal modes rather than geometric optics may determine the wave coupling to the plasma. Our results are not changed significantly by a small amount of dissipation. The level of density fluctuations in modern tokamaks, on the other hand, may cause enough $k_{\|}$ scattering to mask the toroidicity effects. Nevertheless, it is shown that a wide $k_{\|}$ spectrum excited by a monochromatic pump will persist even with vanishing fluctuation level.

* Naval Research Laboratory, Washington, D. C. 20375

I. Introduction

The wave propagation in the lower hybrid frequency regime across magnetized plasmas, with all the related phenomena, has lately been the subject of extensive study[1−5] in connection with plasma heating[6,7] and current drive[8] during tokamak operation near thermonuclear conditions. The long wavelength slow electromagnetic mode, also identified as fast magnetosonic or extraordinary X-mode in case $k_{\|} = 0$, is mode converted to a short wavelength electrostatic mode near the layer x_0 such that $\omega^2 = \omega_{\mathrm{LH}}^2(x_0)$ where ω_{LH} defined by

$$\omega_{\mathrm{LH}}^2 = \omega_{pi}^2 \left[1 + \omega_{pe}^2/\Omega_{ce}^2\right]^{-1}$$

corresponds to the lower hybrid resonance encountered in the cold plasma description.

The procedure is now well understood in case the inhomogeneity is limited in one-dimension with the dielectric properties of the medium varying in a direction perpendicular to the magnetic field. The original analysis by Stix,[1] carried out by expanding the warm plasma dielectric tensor to first order in $k_{\perp}^2 \rho_L^2$ and then simply replacing k_x by $-i(d/dx)$, leads to a fourth order differential equation that demonstrates a complete mode conversion of the incoming slow electromagnetic mode to a fast electrostatic (ES) lower hybrid mode, the group velocity of which is directed back toward the low density regime. A more systematic treatment of the inhomogeneity performed by Wong and Tang[5] by utilizing the non-uniform medium dielectric response[9−10] essentially confirms the results of the previous simpler treatment. It was also shown by the same authors that somewhere between the first mode conversion layer and the plasma boundary the fast ES mode encounters a second regime of a complete mode conversion, emerging as a short wavelength slow ES mode with a group velocity directed towards the plasma center (see Fig. 1).

The accessibility condition[11−12] for the lower hybrid mode conversion regime is

$$k_{\|}^2 c^2/\omega^2 > 1 + \omega_{pe}^2/\Omega_{ce}^2. \tag{1}$$

For shorter parallel wavelengths $k_{\|}$, the incoming slow electromagnetic wave is backscattered into the fast electromagnetic branch and propagation towards the mode conversion regime is prohibited. Both wave penetration and energy deposition in the plasma core depend on $k_{\|}$, therefore any changes in $k_{\|}$ caused by toroidal effects will have an impact on the two most important issues for lower hybrid heating.

One approach in modeling the toroidal effects is to examine the trajectory of a wave package determined by the local group velocity $V_g(\mathbf{x}) = (\partial D/\partial \mathbf{k})(\partial D/\partial \omega)^{-1}$ with $D(\omega, \mathbf{k}; \mathbf{x})$ the local dispersion relation. The ray tracing method has been employed by Bonoli and Ott[13] in examining the wave propagation with toroidicity to address the question of accessibility of the lower hybrid mode conversion layer. It was found that even for small values of ϵ the onset of ergodic ray trajectories allows large changes in $k_{\|}$ thus converting originally inaccessible waves into ones that satisfy the accessibility condition. However, the mode conversion process in itself cannot be studied using this method as the WKB approximation on which ray tracing relies breaks down[14] near a mode conversion regime. Furthermore, after a few wave reflections between cut-off and the plasma edge the boundary effects come into play as the wave spreads through the plasma volume. In this case it appears that excitation of global toroidal eigenmodes should be considered in examining the coupling to the plasma.

When the plasma parameters change in the poloidal direction, the toroidal eigenmodes are expected to be non-monochromatic in θ due to the interaction among the different poloidal Fourier harmonics of the wave and the dielectric and metric tensor. As $k_{\|}$ depends on both the poloidal and toroidal wave numbers m and n, the poloidal spectral width Δm generates a natural width $\Delta k_{\|}$ of a single frequency eigenmode that remains to be examined. So far, the experimentally observed wide spectrum in $k_{\|}$ has been investigated only in connection with wave scattering[13,15,16] from density fluctuations.[17,18] A solution for the wave propagation equation with toroidal effects is therefore necessary to address both issues of the mode conversion and the natural spectral width for a single mode of frequency $\omega \cong \omega_{\rm LH}$.

The amount of mathematical complexity involved in the one-dimensional analysis is indicative of the difficulty that one faces when the effects of the inhomogeneity in a second direction, produced by toroidicity, have to be examined. In principle, the two-dimensional solutions can be built around the one-dimensional ones using perturbation theory, where from now on and for the sake of simplicity, the terms one- and two-dimensional will signify the number of degrees of inhomogeneity. However, the usual calculation by means of solving differential equations to increasing order in the perturbation parameter gets immediately complicated due to the large number of terms resulting from the dependence of the differential coefficients on two variables.

In this paper we use a perturbation technique[19] that circumvents direct integration by formulating the wave propagation equation as an eigenvalue problem

$$\overleftrightarrow{D}\underline{E} - \frac{\omega^2}{c^2}\overleftrightarrow{K}^{(0)}\underline{E} = 0 \tag{2}$$

where the 3×3 differential operator $\overleftrightarrow{D}$ incorporates the $\nabla \times \nabla \times$ operator as well as the warm plasma effects, $\overleftrightarrow{K}^{(0)}$ is the cold plasma contribution to the dielectric tensor and $\underline{E}$ is the vector eigenfunction for the electric field. Then if certain properties such as orthogonality and completeness are satisfied by the set of the one-dimensional solutions and if the non-uniformity in the second direction is small, the two-dimensional solutions can be constructed by a linear superposition of the one-dimensional eigenmodes. The expansion coefficients are determined by evaluating the integral projections among the known one-dimensional eigenmodes in respect to some weighting matrix determined by the perturbation, thus our method is the electromagnetic analog of the quantum mechanical perturbation theory.

The above techniques are applicable through the entire range of the eigenvalue spectrum ω^2 of Eq. (2). However, the regime near the lower hybrid frequency is of particular importance as it contains very closely spaced eigenfrequencies $\omega^2(m,n)$ with m,n the poloidal and toroidal wave numbers, thus the subset of eigenmodes with $\left|\omega^2(m,n) - \omega^2_{\rm LH}(x_0)\right| / \omega^2_{\rm LH}(x_0) \ll \epsilon$ where ϵ is the inverse aspect ratio must be treated according to the degenerate perturbation technique. It follows that although the resulting correction in frequency $\Delta\omega^2$ is small $0(\epsilon)$ the modification in the poloidal dependence is large as the two-dimensional modes are constructed from a zeroth order mixing of almost degenerate eigenmodes with different poloidal wave numbers. To put it differently, even for ϵ very small, the new eigenmodes generically contain a strong spectral contribution from a wide band of poloidal wave numbers m and therefore $k_{\parallel}$. Given that the band of the one-dimensional eigenmodes contributing into a single two-dimensional solution has a relative spectral width $\Delta\omega^2/\omega_{\rm LH}^2$ much less than ϵ, say ϵ^2, an estimate for the spread in the parallel wave number is

$$\frac{\Delta k_{\|}}{k_{\|}} \sim \epsilon^2 \left(\frac{\omega_{\rm LH}}{k_{\|}}\right) \Big/ \left|\frac{\partial \omega}{\partial k_{\|}}\right| \sim \epsilon^2 \frac{(v_{ph})_{\|}}{(v_{gr})_{\|}}.$$

Consequently, $\Delta k_{\|}$ can be large whenever the parallel group velocity falls much below the parallel phase velocity. This situation arises generally during linear mode conversion characterized by $\partial\omega/\partial \underline{k} = 0$. We focus on the case of the lower hybrid mode conversion, but our method is also applicable to the case of the two-ion hybrid mode conversion[20] during ion cyclotron heating in plasmas with two-ion species.

The structure of the toroidal modes as a linear superposition of one-dimensional modes suggests that the accessibility and mode conversion are determined by the location of the spectral band $\Delta k_{\|}$ centered around $k_{\|}^0$ relative to the minimum accessible wavelength $k_{\rm min}$. For $\left|k_{\|}^0 - k_{\rm min}\right| < \Delta k_{\|}/2$, there will always be a partial reflection at the cut-off and a partial conversion near the lower hybrid layer. Strictly speaking, complete accessibility is unattainable as $\Delta k_{\|}$ may tend to infinity, however for practical energy absorption purposes only the contribution $\Delta k_{\|}^0$ from the degenerate modes into a toroidal mode matters and must be considered.

The above conclusions have been drawn without the inclusion of dissipation or density fluctuations. It will be seen that the inclusion of finite dissipation does not change the picture significantly as long as the damping rate $\gamma/\omega_{\rm LH}$ remains much smaller than the inverse aspect ratio ϵ. Of greater importance are the density fluctuations near the plasma edge depending on whether the length of the turbulent layer is comparable to the scattering length for $k_{\|}$. The main outcome in our analysis is that even in a relatively quiet plasma of low level turbulence, a wide spectrum in $k_{\|}$ will be naturally excited by a monochromatic antenna. For relatively high levels of edge turbulence, the effect of degeneracy will be masked by the broadening in the $k_{\|}$ spectrum due to scattering off fluctuations. However, the eigenmode approach remains relevant in any case that the launched waves eventually occupy most of the plasma volume. The effect of fluctuations can be accounted for by the introduction of stochastic terms in the deterministic wave equation (2) and solving a stochastic eigenvalue problem for $\langle\omega^2\rangle$ and $\langle\Delta k_{\|}\rangle$. Although the methodology for solving stochastic eigenvalue equations (with stochastic boundary condition) exists, it is beyond the scope of the present work.

The rest of this paper is organized as follows: In Sec. II we establish the symmetry of the propagation operator and the local orthogonality among nearby eigenmodes in case of one-dimensional non-uniformity. A new set of completely orthonormal modes that approximate the exact solutions in the frequency regime under consideration is selected as the appropriate expansion basis. In Sec. III the effects of toroidicity are introduced to the lowest order in the wave propagation equation. The two-dimensional eigenmodes are constructed by a linear superposition of the one-dimensional solutions applying the degenerate perturbation technique. It is then shown that a wide spectrum in $k_{||}$ is a generic characteristic of the toroidal modes due to frequency accumulation in the spectrum. We conclude and summarize our results in Sec. IV.

II. Symmetry and Orthogonality with Non-Uniformity in One Direction

We use slab geometry to model the situation as the wavelength λ is much shorter than the minor radius r with x in the radial direction and the magnetic field B on the yz plane making an angle $\alpha = \epsilon/q$ relative to the z direction, q the safety factor. Periodic boundary conditions apply along y and z with periods $L_y = 2\pi r$ and $L_z = 2\pi R$, implying $k_y = m/r$ and $k_z = n/R$. The wave propagation equation for the one-dimensional case with the non-uniformity along x is given by

$$D_{ij}E_j - \frac{\omega^2}{c^2}K_{ij}^{(0)}E_j = 0 \qquad (3a)$$

$$D_{ij} = L_{ij} + \left[K_{ij}^{(2)}\partial_x^2 + K_{ij}^{(1)}\partial_x\right] \qquad (3b)$$

with $K_{ij}^{(p)}(\omega, k_y, k_z; x)$ the elements of the non-uniform plasma dielectric tensor given in Ref. 5. Equation (3) is a slight modification of Eq. (A2) in the above reference in order to include the finite shear angle α using the field aligned representation L of the $\nabla\times\nabla\times$ operator

$$L_{ij} = [T(\alpha)(\nabla\times\nabla\times)T(\alpha)]_{ij}, \qquad (4)$$

where $T(\alpha)$ is the rotation matrix around the x direction defining the new field aligned basis vectors

$$\begin{pmatrix}\hat{e}_1\\ \hat{e}_2\\ \hat{e}_3\end{pmatrix} = T(\alpha)\begin{pmatrix}\hat{e}_x\\ \hat{e}_y\\ \hat{e}_z\end{pmatrix}, \quad T(\alpha) = \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\alpha & -\sin\alpha\\ 0 & \sin\alpha & \cos\alpha\end{pmatrix} \qquad (5)$$

shown in Fig. 2. Note that $\hat{e}_1 \equiv \hat{e}_x$, $\hat{e}_2 \equiv \hat{e}_\perp$, $\hat{e}_3 \equiv \hat{e}_\parallel$ with $\hat{e}_\perp$, $\hat{e}_\parallel$ the unit vectors on the yz plane perpendicular and parallel respectively to the local magnetic line direction.

We order the small parameters v_i/c, $n_z = k_z c/\omega$, $\alpha = \epsilon/q$, $(m_e/m_i)^{1/2} \sim 0(\zeta) \ll 1$. Then in the vicinity of the lower hybrid regime $|\omega - \omega_{\rm LH}|/\omega_{\rm LH} \sim \zeta$ the wave propagation equation is given to order ζ^2 by

$$\overleftrightarrow{D}(\omega_{\rm LH})\underline{E} - \left(\frac{\omega^2}{c^2}\right)\overleftrightarrow{K}^{(0)}(\omega)\underline{E} = 0. \qquad (6)$$

Utilizing the following relations among the warm plasma dielectric tensor components

$$\begin{aligned} K_{ii}^{(1)} &= \left[K_{ii}^{(2)}\right]', \quad \left[K_{ii}^{(p)}\right]^* = K_{ii}^{(p)}, \qquad p = 1,2\\ K_{ij}^{(2)} &= \left[K_{ji}^{(2)}\right]^*, \quad K_{21}^{(1)} = K_{12}^{(1)} + 2\left[K_{21}^{(2)}\right]', \end{aligned} \qquad (7)$$

dropping terms $\sim (v_i/c)^2 \left[\omega_p^2(x)\right]'' \sim 0\left(\zeta^3\right)$ and with $|\underline{B}|$ considered constant, $\overleftrightarrow{D}(\omega)$ takes the form

$$
\begin{aligned}
D_{11} &= -\partial_{\|}^2 - \partial_{\perp}^2 - \partial_x K_{11}^{(2)}(\omega)\partial_x \\
D_{12} &= \partial_x\partial_{\perp} - \left[K_{12}^{(2)}(\omega)\partial_x^2 + K_{12}^{(1)}(\omega)\partial_x\right] \\
D_{13} &= \partial_x\partial_{\|} \\
D_{21} &= \partial_{\perp}\partial_x + \left[\partial_x^2 K_{12}^{(2)}(\omega) - K_{12}^{(1)}(\omega)\partial_x\right] \\
D_{22} &= -\partial_{\|}^2 - \partial_x^2 + (\alpha')^2 - \left[\partial_x K_{22}^{(2)}(\omega)\partial_x\right] \\
D_{23} &= -\partial_{\perp}\partial_{\|} + 2\alpha'\partial_x + \alpha'' \\
D_{31} &= \partial_{\|}\partial_x \\
D_{32} &= -\partial_{\|}\partial_{\perp} - (2\alpha')\partial_x - \alpha'' \\
D_{33} &= -\partial_{\perp}^2 - \partial_x^2 + (\alpha')^2 - \left[\partial_x K_{33}^{(2)}(\omega)\partial_x\right]
\end{aligned}
\tag{8}
$$

with

$$
(') \equiv \partial_x, \quad \partial_{\|} = \cos\alpha\partial_z + \sin\alpha\partial_y, \quad \partial_{\perp} = \cos\alpha\partial_y - \sin\alpha\partial_z
$$

$$
\begin{aligned}
K_{11}^{(2)} &= \sum_a \frac{3\omega^2\omega_{pa}^2}{\left(\omega^2 - \Omega_a^2\right)\left(\omega^2 - 4\Omega_a^2\right)}\frac{v_a^2}{c^2} \\
K_{22}^{(2)} &= \sum_a \frac{\left(\omega^2 + 8\Omega_a^2\right)\omega_{pa}^2}{\left(\omega^2 - \Omega_a^2\right)\left(\omega^2 - 4\Omega_a^2\right)}\frac{v_a^2}{c^2} \\
K_{33}^{(2)} &= \sum_a \frac{\omega_{pa}^2}{\omega^2 - \Omega_a^2}\frac{v_a^2}{c^2} \\
K_{12}^{(2)} &= i\sum_a \frac{6\Omega_a\omega\omega_{pa}^2}{\left(\omega^2 - \Omega_a^2\right)\left(\omega^2 - 4\Omega_a^2\right)}\frac{v_a^2}{c^2} \\
K_{12}^{(1)} &= i\sum_a \frac{\omega\left(\omega^2 + 2\Omega_a^2\right)\left(\omega_{pa}^2\right)'}{b\Omega_a\left(\omega^2 - \Omega_a^2\right)\left(\omega^2 - 4\Omega_a^2\right)}\frac{v_a^2}{c^2}.
\end{aligned}
\tag{9}
$$

The operators $\partial_{\|}$, $\partial_{\perp}$ act on the wave function

$$
\begin{aligned}
\underline{E} &= \underline{E}_0(x)e^{i\left(\frac{m}{r}y + \frac{n}{R}z - \omega t\right)} \\
\underline{E}_0(x) &= \left(E_x(x),\ E_{\perp}(x),\ E_{\|}(x)\right)
\end{aligned}
\tag{10}
$$

according to $\partial_{\|}\underline{E} = ik_{\|}\underline{E}$, $\partial_{\perp}\underline{E} = ik_{\perp}\underline{E}$ with $k_{\|}(x) = \cos\alpha\frac{n}{R} + \sin\alpha\frac{m}{r}$, $k_{\perp}(x) = \cos\alpha\frac{m}{r} - \sin\alpha\frac{n}{R}$.

We now review some properties of Eq. (6). As the wave propagates towards resonance it becomes progressively electrostatic. The picture is simplified by assuming that $k_\perp \ll k_\|, k_x$. Then ignoring all terms of order v_i^2/c^2 except in $K_{11}^{(2)}$ and setting $X \equiv E_x$, $Y \equiv E_\perp$, $Z \equiv E_\|$, the following equation results for the parallel component of the electrostatic field

$$\begin{gathered} K_{11}^{(2)} \frac{d^4 Z}{dw^4} + 2\left\{ \left(K_{11}^{(2)}\right)' + \frac{K_{11}^{(2)} \left(K_{11}^{(0)}\right)'}{n_\|^2 - K_{11}^{(0)}} \right\} \frac{d^3 Z}{dw^3} + K_{11}^{(0)} \frac{d^2 Z}{dw^2} \\ + \frac{n_\|^2 \left(K_{11}^{(2)}\right)'}{n_\|^2 - K_{11}^{(2)}} \frac{dZ}{dw} - \left(n_\|^2 - K_{11}^{(2)}\right) Z = 0 \end{gathered} \tag{11a}$$

with $n_\| = k_\| c/\omega$ and $w = \frac{\omega}{c} x$. Equation (11a) contains a resonance cut-off pair at x_R, x_c respectively defined by $K_{11}^{(0)} \left[\frac{\omega}{c} x_R\right] = 0$, $K_{11}^{(0)} \left[\frac{\omega}{c} x_c\right] = n_\|^2 \left[\frac{\omega}{c} x_c\right]$. The region between x_c and x_R is a non-propagation regime for x_c located on the low density side of x_R. Consequently, the resonant layer can be accessed if there is no cut-off between the plasma edge and x_R guaranteed by the accessibility condition Eq. (1). In this case, the third order derivative is eliminated by a linear transformation of the dependent variable Z and a linear expansion around the mode conversion point x_0 near x_R leads to the following differential equation in $\xi = (x - x_0)\frac{\omega}{c}$

$$\mu^2 \frac{d^4 Z'}{d\xi 4} + \left[\xi \frac{d^2 Z'}{d\xi^2} + \frac{dZ'}{d\xi} + \nu Z'\right] = 0 \tag{11b}$$

with μ and ν parameters, exhibiting a full mode conversion near $\xi \cong 0$. The approximate dispersion relation is obtained from the uniform medium electrostatic approximation $K_{11}^{(0)} + K_{11}^{(2)} = 0$,

$$\begin{gathered} \omega^2(m,n) = \omega_{\rm LH}^2 + \frac{3}{4} \Omega_c^2 \left[k_x^2 + k_\perp^2\right] \rho^2 \\ k_\perp^2 = \left(\cos\alpha \frac{m}{r} - \sin\alpha \frac{n}{R}\right)^2 . \end{gathered} \tag{12}$$

It is reasonable to assume that Eq. (12) that does not include the effects of nonuniformity still provides the approximate separation between nearby frequencies in the nonuniform case with density scale length much longer than the wavelength of lower hybrid waves. Going back to the non-uniform plasmas, if $\omega_{\rm LH}^2$ in Eq. (12) is the lower hybrid frequency at $x = x_0$, $\omega_{\rm LH}^2(x_0) \equiv \omega_0^2$, then each of the frequencies $\omega^2(m,n)$ also matches the local lower hybrid frequency at some nearby layer $x = x_{mn}$ defined by $\omega_{\rm LH}^2(x_{mn}) = \omega^2(m,n)$. Therefore, there is a sequence of closely spaced mode conversion layers each one associated with some eigenvalue ω^2 located within the band $\min\{\omega_{\rm LH}^2\} \le \omega^2 \le \max\{\omega_{\rm LH}^2\}$. Were it not for the thermal effects that resolve the singularity at $\xi = 0$ and quantize ω^2, this set of lower hybrid modes would become a continuum similar in structure with the MHD continuum, resulting from Eq. (11b) with $\mu = 0$.

We now want to obtain an orthonormal set of eigenmodes related to the solutions of Eq. (13). The following relation holds between any two eigenfunctions of Eq. (6).

$$\int_a^b dx \int_0^{Ly} dy \int_0^{Lz} dz \left(\underline{E}_1^* \overleftrightarrow{D}\, \underline{E}_2 - \underline{E}_2 \overleftrightarrow{D}^* \underline{E}_1 \right)$$
$$= \frac{\omega_2^2}{c} \left\langle \underline{E}_1 \middle| \overleftrightarrow{K}^{(0)}(\omega_2) \middle| \underline{E}_2 \right\rangle - \frac{(\omega_1^2)}{c^2} \left\langle \underline{E}_1 \middle| \overleftrightarrow{K}^{(0)}(\omega_1) \middle| \underline{E}_2 \right\rangle, \tag{13}$$

where the Hermiticity of the cold plasma dielectric tensor $K^{(0)} = \left(K_T^{(0)}\right)^*$ with $K_T^{(0)}$ the transpose of $K^{(0)}$, was used in shaping the rhs of Eq. (13) and the inner product is defined by

$$\left\langle \underline{E}_1 \middle| \overleftrightarrow{K}^{(0)}(\omega_2) \middle| \underline{E}_2 \right\rangle = \int_a^b dx \int_0^{Ly} dy \int_0^{Lz} dz \underline{E}_1^* \overleftrightarrow{K}^{(0)}(\omega_2) \underline{E}_2. \tag{14}$$

The operator $\overleftrightarrow{D}$ is by definition symmetric if there exists a set of boundary conditions such that the lhs of Eq. (13) becomes zero. This is trivially proven for $m_1 \ne m_2$ or $n_1 \ne n_2$. For $m_1 = m_2$, $n_1 = n_2$ but $\omega_1^2 \ne \omega_2^2$ it is straight forward to show after some lengthy algebra that the integrand is an exact differential

$$\begin{aligned}
\frac{\omega_{\rm LH}^2}{c^2}\frac{d}{dx}\Big\{ &\left(X_1^* K_{11}^{(2)} \partial_x X_2 - X_2 K_{11}^{(2)} \partial_x X_1^*\right) \\
&+ \left(Y_1^* K_{22}^{(2)} \partial_x Y_2 - Y_2 K_{22}^{(2)} \partial_x Y_1^*\right) \\
&+ \left(Z_1^* K_{33}^{(2)} \partial_x Z_2 - Z_2 K_{33}^{(2)} \partial_x Z_1^*\right) \\
&+ i\left[k_\parallel \left(X_1^* Z_2 + Z_1^* X_2\right) + k_\perp \left(X_1^* Y_2 + Y_1^* X_2\right)\right] \\
&+ 2\alpha \left(Y_1^* Z_2 - Z_1^* Y_2\right) \\
&+ K_{12}^{(1)} \left(X_1^* Y_2 + Y_1 X_2\right)
\end{aligned}$$

$$+\left[K_{12}^{(2)}\left(X_1^*\partial_x Y_2 - Y_2\partial_x X_1^*\right)\right.$$

$$\left.\left. + K_{21}^{(2)}\left(Y_1^*\partial_x X_2 - X_z\partial_x Y_1^*\right)\right]\right\},$$

and the integral on the lhs vanishes under a set of general boundary conditions that include perfectly conducting boundaries as a special case, i.e.

$$Y,\ Z,\ \omega_{pa}^2 \quad \text{zero at} \quad x = b \text{ (boundary)}$$

$$X,\ Y,\ Z \quad \text{zero at} \quad x = a = 0 \text{ (``axis'')}.$$

For ω_1 and ω_2 very close to $\omega_0 \equiv \omega_{\rm LH}(x_0)$ setting $K^{(0)}(\omega_1) \cong K^{(0)}(\omega_2) \cong K^{(0)}(\omega_0)$ in Eq. (13) and using the Hermiticity of $K^{(0)}$ leads to

$$\left(\omega_1^2 - \omega_2^2\right)\left\langle \underline{E}_1 \left| \overleftrightarrow{K}^{(0)}(\omega_0) \right| \underline{E}_2 \right\rangle \cong 0. \tag{15}$$

Thus, given that the cold plasma dielectric tensor $\overleftrightarrow{K}^{(0)}$ is Hermitian, the symmetry of the operator $\overleftrightarrow{D}$ is the necessary and sufficient condition that eigenfunctions corresponding to neighboring ωs are almost orthogonal. This motivates us to replace $\overleftrightarrow{K}^{(0)}(\omega)$ by $\hat{K}^{(0)} \equiv \overleftrightarrow{K}^{(0)}(\omega_0)$ in Eq. (6) as the solutions $\underline{\hat{E}}$ of the modified equation

$$\overleftrightarrow{D}\underline{\hat{E}} - \frac{\omega^2}{c^2}\hat{K}^{(0)}\underline{\hat{E}} = 0 \tag{16}$$

constitute an exactly orthonormal set of eigenmodes that in the vicinity of $\omega^2 \cong \omega_{\rm LH}^2(x_0)$ are good approximations of the solutions $\underline{E}$ of the exact Eq. (6).

We assume completeness without proving it. It has been proved in the simple case of wave propagation in the MHD regime with a linear density profile, when the behavior in the x direction is determined by a modified Bessel equation.[21] In the case under consideration, the completeness will depend on the properties of the solutions of differential equations of the type (11a), a subject left for further research.

The Landau damping terms have been omitted from the beginning in our description and both Eqs. (6) and (16) describe wave propagation without dissipation, therefore they can not admit complex eigenvalues for ω. Thus we wish to point out that the symmetry property of $\overleftrightarrow{D}$ should be expected from physical reasoning since, according to Eq. (13)

with $\omega_1^2 = \omega_2^2$ and $E_2 = E_1$, it is a necessary condition for real eigenvalues ω^2. It is also a sufficient condition for real ω^2, provided $\left\langle \underline{E}_1 \middle| \overleftrightarrow{K}^{(0)} \middle| \underline{E}_1 \right\rangle$ is non-zero. Had the dissipative terms been kept, they would add a non-Hermitian contribution to the weighting matrix $K^{(0)}$. This in turn would allow complex ω^2 for real $\vec{k}$ regardless of the symmetry of $\overleftrightarrow{D}$, as in case of non-Hermitian $K^{(0)}$ one can show that symmetry does not guarantee real ω^2. It is also worth noting that the demonstrated symmetry is brought out by employing the non-uniform plasma response in deriving $\overleftrightarrow{D}$ while a mere substitution of $-d^2/dx^2$ in place of k_x^2 leads to an asymmetric operator.

Inclusion of dissipation while keeping ω^2 real as in case of excitation by external source would lead to an imaginary part in the wave number $\vec{k}$. The exact symmetry of the operator $\overleftrightarrow{D}$ would be violated resulting in violation of the exact orthogonality. However, it can be shown using similar arguments as in Ref. 19 that the departure from orthogonality

$$\left\langle \underline{E}_i \middle| \overleftrightarrow{K}^{(0)} \middle| \underline{E}_j \right\rangle \Big/ \left\langle \underline{E}_i \middle| \overleftrightarrow{K}^{(0)} \middle| \underline{E}_i \right\rangle$$

scales as $\mathrm{Im}(k)/\mathrm{Re}(k)$. Therefore, corrections due to dissipation do not modify the lowest order result, provided $\mathrm{Im}(k)/\mathrm{Re}(k) \sim 0\left(\epsilon^2\right)$ or higher.

Finally, it should be noted that the boundary effects that discretize the frequency spectrum become unimportant if the absorption length L_{abs} becomes much shorter than the distance between boundaries $b-a$, converting the spectrum in Eq. (12) into a continuum set. As the incident magnetosonic mode is only weakly damped, this situation can arise only in case of a close to 100% mode conversion of the incident wave to the short strongly damped short wavelength ES mode. That, in turn, requires that the launched $k_\parallel^0$ be much higher than the accessibility threshold of Eq. (1) according to the new criterion for full accessibility given in Sec. IV. Similar insensitivity to the boundary conditions may result from scattering off density fluctuations at the plasma edge (case of "diffusive" boundaries) if the turbulent layer width is larger than the scattering length.[15,16]

III. Construction of eigenmodes including toroidal effects

We now proceed to introduce the toroidal effects into Eq. (6). We model the toroidal perturbation by a periodic variation in the y direction of the density, the toroidal, and the poloidal component of the magnetic field $\rho \equiv \rho\,(x, \epsilon\cos(y/r))$, $B_z \equiv B_z\,(x, \epsilon\cos(y/r))$, $B_y \equiv B_y\,(x, \epsilon\cos(y/r))$. The elements L_{ij} of the field aligned $\underline{\nabla}\times\underline{\nabla}\times$ operator depend on y through the shear angle $a \sim 0(\epsilon)$ thus the effect of the toroidal perturbation is of order ϵ^2. The correction in $K_{ij}^{(1)}$ and $K_{ij}^{(2)}$ is of order $\zeta^2\epsilon$ with ζ the small parameter introduced earlier, while the spectral decomposition of $K^{(0)}\,(x, \epsilon\cos(y/r))$ leads to a Fourier series with coefficients ordered in ϵ (see Ref.).

$$\overleftrightarrow{K}^{(0)}(x, \epsilon\cos(y/r)) = \overleftrightarrow{K}^{(0)}(x) + \sum_{\ell=1}^{\infty} \epsilon^{\ell}\,\overleftrightarrow{K}_n^{(0)}(x)\cos(\ell y/r).$$

Thus, omitting terms of order higher than ϵ^2 or $\zeta^2\epsilon$ we obtain the lowest order toroidal version of Eq. (6)

$$\overleftrightarrow{D}\,\underline{\mathcal{E}} + \frac{\omega^2}{c^2}\left[\overleftrightarrow{K}^{(0)}(x) + \epsilon\,\overleftrightarrow{K}_1^{(0)}(x)\cos(y/r)\right]\underline{\mathcal{E}} = 0 \tag{17a}$$

with

$$\underline{\mathcal{E}} = \underline{\mathcal{E}}_0 e^{i\left(\frac{n}{R}z - wt\right)}$$

$$\underline{\mathcal{E}}_0 = \left(X(x,y),\ Y(x,y),\ Z(x,y)\right). \tag{17b}$$

We are primarily interested in the narrow band of frequencies ω around $\omega_0 \equiv \omega_{\mathrm{LH}}\,(x_0)$ such that

$$|\omega - \omega_0| \Big/ \omega_0 \ll \epsilon, \tag{18}$$

as this part of the spectrum will furnish the dominant contribution in the expansion of $\underline{\mathcal{E}}$ into the eigenmodes $\hat{\underline{E}}$ of Eq. (16). Let $\omega^2\,(m_i, n)$, $\quad i = 1, 2 \ldots N$ be the set of the above almost degenerate $\hat{\underline{E}}\,(\omega^2; m_i, n)$.

The first order expansion for E and ω is

$$\begin{aligned} \underline{\mathcal{E}} &= \sum_{i=1}^{N} c_i \hat{\underline{E}}\,(\omega_0^2; m_i) + \epsilon \sum_{\omega^2 \neq \omega_0^2} \sum_{m'} c\,(\omega^2, m')\,\hat{\underline{E}}\,(\omega^2, m') \\ \omega^2 &= \omega_0^2 + \epsilon\Delta\omega^2 \end{aligned} \tag{19}$$

where, because of inequality (18) we have set $\omega^2(m_i, n) \cong \omega_0^2$ for all the almost degenerate modes while n remains fixed and has been dropped as an index since the z-independent perturbation does not mix different ns.

By expanding the matrices $\overleftrightarrow{K}^{(0)}(\omega)$ and $\overleftrightarrow{K}_1^{(0)}(\omega)$ around ω_0 and exploiting the orthogonality of $\hat{E}$ in respect to $K^{(0)}(\omega_0)$ we find that the coefficients c_i are solutions of the uniform system

$$\overleftrightarrow{A}\,\underline{C} + \left(\overleftrightarrow{B} - \overleftrightarrow{I}\right)\lambda \underline{C} = 0 \tag{20}$$

with

$$\begin{aligned}
\underline{C} &= (c_1, c_2, \ldots, c_N) \\
A_{ij} &= \omega_0^2 \left\langle \hat{E}(m_i) \middle| \overleftrightarrow{K}_1^{(0)} \cos\left(\frac{y}{r}\right) \middle| \hat{E}(m_j) \right\rangle \\
B_{ij} &= \left\langle \hat{E}(m_i) \middle| \left(\partial \overleftrightarrow{K}^{(0)} \middle/ \partial \omega^2 \right) |_{\omega=\omega_0} \middle| \hat{E}(m_j) \right\rangle
\end{aligned}$$

while the first order frequency correction $\Delta\omega^2$ is determined by the eigenvalues λ of Eq. (20),

$$\det\left|\overleftrightarrow{A} + (\overleftrightarrow{B} - \overleftrightarrow{I})\lambda\right| = 0. \tag{21}$$

In general there exist N coefficient vectors $C^{(k)} = [c_1, c_2, \ldots, c_N]^{(k)}$ $k = 1, 2 \ldots N$ associated with N real eigenvalues for $\Delta\omega_{(k)}^2$ as the matrix $A + \lambda B$ is Hermitian.

The diagonalization is particularly easy in case when $K_1^{(0)}$ has only one spectral component, $\frac{1}{2}K_\perp^{(0)}(x)\left[e^{i\frac{y}{r}} + e^{-i\frac{y}{r}}\right]$ as this yields the tridiagonal matrix

$$[A + (B - I)\lambda]_{ij} = \delta_{i,j+1}\bar{A}_{ij} + \delta_{ij}\left(\bar{B}_{ij} - 1\right)\lambda + \delta_{i,j-1}\bar{A}_{ij} \tag{22}$$

with

$$\begin{aligned}
\bar{A}_{ij} &= -\frac{1}{2}\omega_0^2 \int_\alpha^b dx \underline{E}_0(x; m_i)\, \overleftrightarrow{K}_1^{(0)}(x) \underline{E}_0(x; m_j) \\
\bar{B}_{ij} &= -\frac{1}{2}\omega_0^2 \int_\alpha^b dx \underline{E}_0(x; m_i)\, \frac{\partial \overleftrightarrow{K}^{(0)}(x)}{\partial \omega^2} \underline{E}_0(x; m_j).
\end{aligned}$$

Thus, owing to degeneracy, the spatial structure of a two-dimensional mode of single frequency ω^2 near $\omega_{\rm LH}^2(\omega_0)$ is obtained by a zeroth order superposition of the one-dimensional modes $\underline{\hat{E}}$ with frequencies ω^2 near $\omega_{\rm LH}^2(x_0)$ and mode conversion layers x near

x_0. These modes $\underline{\hat{E}}$, chosen for their orthonormality, are also good approximations to the exact solutions $\underline{E}$ of equation (6) in the neighborhood $\omega^2 \cong \omega_0^2$. In conclusion, the toroidal modes are mainly composed by equal order contributions from the one-dimensional modes of nearby frequencies and mode conversion layers. The modes "far" from ω_0^2 contribute to order ϵ in the wave function and ϵ^2 in the frequency correction. Actually the separation between almost degenerate and non-degenerate modes is not clear cut; one should diagonalize matrices of increasing dimension until further increase in dimension has a small effect on the solutions.

The toroidal modes are non-monochromatic in the poloidal direction (y in our slab model). Using Eq. (12) one can find that the poloidal spectral width Δm associated with a frequency regime $\Delta\omega$ is given for fixed k_x and n by

$$\Delta m \cong \left(\frac{r}{\rho}\right)\left(\frac{\omega_{\rm LH}}{\Omega_i}\right)(\cos\alpha)^{-1}\left(\frac{\Delta\omega}{\omega_{\rm LH}}\right)^{1/2} \tag{23a}$$

yielding

$$\Delta k_{\parallel} \cong \tan\alpha\left(\frac{\omega_{\rm LH}}{v_i}\right)\left(\frac{\Delta\omega}{\omega_{\rm LH}}\right)^{1/2} . \tag{23b}$$

Although the spectral width for degeneracy is small, limited by $\left(\Delta\omega/\omega_{\rm LH}\right)^{1/2} \ll \epsilon$, the spread in parallel wave number for a single solution $\Delta k_{\parallel}/k_{\parallel}$ can be large and of order 1 for ion thermal velocity much smaller than the parallel phase velocity of the wave $v_i \ll \omega_{\rm LH}/k_{\parallel}$. Equation (23b) shows that the spread in the parallel wave number increases with increasing density (increasing $\omega_{\rm LH}$). This tendency is consistent with the recent experimental observations[18] in Alcator-C. Increasing density tends to broaden the $k_{\parallel}$ spectrum through both toroidal effects and scattering off density fluctuations.

IV. Conclusion and Summary

The main outcome of our analysis is that the toroidal modes generically contain a wide spectrum in $k_{\|}$ owing to the frequency accumulation in the lower hybrid regime and that toroidal effects of order ϵ can produce a spectral width $\Delta k_{\|}/\langle k_{\|}\rangle$ of order ϵ^0 for small $\langle k_{\|}\rangle$. We clarify that although the degenerate set contains a narrow frequency band $\Delta\omega$ around $\omega_{\mathrm{LH}}(x_0)$, the two-dimensional modes are monochromatic in time oscillating with the corrected frequency

$$\omega_{(k)}^2 = \omega_0^2 + \epsilon\Delta\omega_{(k)}^2, \qquad k = 1, 2 \ldots, N.$$

The mixing of N one-dimensional modes with frequencies ω_i very close to $\omega_{\mathrm{LH}}(x_0)$, in the sense $\left|\omega_i^2 - \omega_0^2\right| \Big/ \omega_0^2 \ll \epsilon$, produces N two-dimensional modes of different frequencies $\omega_{(k)}$ with frequency separation of order ϵ. However one can generally create a set of two-dimensional modes, the corrected frequencies of which fall close to a given ω_0', $\left|\omega_{(k)}^2 - (\omega_0')^2\right| \Big/ (\omega_0')^2 \ll \epsilon$, as follows: one of the frequencies $\omega_{(k)}^2$ of the toroidal mode produced by mixing one-dimensional modes around $\omega_i \cong \omega_0$ can be very close to one of the frequencies $\omega_{(\ell)}^2$ of the toroidal mode produced by mixing one-dimensional modes around $\bar{\omega}_i \cong \bar{\omega}_0$ with $\bar{\omega}_0$ different than ω_0,

$$\omega_{(k)}^2 = \omega_0^2 + \epsilon\Delta\omega_{(k)} \cong \omega_{(\ell)}^2 = \bar{\omega}_0^2 + \epsilon\Delta\bar{\omega}_{(\ell)} \cong (\omega_0')^2 .$$

In a sense, the perturbation splits the old "degeneracy" between the one-dimensional modes and creates a new between the two-dimensional modes so that the new spectrum remains dense in ω^2.

For ω^2 kept fixed while increasing ϵ from zero to a finite value we have a shift of the mode conversion layer from its initial position x_0 to x_0' given by

$$\omega_{\mathrm{LH}}^2(x_0') = \omega_{\mathrm{LH}}^2(x_0) - \epsilon\Delta\omega^2. \tag{24}$$

Let us assume, for illustrative purposes, the simple picture of a plasma being excited by an antenna of frequency ω^2 and single helicity (m_k, n) located at $x = c$, $a < c < b$. The solution must be expressed as a superposition of toroidal modes

$$\underline{S} = \sum_{i=1}^{N} g_i \underline{\mathcal{E}}^{(i)}\left(x, y, z; \omega^2\right) \tag{25}$$

that at $x = c$ have their polarization vector parallel to the antenna, $\underline{\mathcal{E}}^{(i)}(x = c) \| \hat{e}_A$. Thus, using the zeroth order terms in Eq. (19) for $\underline{\mathcal{E}}$, Eq. (25) becomes near $x = c$

$$\underline{S} = \hat{e}_A \sum_j \sum_i e^{i(m_j y/r + nz/R)} f_{ji}(c)$$

with (26)

$$f_{ji}(x) = \sum_i g_i c_j^{(i)} \hat{E}_0 \left(x; m_j, n, \omega_i^2\right) .$$

In the vicinity of $x = c$, <u>and only there</u>, we expect the solution S to behave as monochromatic $S \sim e^{i(m_k y/r + nz/R)}$ in order to satisfy the local matching conditions with the antenna; indeed the coefficients g_i can be determined uniquely in a way such that

$$f_{ji}(x = c) = \delta_{jk} \tag{27}$$

by solving the system $j = 1, \ldots N$

$$\sum_{i=1}^{N} G_{ji} g_i = \delta_{jk}$$
$$G_{ji} = c_j^{(i)} \hat{E}_0 \left(x = c; m_j, n, \omega_i^2\right) . \tag{28}$$

In other words, solutions of the form Eq. (25) exist that start with nearly monochromatic dependence close to the exciting antenna but pick a wide spectrum as they get further inside the plasma, since the relation (27) cannot hold for $x \neq c$ once the coefficients g_i have been determined by Eq. (28). A way to visualize this multiple spectrum in $k_\|$ is to consider the local parallel "wave number" $k_\|(x, y) = \left(\hat{e}_\| \cdot \nabla\right) |\underline{S}|/i|\underline{S}|$ with $\underline{S}$ a solution of the form (25). As the different components $\mathcal{E}^i$ oscillate with different wavelength in x and y, $k_\|$ changes across the plasma volume as expected from the WKB theory.

The effect of the spread in $k_\|$ on the efficiency of the mode conversion process can now be addressed in brief. Each toroidal mode of a single frequency ω is composed by a linear superposition of one-dimensional modes, each one with different $k_\|(m)$. The asymptotic behavior of any one-dimensional mode is determined by Eqs. (6) and (11). Accordingly, each individual mode $\underline{\hat{E}}(m, n; \omega)$ will be mode converted, provided its parallel wave number $k_\|(m)$ satisfies the accessibility condition Eq. (1), or otherwise it will be backscattered before reaching the mode conversion layer. Therefore in the toroidal case, a gradual depletion of the energy carried by a single frequency mode may occur during the wave propagation towards the mode conversion layer proportional to the relative width of the inaccessible portion in the $k_{\parallel}$ spectrum. Conversely, an antenna that in cylindrical geometry would excite a single wavelength $k_{\parallel}^{0}$ below accessibility, in the toroidal case will excite additional spectral components above $k_{\parallel}^{0}$ for which the mode conversion layer is accessible. It is here that our picture differs from the WKB approach in the following sense: a toroidal mode centered around $k_{\parallel}^{0}$ in the neighborhood of accessibility is a steady state, characterized by partial mode conversion and partial reflection coefficients, while a wave package with $k_{\parallel}^{0}$ below accessibility has to bounce off a number of times between the cut-off and the plasma edge before the necessary change in $k_{\parallel}^{0}$ to allow accessibility occurs according to the WKB treatment.

When a considerable number of reflections with energy spread over the plasma volume occurs, the global mode approach is more relevant to the situation as the state of the system is described by the superposition of a few toroidal modes. On the other hand, a complete accessibility in "one pass" of the wave when $k_{\parallel}^{0}$ is well above threshold is easier described by ray optics, the reproduction of which would require a superposition of a considerable number of toroidal modes.

Our approach recognizes the role of the broad generic spread $\Delta k_{\parallel}$ around $k_{\parallel}^{0}$ in yielding partial reflection and partial conversion. The WKB counterpart would be the splitting of a single wave package in two independently propagating packages due to partial reflection not accounted for in standard ray tracing methods. On the other hand, the ray tracing method offers the advantage of treating the nonlinear effects, such as the change of the average wave number $< m >$ at a given point x and the onset of ergodic ray orbits, while $< m >$ is conserved in the linear approach.

For a toroidal mode with parallel spectrum centered around $k_{\parallel}^{0}$ and of width $\Delta k_{\parallel}$, the condition for full accessibility becomes

$$\frac{\omega^2}{c^2}\left(k_{\parallel}^{0}-\frac{\Delta k_{\parallel}}{2}\right)^2 > 1+\frac{\omega_{pe}^2}{\Omega_{ce}^2}. \tag{29}$$

$\Delta k_{\parallel}$ can be estimated from Eqs. (23) or more accurately by determining the expansion coefficients in Eq. (20). Strictly speaking, if terms of order ϵ or higher are included in the expansion Eq. (19), the spread in $\Delta k_{\|}$ becomes very large and complete mode conversion should not occur. For practical purposes, however, it will suffice to consider the width $\Delta k_{\|}^0$ of the degenerate part of the spectrum in order to determine absorption, since the energy carried by the rest of the expansion is of order ϵ^2. In case $\Delta k_{\|}^0$ does not satisfy the condition Eq. (29), the wave covers the plasma volume due to partial reflection combined with broad $\Delta k_{\|}^0$. Boundary conditions become important and the solutions behave as the global modes described by Eq. (19), rather than thin pencils of rays obeying geometrical optics. In the absence of excitation and dissipation, our toroidal modes are steady state solutions that can be thought of as wave trains of infinite duration in time $\tau\omega_0 \to \infty$, τ the duration of the rf pulse, the frequency spectrum becoming a δ-function $\Delta\omega \to 0$, while $\Delta k_{\|}$ still remains large and finite due to toroidicity.

In conclusion, there is a natural and significant spectral width $\Delta k_{\|}$ associated with each toroidal mode of single frequency ω. This should be partly responsible for the wide spectrum in $k_{\|}$ observed during lower hybrid heating experiments together with scattering off density fluctuations, the importance of which is not to be downplayed. Our analysis shows that a wide spectrum should persist in the limit that the thickness of the turbulent layer near the tokamak edge is much shorter than the characteristic length for wave scattering off fluctuations. It also suggests a method for the calculation of the efficiency of the lower hybrid mode conversion in toroidal geometry in the above situation.

It appears that the width of the turbulent layer in modern experiments such as the Alcator-A and Alcator-C is enough to justify extensive scattering in $k_{\|}$,[17,18] masking the effects of degeneracy discussed so far. We take the opportunity to emphasize that fluctuations in general do not invalidate the eigenmode approach which we consider appropriate for waves occupying a large section of the torus. Stochastic terms due to fluctuations can be added to the wave propagation Eq. (2) that then must be solved as a stochastic eigenvalue problem[21] for $\langle\omega^2\rangle$, $\langle\Delta\omega^2\rangle$ yielding $\left\langle k_{\|}^2\right\rangle$ and $\langle\Delta k_y^2\rangle$.

References

1. T. H. Stix, Phys. Rev. Lett. **15**, 878 (1965).
2. P. Bellan and M. Porkolab, Phys. Fluids **17**, 1952 (1974).
3. S. Puri and M. Tutter, Nucl. Fusion **14**, 93 (1974).
4. B. N. Moore and M. E. Oakes, Phys. Fluids **15**, 144 (1972).
5. K. C. Wong and Ting-wei Tang, Plasma Phys. **18**, 991 (1976).
6. C. F. F. Karney, Phys. Fluids **21**, 1584 (1978).
7. C. F. F. Karney, Phys. Fluids **22**, 2188 (1979).
8. N. J. Fisch, Phys. Rev. Lett. **41**, 873 (1978).
9. Ting-wei Tang and A. Sivasubramanian, Phys. Fluids **15**, 1534 (1972).
10. H. Derfler and F. Leuterer, Plasma Phys. **14**, 473 (1972).
11. T. H. Stix, *Theory of Plasma Waves*, McGraw-Hill, New York (1962).
12. V. E. Golant, Soviet Phys. Tech. Phys. **16**, 1980 (1972).
13. P. T. Bonoli and E. Ott, Phys. Fluids **25**, 359 (1982).
14. Tang-wei Tang in *Research Topics in Electromagnetic Wave Theory*, J. A. Kong editor, Wiley Interscience, New York (1980), p. 38.
15. P. L. Andrews and F. W. Perkins, Phys. Fluids **26**, 2537 (1983).
16. P. L. Andrews and F. W. Perkins, Phys. Fluids **26**, 2546 (1983).
17. R. L. Watterson, R. E. Slusher and C. M. Surko, Phys. Fluids **28**, 2857 (1985).
18. R. L. Watterson, Y. Takase, P. T. Bonoli, M. Porkolab, R. E. Slusher, and C. M. Surko, Phys. Fluids **28**, 2622 (1985).
19. S. Riyopoulos and S. Mahajan, Phys. Fluids **29**, 731 (1986).
20. D. G. Swanson, Phys. Rev. Lett. **36**, 316 (1976).
21. J. A. Tataronis, J. Plasma Phys. **13**, 87 (1975).

Figure Captions

Fig. 1 Dispersion relation n_x^2 vs. distance from the boundary for constant $k_{\|}$. Fast electromagnetic mode FEM (dashed line) and slow electromagnetic mode SEM (dot-dash) are the two branches in cold plasma theory. The warm plasma dispersion for SEM (solid line) shows the two mode conversions at $x = x_{c1}$ and $x = x_{c2}$.

Fig. 2 Field aligned geometry for the wave propagation equations. Local magnetic field is along $\hat{e}_{\|}$.