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Renormalized Turbulence Theory of Ion Pressure Gradient Driven Drift Modes

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Abstract

From the nonlinear gyrokinetic equation we formulate the renormalized turbulence equation for the η_i-mode drift wave instability. The study shows that the dominant nonlinear damping mechanism is from the $\mathbf{E} \times \mathbf{B}$ convection of the pressure fluctuation and that the kinetic modifications to the fluid $\mathbf{E} \times \mathbf{B}$ mode coupling, studied earlier, shift the spectrum toward the shorter wavelengths. Balancing the linear growth rate with the nonlinear damping rate at the linearly most unstable region, we calculate the anomalous ion thermal conductivity which exceeds the neoclassical plateau formula and gives a value of the same order as that previously computed by Horton, Choi and Tang [Phys. Fluids **26**, 1077 (1981)], but with a kinetic enhancement factor. Also, the thermal conductivity formula remains finite for vanishing density gradient.

I. Introduction

In the recent pellet fueling experiments[1] in the Alcator C tokamak, the global energy confinement times were longer than in the discharges fueled by gas puffing. Such an improved energy confinement is attributed to the reduced ion heat conduction which has a large anomaly with gas fueling. The main mechanism of degrading ion confinement in the gas puffing experiment is suspected to be the ion pressure gradient driven drift mode, often called the η_i-mode, which is probably stabilized when the plasma density profile becomes peaked as observed in the pellet fueling experiments. Also in the edge regions of tokamaks, the instability parameter $\eta_i \left(= d\ell n T_i / d\ell n N_i\right)$ for this mode is large and sometimes exceeds by an order of magnitude the critical $\eta_c \approx 1$ required for instability, indicating the η_i-mode is strongly unstable in this region.[2] Coppi[3] et al. use this drift mode to explain the anomalously fast inward particle transport and called it the "mixing mode." Earlier studies of ion power balance during neutral beam injection in the TFR tokamak[4] show that the ion thermal conductivity must exceed the neoclassical values by a substantial margin to obtain consistent $T_i(r)$ profiles and post injection decay rates of the central ion temperature. Therefore the relevance of the η_i-mode for the tokamak experiments is well recognized and the characteristics of this mode need to be investigated in detail.

Numerous works have been done on this mode in the linear phase of growth both in the electrostatic[3,5,6] and the electromagnetic[7,8] regimes. The studies show that except near marginal stability where the kinetic resonances are important, the mode characteristic is a fluid-like instability, with $\gamma_k > \omega_k > \omega_{Di}$, $k_{\|} v_i$, and $k_{\perp} \rho_i < 1$ for the parameter range η_i well past the threshold value[8] of $\eta_c \cong 1$. From fluid theory Horton et al.[5] analyze the toroidal η_i-mode using the ballooning formulation and show that the fast growing mode balloons significantly to the outside of the torus and is driven by the unfavorable magnetic curvature for typical tokamak parameters. Also they construct a mode-coupling theory based on the renormalized turbulence formulation of Horton and Choi[9] and compute the saturation spectrum, the saturation level and the anomalous ion thermal conductivity.

Since $\omega_k \ll \gamma_k$ in the strongly unstable regime in toroidal geometry, the study of the η_i-mode is outside the scope of the standard weak turbulence theory[10] and the renormalized treatment is required. The renormalized turbulence equations are found in numerous works.[9,11–13] The basic structure of these works may be summarized as follows. The renormalization reduces through statistical closure the original nonlinear dynamical equations into two coupled equations for two unknowns: the nonlinear response function ϵ_k and the spectral distribution I_k. In the complete renormalization theory, both the wave-particle propagator and the vertex interaction become dressed or renormalized by the turbulent fluctuation.

Recent nonlinear studies include the Waltz and Dominguez's study[14] of the drift wave problem using the renormalized turbulence equations for a sheared slab and solving reduced equations numerically. The Similon and Diamond[15] and Diamond et al.[16] study the various nonlinear problems using the coherent approximation to the direct interaction approximation (DIA)[17] within the one-dimensional ballooning mode representation of the fluctuations. The quasilinear study of Migliuolo[18] calculates the saturation amplitude of the most unstable η_i-mode as an expansion about marginal stability.

The present work extends the earlier studies by including a systematic development and reduction of the renormalized propagator and vertex renormalization of the η_i-mode turbulence and using a WKB formulation of the fluctuations in the nonlinear gyrokinetic equation. By taking the fluid limit of the fully renormalized kinetic response functions the resulting cancellations in the vertex and propagator renormalizations are found to retrieve the results of renormalized fluid turbulence theory applied to the η_i-mode.[5] In the process of making the fluid reduction we find the important kinetic theory modifications to the nonlinear damping.

The organization of this paper is as follows. In Sec. II, we formulate the renormalized turbulence equation and obtain the reduced set of equations. In Sec. III, we calculate the renormalized response function ϵ_k for the nonlinear evolution of the η_i-mode by taking the fluid limit of the response functions. The reduction of the response function is utilized in the computation of the nonlinear growth rate in Sec. IV and the dominant mechanism of the nonlinear damping is shown to be the $\mathbf{E} \times \mathbf{B}$ convection of the pressure fluctuation. In addition to the fluid contribution we also obtain the kinetic correction to the nonlinear mechanism whose sign changes as the value of wavenumber varies in contrast to the fluid contribution of damping. The kinetic corrections are a stabilizing effect to the long wavelengths and destabilizing effect to the short wavelengths. By balancing the linear growth rate with the nonlinear damping rate at $k \sim k_0$ where the linear growth rate has a maximum value, we calculate the anomalous ion thermal conductivity. The result is compared with the previous formula.[5] In Sec. V, we summarize our findings. In Appendix A, the details of the renormalization of the gyrokinetic equation are given. For a clearer understanding of the physical contents of the nonlinear terms, we review in Appendix B the renormalization of the fluid model developed by Horton, Choi and Tang[5] and compare with the kinetic calculations of the main text.

II. Formulation

Recently, many nonlinear problems of various plasma instabilities are studied from different view points. One of the most successful approaches is based on the renormalization theory. In this section, we renormalize the nonlinear gyrokinetic equation for the low frequency, electrostatic perturbations and obtain the renormalized turbulence equations.

We consider the toroidal confinement geometry with a strong toroidal magnetic field such as a tokamak. Since in the plasma motion there exists two spatial scales, the slow variation in the parallel direction and the rapid variation perpendicular to the magnetic field, the WKB ansatz is employed to express the fluctuations as

$$\delta f(\mathbf{x},\mathbf{v}) = \sum_{\mathbf{k}_\perp} \delta \bar{f}\left(\mathbf{X},\mathbf{V};\mathbf{k}_\perp\right) \exp\left[i\int^{\mathbf{X}_{\perp 0}} \mathbf{k}_\perp \cdot d\mathbf{X}_\perp - iL\left(\mathbf{k}_\perp\right)\right],$$

with $(\mathbf{X},\mathbf{V})$ and $(\mathbf{x},\mathbf{v})$ being the phase space coordinates of the guiding center and the particle, where $L(\mathbf{k}_\perp) = \mathbf{k}_\perp \cdot \mathbf{v} \times \hat{e}_\parallel / \Omega$, and $\delta \bar{f}$ as well as $\mathbf{k}_\perp$ contain slow spatial variations.

The nonlinear gyrokinetic equation[19] for low frequency, electrostatic perturbations in toroidal geometry of the fluctuating part of the ion distribution function is given by

$$f_k = -\frac{e}{T}F_0\phi_k + h_k e^{iL} \tag{1}$$

where the non-adiabatic part of the distribution function h_k satisfies

$$\left(\omega - k_{\|}v_{\|} - \omega_{Di}\right) h_k = \frac{e}{T_i}\left(\omega - \omega_{*t}^{i}\right) F_0 \phi_k J_0(\gamma) + \sum_{k_1+k_2=k} i\frac{c}{B}\left(\hat{e}_{\|}\cdot \mathbf{k}_1 \times \mathbf{k}_2\right) \phi_{k_1} J_0\left(\gamma_1\right) h_{k_2} \tag{2}$$

with

$$k_{\|} = \frac{-i}{Rq}\frac{\partial}{\partial\theta},$$

$$\omega_{Di} = 2\epsilon_n\omega_{*i}(\cos\theta + s\theta\sin\theta)\left(v_{\|}^2 + v_{\perp}^2/2\right)/v_i^2,$$

$$\omega_{*t}^{i} = \omega_{*i}\left[1 - \frac{3}{2}\eta_i + \frac{\eta_i}{2}\frac{m_i}{T_i}\left(v_{\|}^2 + v_{\perp}^2\right)\right],$$

$$\omega_{*i} = k_\theta\frac{cT_i}{eB}\frac{1}{N_i}\frac{dN_i}{dr}, \quad \epsilon_n = \frac{r_n}{R}, \quad r_n = -\left(\frac{1}{N_i}\frac{dN_i}{dr}\right)^{-1}, \quad q = \frac{rB_0}{RB_\theta},$$

$$s = rq'/q, \quad v_i^2 = \frac{2T_i}{m_i}, \quad \text{and} \quad \gamma = k_\perp v_\perp/\Omega.$$

The background ion distribution F_0 is taken as Maxwellian with $\eta_i = d\ell nT_i/d\ell nN_i$, and J_0 is zeroth order bessel function. In Eq. (1)–(2) and hereafter we adopt the notation $k = (\mathbf{k}, \omega)$. All nonlinearities in the problem arise from the $\mathbf{E}\times\mathbf{B}$ convection of the particles contained in Eq. (2).

For electrons we take them as adiabatic and the Poisson equation becomes

$$\phi_k = -\frac{4\pi N_0 e}{k^2}\left(\frac{e}{T_i} + \frac{e}{T_e}\right)\phi_k + \frac{4\pi N_0 e}{k^2}\int d\mathbf{v} h_k J_0(\gamma). \tag{3}$$

Next, we set up the renormalized turbulence equations from Eqs. (2) and (3). The simplest procedure is to follow that of Kadomtsev[11] as briefly explained below and the details of the derivation are given in Appendix A. We isolate the "self-action" term proportional to h_k and ϕ_k among the nonlinear term of Eq. (2) and write Eq. (2) as

$$\left(\omega - k_{\|}v_{\|} - \omega_{Di} + d_k\right) h_k = \left(S_k + \xi_k\right)\phi_k + \left(\sum_{k_1} V_{kk_1}\phi_{k_1}h_{k_2} + d_k h_k - \xi_k\phi_k\right) \tag{4}$$

with the shorthand notation of

$$S_k = \frac{e}{T_i}\left(\omega - \omega_{*t}^{i}\right) J_0(\gamma) F_0$$

and

$$V_{kk_1} = i\frac{c}{B}\left(\hat{e}_{\|} \cdot \mathbf{k}_1 \times \mathbf{k}_2\right) J_0\left(\gamma_1\right)$$

where $k_2 = k - k_1$.

The resonance broadening and vertex renormalization functions d_k and ξ_k are defined by the requirement that among the terms in the second parentheses of the right-hand side of Eq. (4) only the input from the beat interaction of different modes remain. These beat interaction inputs are considered to be small. So we put $h_k = h_k^{(0)} + h_k^{(1)}$ and $\phi_k = \phi_k^{(0)} + \phi_k^{(1)}$. As a result we obtain

$$\phi_k^{(1)} = -\frac{4\pi N_0 e}{k}\left(\frac{e}{T_i} + \frac{e}{T_e}\right)\phi_k^{(1)} + \frac{4\pi N_0 e}{k^2}\int d\mathbf{v} h_k^{(1)} J_0(\gamma) \tag{5}$$

and

$$\left(\omega - k_{\|}v_{\|} - \omega_{Di} + d_k\right) h_k^{(1)} = \left(S_k + \xi_k\right)\phi_k^{(1)} + \sum_{k_1} V_{kk_1}\phi_{k_1}^{(0)} h_{k_2}^{(0)}. \tag{6}$$

Then, we multiply Eqs. (3) and (4) by ϕ_k^* and average over the random phases of $h_k^{(0)}$ and $\phi_k^{(0)}$. We now substitute $h_k = h_k^{(0)} + h_k^{(1)}$ and $\phi_k = \phi_k^{(0)} + \phi_k^{(1)}$ in the nonlinear term and identify the terms proportional to $\langle h_k \phi_k^* \rangle$ and $\langle \phi_k \phi_k^* \rangle$ as d_k and ξ_k. As a result we have, upon dropping the superscript (0), the renormalized set of equations for the correlations functions $P_k = \langle h_k \phi_k^* \rangle$ and $I_k = \langle \phi_k \phi_k^* \rangle$.

$$P_k = g_k\left(S_k + \xi_k\right) I_k + g_k \sum_{k_1} V_{kk_1} g_{k_2}\left(S_{k_2} + \xi_{k_2}\right) v_{-k,-k_1}\frac{I_{k_1} I_{k_2}}{\epsilon_{-k}}, \tag{7}$$

$$\epsilon_k I_k = \frac{1}{2}\sum_{k_1}\frac{v_{kk_1} v_{-k,-k_1}}{\epsilon_{-k}} I_{k_1} I_{k_2}, \tag{8}$$

where

$$\begin{aligned}\epsilon_k &= 1 + \frac{4\pi N_0 e^2}{k^2}\left(\frac{1}{T_i} + \frac{1}{T_e}\right) - \frac{4\pi N_0 e}{k^2}\int d\mathbf{v} J_0(\gamma) g_k S_k \\ &\quad - \frac{4\pi N_0 e}{k^2}\int d\mathbf{v} J_0(\gamma) g_k \sum_{k_1} V_{kk_1} g_{k_2} V_{k_2 k} P_{-k_1} - \sum_{k_1}\frac{v_{kk_2} v_{k_2 k}}{\epsilon_{k_2}} I_{k_1},\end{aligned} \tag{9}$$

$$g_k = \left(\omega - k_{\|}v_{\|} - \omega_{Di} + d_k\right)^{-1}, \tag{10}$$

$$d_k = -\sum_{k_1} V_{kk_1} g_{k_2} V_{k_2 -k_1} I_{k_1}, \tag{11}$$

$$\xi_k = \sum_{k_1} V_{kk_1} g_{k_2} V_{k_2 k} P_{-k_1}$$

$$+ \sum_{k_1} V_{kk_1} g_{k_2} \left(S_{k_2} + \xi_{k_2}\right) \left(\frac{v_{k_2 k}}{\epsilon_{k_2}} I_{k_1} + \frac{v_{k_1 k}}{\epsilon_{k_1}} I_{k_2}\right), \quad (12)$$

and

$$v_{kk_1} = \frac{4\pi N_0 e}{k^2} \int d\mathbf{v} J_0(\gamma) g_k \left[V_{kk_1} g_{k_2} \left(S_{k_2} + \xi_{k_2}\right) + (1 \leftrightarrow 2)\right]. \quad (13)$$

The primitive form of mode-coupling equations contain the usual "free" propagator $g_k^{(0)} = \left(\omega - k_{\|} v_{\|} - \omega_{Di}\right),^{-1}$ vertex V_{kk_1} and source of instability S_k with the two equations determining h_k and ϕ_k. In the renormalization, the selection of dominant interactions[9,11,12] and ensemble averaging lead to the seven equations of Eqs. (7)–(13) for seven unknowns P_k, I_k, ϵ_k, g_k, d_k, ξ_k and v_{kk_1}. These equations may be viewed as determining the unknowns in two steps. For given I_k and P_k, Eqs. (10)–(11) determine the renormalized propagator g_k and the resonance broadening d_k. For given I_k, P_k and g_k, then Eqs. (12)–(13) determine the renormalized wave-particle vertex ξ_k and the renormalized wave mode-coupling vertex v_{kk_1}. Finally, Eqs. (7)–(9) determine the wave-particle correlation function P_k, the wave–wave correlation function I_k and nonlinear response function ϵ_k.

To make the problem tractable, we take the lowest order contributions for the wave-particle correlation P_k and the renormalized wave-particle vertex ξ_k as $P_k = g_k \left(S_k + \xi_k\right) I_k$ and $\xi_k = \sum_{k_1} V_{kk_1} g_{k_2 k} V_{k_2 k} P_{-k_1}$. These simplifications amount to retaining terms of the lowest power in I_k for P_k and in g_k for ξ_k. Equivalently, the approximation drops the correction terms due to the renormalized wave–wave coupling v_{kk_1} within the wave-particle correlation function P_k and the vertex renormalization ξ_k. Then, the truncated equations are

$$\epsilon_k I_k = \frac{1}{2} \sum_{k_1} \frac{v_{kk_1} v_{-k,-k_1}}{\epsilon_{-k}} I_{k_1} I_{k_2} \quad (14)$$

with

$$\epsilon_k = 1 + \frac{4\pi N_0 e^2}{k^2} \left(\frac{1}{T_i} + \frac{1}{T_e}\right) - \frac{4\pi N_0 e^2}{k^2 T_i} \int d\mathbf{v} g_k \left(\omega - \omega_{*t}^i + b_k\right) F_0 J_0^2(\gamma)$$

$$- \sum_{k_1} \frac{v_{kk_2} v_{k_2 k}}{\epsilon_{k_2}} I_{k_1}, \quad (15)$$

$$v_{kk_1} = \frac{4\pi N_0 e^2}{k^2 T_i} i \frac{c}{B} \int d\mathbf{v} g_k J_0(\gamma) J_0(\gamma_1) J_0(\gamma_2) \left[(\mathbf{k}_1 \times \mathbf{k})_{\|} g_{k_2} \left(\omega_2 - \omega^i_{*t,2} + b_{k_2}\right) + (1 \leftrightarrow 2) \right] F_0, \tag{16}$$

$$d_k = -\sum_{k_1} \frac{c^2}{B^2} (\mathbf{k}_1 \times \mathbf{k})^2_{\|} g_{k_2} J_0^2(\gamma_1) I_{k_1}, \tag{17}$$

and

$$b_k = -\sum_{k_1} \frac{c^2}{B^2} (\mathbf{k}_1 \times \mathbf{k})^2_{\|} g_{k_2} g_{-k_1} \left(-\omega_1 + \omega^i_{*t,1} + b_{-k_1}\right) J_0^2(\gamma_1) I_{k_1} \tag{18}$$

where we define b_k by

$$\xi_k = \frac{e}{T_i} F_0 J_0(\gamma) b_k.$$

With these reductions we have six equations for six unknowns I_k, ϵ_k, g_k, d_k, b_k and v_{kk_1}. Dupree and Tetreault[20] argue in the drift wave problem that the resonance broadening d_k alone does not conserve energy so that the inclusion of b_k in the renormalization is important to conserve energy. Thayer and Molvig[21], however argue that the inclusion of b_k is to be taken as a constraint on the renormalization and is to be distinguished from the traditional energy conservation relation where the average particle distribution equation is employed.

The equations (14)–(18) also follow from the selective summation to all orders of the most secular contributions in the small ϕ_k expansion of the mode-coupling equation as given by Horton and Choi.[9] The selective summation for the vertex renormalization and the energy conserving clump kinetic equation are given by Balescu and Misguich.[22]

III. Reduction of the renormalized turbulence theory for the η_i-mode

In this section, we study the nonlinear evolution of the η_i-mode. Consider the renormalized response function ϵ_k which we write as

$$\epsilon_k = \epsilon_k^{(1)} + \epsilon_k^{(2)}, \tag{19}$$

where

$$\epsilon_k^{(1)} = 1 + \frac{4\pi N_0 e^2}{k^2 T_e}\left[1 + \frac{T_e}{T_i} - \frac{T_e}{T_i}\int d\mathbf{v} g_k\left(\omega - \omega_{*t}^i + b_k\right) J_0^2(\gamma) F_0\right] \tag{20}$$

and

$$\begin{aligned}\epsilon_k^{(2)} = & -\frac{N_0^2\left(4\pi e^2\right)^2}{k^2 T_i^2}\frac{c^2}{B^2}\sum_{k_1}(\mathbf{k}_1\times\mathbf{k})_{\parallel}^2 \frac{I_{k_1}}{k_2^2\epsilon_{k_2}} \times \int d\mathbf{v} g_k J_0(\gamma) J_0(\gamma_1) J_0(\gamma_2) \\ & \times\left[g_{k_1}\left(\omega_1 - \omega_{*t,1}^i + b_{k_1}\right) - g_{k_2}\left(\omega_2 - \omega_{*t,2}^i + b_{k_2}\right)\right] F_0 \times \int d\mathbf{v} g_{k_2} J_0(\gamma) J_0(\gamma_1) J_0(\gamma_2) \\ & \times\left[g_{-k_1}\left(-\omega_1 + \omega_{*t,1}^i + b_{-k_1}\right) - g_k\left(\omega - \omega_{*t}^i + b_k\right)\right] F_0.\end{aligned} \tag{21}$$

Here, $\epsilon_k^{(1)}$ contains the renormalized propagator g_k and the vertex renormalization b_k while $\epsilon_k^{(2)}$ contains the induced scattering contribution from the beat-wave mode couplings.

We first investigate the lowest order response function $\epsilon_k^{(1)}$. Since we are interested in $\omega \gg k_{\parallel}v_{\parallel}$, we expand g_k in power series of $k_{\parallel}v_{\parallel}/\left(\omega - \omega_{Di}\right)$. With the assumption of $d_k/\omega < 1$, $b_k/\omega < 1$ and quasi-neutral limit of $k^2\lambda_{De}^2 \ll 1$, the response function $\epsilon_k^{(1)}$ reduces to

$$\begin{aligned}\epsilon_k^{(1)} = \frac{4\pi N_0 e^2}{k^2 T_e}\Bigg[& 1 + \tau - \tau\int d\mathbf{v}\frac{\omega - \omega_{*t}^i}{\omega - \omega_{Di}} J_0^2(\gamma) F_0 \\ & - \tau k_{\parallel}^2\int d\mathbf{v}\frac{v_{\parallel}^2}{\left(\omega - \omega_{Di}\right)^2}\frac{\omega - \omega_{*t}^i}{\omega - \omega_{Di}} J_0^2(\gamma) F_0 \\ & + \tau\int d\mathbf{v}\left(\frac{\omega - \omega_{*t}^i}{\omega - \omega_{Di}}\frac{d_k}{\omega - \omega_{Di}} - \frac{b_k}{\omega - \omega_{Di}}\right) J_0^2(\gamma) F_0\Bigg],\end{aligned} \tag{22}$$

where we put $\tau = T_e/T_i$.

In the fluid limit of $k_{\perp}\rho_i < 1$ and $\omega_{Di}/\omega < 1$, we perform the velocity space integral in Eq. (22) and obtain

$$\epsilon_k^{(1)} = \frac{4\pi N_0 e^2}{k^2 T_e}\left[\left(1 - \frac{\omega_{*e}}{\omega}\right) + k_{\perp}^2\left(1 - \frac{\omega_{*p}^i}{\omega}\right) + \frac{\omega_{*e}}{\omega}2\epsilon_n(\cos\theta + s\theta\sin\theta)\left(1 - \frac{\omega_{*p}^i}{\omega}\right)\right.$$

$$+\frac{\epsilon_n^2}{\omega^2 q^2}\left(1-\frac{\omega_{*p}^i}{\omega}\right)\frac{\partial^2}{\partial\theta^2}$$

$$+\sum_{k_1}(\mathbf{k}_1\times\mathbf{k})_{\parallel}^2\frac{I_{k_1}}{\omega\omega_2}\left\{\frac{D_k^\ell}{\omega^2}-\frac{D_{k_1}^\ell}{\omega_1^2}-k^2\left(1-\frac{\omega_{*p,1}^i}{\omega_1}\right)+k_1^2\left(1-\frac{\omega_{*p}^i}{\omega}\right)\right.$$

$$\left.\left.+2\epsilon_n\left(\frac{\omega_{*e}}{\omega}+\frac{\omega_{*e,2}}{\omega_2}\right)\left(\frac{\omega_{*p,1}^i}{\omega_1}-\frac{\omega_{*p}^i}{\omega}\right)\right\}\right], \tag{23}$$

with

$$D_k^\ell=\omega\left(\omega-\omega_{*e}\right)+k^2\omega\left(\omega-\omega_{*p}^i\right)+2\epsilon_n\omega_{*e}\left(\omega-\omega_{*p}^i\right).$$

Here, $k_\perp^2=k^2\left(1+s^2\theta^2\right)$, $\omega_{*e}=k$, $\omega_{*p}^i=-\frac{k(1+\eta_i)}{\tau}$ and $\omega_{*p,1}^i=-\frac{k_1(1+\eta_i)}{\tau}$, and we used dimensionless variables given by the azimuthal wavenumber $k=k_\theta\rho_s$ with $\rho_s=c\left(m_iT_e\right)^{1/2}/eB$, the frequency $\omega=\omega r_n/c_s$ with $c_s=\left(T_e/m_i\right)^{1/2}$ and the potential $\Phi_k=\frac{T_e}{e}\frac{\rho_s}{r_n}\phi_k$.

The toroidal η_i-modes are localized to the outside of the torus[5] where θ is small, and we solve $\epsilon_k^{(1)}\phi_k=0$ which is a Weber equation in θ with the fundamental $(n=0)$ solution of the form

$$\phi_k(\theta)=\phi_k\exp\left(-\frac{1}{2}\mu_k\theta^2\right), \tag{24}$$

where μ_k is given by

$$\mu_k=\frac{\omega q}{\epsilon_n}\left[\frac{\omega_{*e}}{\omega}2\epsilon_n\left(\frac{1}{2}-s\right)-k^2s^2\right]^{1/2}$$

valid for $\left\langle\theta^2\right\rangle\simeq 1/\mu_k<1$ and with $\left\langle k_x^2\right\rangle=k^2s^2\left\langle\theta^2\right\rangle$. The eigenfrequency for $\phi_k(\theta)$ is the solution of

$$\epsilon_k^{(1)}=\frac{4\pi N_0e^2}{k^2T_e\omega^2}\left[\left(1+k^2\right)\omega^2-\left(1-k^2\frac{1+\eta_i}{\tau}-2\epsilon_n\right)k\omega+2\epsilon_n\frac{1+\eta_i}{\tau}k^2\right.$$

$$-i\frac{\mu_k\epsilon_n}{q^2}\left(1+\frac{k}{\omega}\left(\frac{1+\eta_i}{\tau}\right)\right)+\sum_{k_1}(\mathbf{k}_1\times\mathbf{k})_{\parallel}^2 I_{k_1}\frac{\omega}{\omega_2}$$

$$\times\left\{\frac{D_k^\ell}{\omega}-\frac{D_{k_1}^\ell}{\omega_1}-k^2+k_1^2-\frac{1+\eta_i}{\tau}\left(k^2\frac{k_1}{\omega_1}-k_1^2\frac{k}{\omega}\right)\right.$$

$$\left.\left.+2\epsilon_n\frac{1+\eta_i}{\tau}\left(\frac{k}{\omega}+\frac{k_2}{\omega_2}\right)\left(\frac{k}{\omega}-\frac{k_1}{\omega_1}\right)\right\}\right]=\frac{4\pi N_0e^2}{k^2T_e\omega^2}\left[D_k^\ell(\omega)+X_k(\omega)\right]=0. \tag{25}$$

In writing Eq. (25), we have defined

$$D_k^\ell(\omega) = \left(1+k^2\right)\omega^2 - k u_k \omega + k^2\gamma_0^2$$

with

$$u_k = 1 - 2\epsilon_n - k^2\left(1+\eta_i\right)/\tau \quad , \quad \gamma_0^2 = 2\epsilon_n\left(1+\eta_i\right)/\tau \tag{26}$$

and the remaining terms by $X_k(\omega)$. The lowest order linear eigenmodes are given by $D_k^\ell(\omega) = 0$ and the kinetic integral correction are given in Ref. 8.

Now we consider the second part of the response function $\epsilon_k^{(2)}$ from the beat-wave mode coupling where we take the lowest order term in $k_\| v_\| / \left(\omega - \omega_{Di}\right)$,

$$\begin{aligned} \epsilon_k^{(2)} = & -\frac{\left(4\pi e^2\right)^2 N_0^2 c^2}{k^2 T_i^2 B^2} \sum_{k_1} \left(\mathbf{k}_1 \times \mathbf{k}\right)_\|^2 \frac{I_{k_1}}{k_2^2 \epsilon_{k_2}} \\ & \times \int d\mathbf{v} J_0(\gamma) J_0\left(\gamma_1\right) J_0\left(\gamma_2\right) \frac{F_0}{\omega - \omega_{Di}} \left(\frac{\omega_1 - \omega_{*t,1}^i}{\omega_1 - \omega_{Di,1}} - \frac{\omega_2 - \omega_{*t,2}^i}{\omega_2 - \omega_{Di,2}}\right) \\ & \times \int d\mathbf{v} J_0(\gamma) J_0\left(\gamma_1\right) J_0\left(\gamma_2\right) \frac{F_0}{\omega_2 - \omega_{Di,2}} \left(\frac{\omega_1 - \omega_{*t,1}^i}{\omega_1 - \omega_{Di,1}} - \frac{\omega - \omega_{*t}^i}{\omega - \omega_{Di}}\right) . \end{aligned} \tag{27}$$

Taking the fluid limit and calculating at $\theta = 0$, $\epsilon_k^{(2)}$ reduces to

$$\begin{aligned} \epsilon_k^{(2)} = & -\frac{4\pi N_0^2 e^2}{k^2 T_e \omega^2} \sum_{k_1} \left(\mathbf{k}_1 \times \mathbf{k}\right)_\|^2 \frac{I_{k_1}}{D_{k_2}^\ell} \omega\omega_2 \\ & \times \left[\frac{D_{k_2}^\ell}{\omega_2^2} - \frac{D_k^\ell}{\omega_1^2} - k k_2 + k k_1 - k\frac{1+\eta_i}{\tau}\left(k_2\frac{k_1}{\omega_1} - k_1\frac{k_2}{\omega_2}\right) + \gamma_0^2\frac{k}{\omega}\left(\frac{k_2}{\omega_2} - \frac{k_1}{\omega_1}\right)\right] \\ & \times \left[\frac{D_k^\ell}{\omega^2} - \frac{D_{k_1}^\ell}{\omega_1^2} - k k_2 - k_1 k_2 - k_2\frac{1+\eta_i}{\tau}\left(k\frac{k_1}{\omega_1} + k_1\frac{k}{\omega}\right) + \gamma_0^2\frac{k_2}{\omega_2}\left(\frac{k}{\omega} - \frac{k_1}{\omega_1}\right)\right] \\ = & \frac{4\pi N_0 e^2}{k^2 T_e \omega^2} Y_k(\omega) \end{aligned} \tag{28}$$

in dimensionless variables, where we approximated $D_{k_2}^\ell$ by ϵ_{k_2} in the denominator. Including $\epsilon_k^{(2)}$, the nonlinear response function now becomes

$$\epsilon_k = \frac{4\pi N_0 e^2}{k^2 T_e \omega^2}\left[D_k^\ell(\omega) + X_k(\omega) + Y_k(\omega)\right] . \tag{29}$$

With the expressions for D_k^ℓ, X_k and Y_k given in Eqs. (25), (26) and (28), we have calculated the reduction in the fluid expansion of the renormalized response function ϵ_k.

IV. Saturation condition and the anomalous ion thermal conductivity

In this section, we balance the linear growth rate with the nonlinear damping rate and from this calculate the anomalous ion thermal conductivity.

A. Radial wavenumber spectrum

We model the radial wavenumber spectrum to reduce the problem to a one-dimensional one. From Sec. III, recognizing the eigenfunction is the same one as the linear case of Ref. 5, we employ the result of that reference where a well defined radial wavenumber for the entire spectrum of azimuthal wavenumber k is given by

$$\left\langle k_x^2 \right\rangle^{1/2} = \bar{k}_r = \frac{(2\epsilon_n)^{1/4} s^{1/2}}{q^{1/2}\left(\frac{1+\eta_i}{\tau}\right)^{1/4}}, \tag{30}$$

which remains finite as $r_n \to \infty$ and satisfies $k_\perp \rho_i \ll 1$. Thus, we are led to investigate the azimuthal mode coupling in a spectrum characterized by its lowest moments in k_x. Taking the spectrum as symmetric in k_x with $I(k_x, k_y) = I\left(k_x^2, k\right)$ and of width $\left\langle k_x^2 \right\rangle$, we write

$$I(k) = \int_{-\infty}^{\infty} dk_x I\left(k_x^2, k\right) = 2\int_0^{\infty} dk_x I\left(k_x^2, k\right),$$

and

$$\left\langle k_x^2 \right\rangle I(k) = 2\int_0^{\infty} dk_x k_x^2 I\left(k_x^2, k\right).$$

The 2D mode-coupling simulations[23] for a reduced fluid model of this mode are consistent with a Lorentzian distribution in k_x with

$$I\left(k_x^2, k\right) \cong \frac{\pi^{-1}\left\langle k_x^2 \right\rangle^{1/2} I(k)}{k_x^2 + \left\langle k_x^2 \right\rangle}.$$

B. Reduced spectral balance equation

Taking $X_k(\omega)$ and $Y_k(\omega)$ of Eqs. (25) and (28) as corrections to D_k^ℓ of Eq. (28) we obtain

$$\omega = \omega_0 + \Delta\omega_1 + \Delta\omega_2$$

with

$$\omega_0 = \frac{k}{2\left(1+k^2\right)}\left[u_k \pm i\sqrt{4\gamma_0^2\left(1+k^2\right) - u_k^2}\right],$$

and the complex frequency shifts $\Delta\omega_1$ and $\Delta\omega_2$ of

$$\Delta\omega_1 = -X_k(\omega)\Big|_{\omega=\omega_0} \Big/ \frac{\partial D_k^\ell}{\partial\omega}\Big|_{\omega=\omega_0}$$

and

$$\Delta\omega_2 = -Y_k(\omega)\Big|_{\omega=\omega_0} \Big/ \frac{\partial D_k^\ell}{\partial\omega}\Big|_{\omega=\omega_0}.$$

We model the frequency spectrum as the evolving Lorentz spectrum, i.e.,

$$I_{\mathbf{k}}(\omega) = I_{\mathbf{k}}(T)\frac{2\nu_k}{\left(\omega-\omega_k\right)^2 + \nu_k^2} \tag{31}$$

with $T = \left(t_1+t_2\right)/2$. Here, ω_k is a nonlinear frequency and ν_k is a decay rate of the two-time correlation function which are determined by the coupled set of nonlinear equation resulting from Eqs. (14)–(18) with Eq. (31) for the ω integration. In this paper, we approximate $\nu_k = \gamma_k^\ell$. The fluid simulations of Brock and Horton[23] show an exponential decay of the two time correlation function consistent with Eq. (31).

Using this frequency spectrum, we carry out the frequency integration and obtain

$$\mathrm{Im}\left(\Delta\omega_1\right) = -\frac{1}{2\gamma_0}\sum_{\mathbf{k}_1<0}\left(\mathbf{k}_1\times\mathbf{k}\right)_\parallel^2 I_{\mathbf{k}_1}\left(\frac{2}{k+k_1} + \frac{k-k_1}{\left(k+k_1\right)^2}\right) \tag{32}$$

and

$$\mathrm{Im}\left(\Delta\omega_2\right) = -\frac{1}{2\gamma_0}\sum_{\mathbf{k}_1>0}\left(\mathbf{k}_1\times\mathbf{k}\right)_\parallel^2 I_{\mathbf{k}_1}\left[\frac{1}{k_1}\frac{k-k_1}{k+k_1} - \frac{k-k_1}{\left(k+k_1\right)^2} - \left(\frac{1+\eta_i}{2\gamma_0\tau}\right)^2\frac{1}{k_1}\left(k-k_1\right)^3\left(k+2k_1\right)\right]. \tag{33}$$

In the calculation of $\Delta\omega_1$, the contribution from the ion acoustic terms to $\mathrm{Im}(\omega)$ is neglected because in the toroidal geometry it is small for $k\gamma_0 > \epsilon_n/q$ as confirmed by numerical integration in Ref. 5. Thus the nonlinear growth rate becomes

$$\gamma_k^{n\ell} = \gamma_k^{\ell} - \frac{1}{2\gamma_0} \sum_{\mathbf{k}>0} (\mathbf{k}_1 \times \mathbf{k})_{\parallel}^2 I_{\mathbf{k}_1} \times \left[\frac{2}{k+k_1} + \frac{1}{k_1}\frac{k-k_1}{k+k_1} - \left(\frac{1+\eta_i}{2\gamma_0\tau}\right)^2 \frac{1}{k}(k-k_1)^3 \cdot (k+2k_1) \right] . \tag{34}$$

In obtaining the expression of Eq. (34), terms underlined in Eqs. (32) and (33) cancel each other. This shows the importance of retaining the vertex and the propagator renormalization up to relevant order since without it, the spurious terms enter in the final balance equation. In Appendix B, we review the renormalization of the fluid model developed by Horton, Choi and Tang.[5] Comparing the kinetic and fluid formulas we see that the first and second term in the bracket of Eq. (34) are from the $\mathbf{E} \times \mathbf{B}$ convection of the pressure fluctuation and it is the dominant term in the fluid limit which is the nonlinearity considered in Ref. 5, while the third term arises from the kinetic contributions. The kinetic contribution gives a weak stabilizing effect for long wavelengths and destabilizing effect for short wavelengths.

For the saturation condition of $\gamma_k^{n\ell} = 0$, we consider the following situation. When the spectrum I_k is negligible, the system is unstable and the waves grow with the linear growth rate γ_k^{ℓ}. As I_k becomes non-negligible, $\gamma_k^{n\ell}$ is reduced overall wavenumber space and only region near k_0 of the maximum linear growth would still remain positive. Further growth of I_k then makes the saturation condition of $\gamma_k^{n\ell} = 0$ which, in terms of the one-dimensional spectrum $I(k)$, requires

$$\gamma_k^{\ell} = \frac{\langle k_x^2 \rangle}{2\gamma_0} \int_{k_{\mathrm{min}}}^{k_{\mathrm{max}}} dk_1 \left(k_1^2 + k^2\right) I(k_1) \frac{1}{k_1} \left[1 - \left(\frac{1+\eta_i}{2\gamma_0\tau}\right)^2 (k-k_1)^3 (k+2k_1) \right] \tag{35}$$

to be satisfied at $k \sim k_0$ where γ_k^{ℓ} has the maximum value. From Eq. (35) we can see that as $k_{\mathrm{max}} \to \infty$, for the integral to have a finite value, $I(k)$ for large k should decrease faster than k^{-6}. In the right-hand side of Eq. (35) the dominant contribution comes from small values of k_1. Calculating the kinetic term at $k \sim k_0 > k_1$, Eq. (35) reduces to the quadratic equation in k. With $\gamma_k = \gamma_0 k$ and balancing at $k = k_0$ gives two constraints or integral equations of $I(k)$

$$\int_{k_{\min}}^{k_{\max}} dk_1 \frac{I(k_1)}{k_1} = \frac{1}{\langle k_x^2 \rangle} \frac{\gamma_0^2}{k_0} \left[1 - \left(\frac{1+\eta_i}{2\gamma_0 \tau} \right)^2 k_0^4 \right]^{-1} \tag{36}$$

and

$$\int_{k_{\min}}^{k_{\max}} dk_1 k_1 I(k_1) = \frac{1}{\langle k_x^2 \rangle} \gamma_0^2 k_0 \left[1 - \left(\frac{1+\eta_i}{2\gamma_0 \tau} \right)^2 k_0^4 \right]^{-1} \tag{37}$$

with

$$k_0 = \left\{ \frac{2(1-2\epsilon_n) + \left[(1-2\epsilon_n)^2 + 24\epsilon_n \left((1+\eta_i)/\tau \right) \right]^{1/2}}{3(1+\eta_i)/\tau} \right\}^{1/2}$$

which remains finite as $r_n \to \infty$. For the toroidal regime of $s < q$, using Eq. (12) of Ref. 5, the condition $\omega_{Di} > k_{\|} v_{\|}$ leads to $(s/q)^{1/2} \left((1+\eta_i)/\epsilon_n \right)^{1/4} < 2^{3/4}$ so that $\epsilon_n > \epsilon_n^{\text{crit}}$ and the kinetic correction term is smaller than the fluid terms. Both of these constraints are useful relations, and in particular Eq. (37) provides the moment needed for the thermal conductivity. Thus, without solving for the $I(k)$ spectrum explicitly, one can obtain the ion thermal conductivity. The explicit solution of the $I(k)$ spectrum is needed for the computations of other physical quantities. We will report the $I(k)$ solution elsewhere from the formulation which includes the right-hand side of Eq. (14), sometimes called the "incoherent" term of mode-coupling equation.

C. Anomalous ion thermal conductivity

The anomalous ion thermal flux is

$$\mathbf{Q} = \langle P_i \mathbf{V}_E \rangle$$

with

$$\mathbf{V}_E = \frac{c}{B}\hat{z} \times \nabla \Phi$$

and

$$P_i = \frac{1}{2} N_i m_i \int d\mathbf{v} v^2 h_k J_0(\gamma) \cong \frac{3}{2} N_i e \left(1 - \frac{\omega_{*pi}}{\omega}\right) \Phi_k .$$

In the dimensionless variables of

$$\phi_k = \frac{r_n}{\rho_s} \frac{e}{T_e} \Phi_k$$

and

$$I(k) = \langle \phi_k \phi_k^* \rangle ,$$

the thermal flux $Q = \hat{e}_x \cdot \mathbf{Q}$ reduces to

$$Q = -\frac{\rho_s}{r_n} \frac{cT_e}{eB} \sum_k k I_k \mathrm{Im} \left(\frac{\omega_{*pi}}{\omega}\right) \frac{3}{2} \frac{N_i T_e}{r_n} \tag{38}$$

and the thermal conductivity is given by

$$\chi_i = -Q \bigg/ \frac{dP_i}{dr} .$$

Using $\omega \cong i\gamma_0 |k|$ in Eq. (38) and with Eqs. (30) and (37), we obtain

$$\begin{aligned} \chi_i &= \frac{\rho_s}{r_n} \frac{cT_e}{eB} \frac{2}{\gamma_0} \int_{k_{\min}}^{k_{\max}} dk k I(k) \\ &\sim 2 \frac{\rho_s}{r_n} \frac{cT_e}{eB} \frac{q}{s} \left(\frac{1+\eta_i}{\tau}\right) k_0 \left[1 - \left(\frac{1+\eta_i}{2\gamma_0 \tau}\right)^2 k_0^4\right]^{-1} \end{aligned} \tag{39}$$

in the toroidal regime in which $\epsilon_n > \epsilon_n^{\mathrm{crit}} \sim 0.05$.

We compare the above formula for the anomalous ion thermal conductivity with the neoclassical plateau formula

$$\chi_i^{nc} = 2.6 \, \frac{\rho_s}{r_n} \frac{cT_e}{eB} q\epsilon_n$$

and the previous formula of Ref. 5

$$\chi_i^{\rm HCT} = \frac{\rho_s}{r_n} \frac{cT_e}{eB} \frac{q}{s} \left(\frac{1+\eta_i}{\tau} \right)^{1/2} \ell n \left(k_{\rm max} / k_{\rm min} \right) .$$

The anomalous transport exceeds the neoclassical plateau transport by the factor

$$\frac{2}{2.6} \frac{1+\eta_i}{s\epsilon_n \tau} k_0 \cdot \left[1 - \left(\frac{1+\eta_i}{2\gamma_0 \tau} \right)^2 k_0^4 \right]^{-1} ,$$

which is typically a factor of order ten, and the previous formula $\chi_i^{\rm HCT}$ by the factor

$$2 \left(\frac{1+\eta_i}{\tau} \right)^{1/2} k_0 \cdot \left[1 - \left(\frac{1+\eta_i}{2\gamma_0 \tau} \right)^2 k_0^4 \right]^{-1} \Big/ \ell n \left(\frac{k_{\rm max}}{k_{\rm min}} \right)$$

which is order one. Without the enhancement from the kinetic correction terms, the main term from the fluid contribution gives the value of same order as in Ref. 5. However, the enhancement factor can be as large as three. Our formula remains finite as $r_n \to \infty$ in contrast to the previous formula which goes to zero. Corrections to ion thermal flux from gyroradius and nonlinear effects can modify the result and will be investigated in the future work.

V. Conclusion

In conclusion, we formulate the renormalized turbulence equations for the ion pressure gradient driven or the η_i-mode from the nonlinear gyrokinetic equation. We show that the linearly unstable η_i-modes are stabilized by the turbulent $\mathbf{E}\times\mathbf{B}$ convection of the pressure fluctuation and the kinetic correction gives a stabilizing effect to the long wavelength and a destabilizing effect to the short wavelengths. The resulting anomalous ion thermal conductivity formula given in Eq. (39) is greater than the neoclassical plateau formula by a factor of up to ten, but gives the value of same order as the previously computed formula of Horton-Choi-Tang with a kinetic enhancement factor that can be as large as three. Also, the formula remains finite as $r_n \to \infty$. The turbulence theory developed here requires the system be sufficiently above threshold $\eta_i > \eta_{\rm crit} \approx 1$ that a number of modes $\Delta k \lesssim k_0$ are unstable. Linear studies[8] show that by $\eta_i \sim 2$ there is a broad spectrum of unstable modes (e.g., Fig. 1 of Ref. 8). The effect of coupling to the electromagnetic components $A_{\|}$ and $\delta B_{\|}$ is not appreciable until β approaches $\beta/\beta_c \sim 0.9$ where $\beta_c = \epsilon_n/q^2(1+\eta_i)$ is the critical plasma pressure for MHD instability.

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Appendix A: Renormalization of the Gyrokinetic Equation

In this appendix, we derive the Eqs. (7)–(13) from Eqs. (2) and (3). Equations (2) and (3) are written as

$$\left(\omega - k_{\|}v_{\|} - \omega_{Di}\right) h_k = S_k \phi_k + \sum_{k_1} V_{kk_1} \phi_{k_1} h_{k_2} \tag{A1}$$

and

$$\phi_k = -\frac{4\pi N_0 e}{k^2}\left(\frac{e}{T_i} + \frac{e}{T_e}\right)\phi_k + \frac{4\pi N_0 e}{k^2}\int d\mathbf{v} h_k J_0(\gamma). \tag{A2}$$

We isolate the "self-action" term proportional to h_k and ϕ_k among the nonlinear term of Eq. (A1) and write Eq. (A1) as

$$\left(\omega - k_{\|}v_{\|} - \omega_{Di} + d_k\right) h_k = \left(S_k + \xi_k\right)\phi_k + \left(\sum_{k_1} V_{kk_1}\phi_{k_1} h_{k_2} + d_k h_k - \xi_k \phi_k\right). \tag{A3}$$

In the second parenthesis of the right-hand side of Eq. (A3), from which we have removed the self-action of each mode, only the input from the beat interaction of different modes remains. These inputs are considered to be small. So we put $h_k = h_k^{(0)} + h_k^{(1)}$ and $\phi_k = \phi_k^{(0)} + \phi_k^{(1)}$. As a result we obtain

$$\phi_k^{(1)} = -\frac{4\pi N_0 e}{k}\left(\frac{e}{T_i} + \frac{e}{T_e}\right)\phi_k^{(1)} + \frac{4\pi N_0 e}{k^2}\int d\mathbf{v} h_k^{(1)} J_0(\gamma) \tag{A4}$$

and

$$\left(\omega - k_{\|}v_{\|} - \omega_{Di} + d_k\right) h_k^{(1)} = \left(S_k + \xi_k\right)\phi_k^{(1)} + \sum_{k_1} V_{kk_1}\phi_{k_1}^{(0)} h_{k_2}^{(0)}. \tag{A5}$$

We need only the nonlinear term on the right-hand side of Eq. (A5) because $d_k h_k^{(0)} - \xi_k \phi_k^{(0)}$ do not contribute in the third-order correlation.

To find the equation for the spectrum, we multiply Eqs. (A1) and (A2) by ϕ_k^* and average over the random phases of $h_k^{(0)}$ and $\phi_k^{(0)}$. This gives

$$P_k = g_k\left(S_k + \xi_k\right) I_k + g_k\left(\sum_{k_1} V_{kk_1}\left\langle \phi_{k_1} h_{k_2} \phi_k^* \right\rangle + d_k P_k - \xi_k I_k\right) \tag{A6}$$

and

$$I_k = -\frac{4\pi N_0 e}{k^2}\left(\frac{e}{T_i}+\frac{e}{T_e}\right) I_k + \frac{4\pi N_0 e}{k^2}\int d\mathbf{v} P_k J_0(\gamma), \qquad (A7)$$

where we defined $P_k = \langle h_k \phi_k^* \rangle$, $I_k = \langle \phi_k \phi_k^* \rangle$ and $g_k = \left(\omega - k_\parallel v_\parallel - \omega_{Di} + d_k\right)^{-1}$.

To evaluate the third-order correlation, we put $h_k = h_k^{(0)} + h_k^{(1)}$ and $\phi_k = \phi_k^{(0)} + \phi_k^{(1)}$, and substitute them into the third-order correlation term. Since the third-order correlation vanishes in the lowest order, we obtain to the next order

$$\langle \phi_{k_1} h_{k_2} \phi_k^* \rangle \cong \left\langle \phi_{k_1}^{(1)} h_{k_2}^{(0)} \phi_k^{*(0)} \right\rangle + \left\langle \phi_{k_1}^{(0)} h_{k_2}^{(1)} \phi_k^{*(0)} \right\rangle + \left\langle \phi_{k_1}^{(0)} h_{k_2}^{(0)} \phi_k^{*(1)} \right\rangle.$$

We substitute $h_k^{(1)}$ and $\phi_k^{(1)}$ from Eqs. (A4) and (A5) and use the random phase approximations of correlations in the form $\langle ABCD \rangle = \langle AB \rangle \langle CD \rangle + \langle AC \rangle \langle BD \rangle + \langle AD \rangle \langle BC \rangle$ and obtain

$$\begin{aligned}\langle \phi_{k_1} h_{k_2} \phi_k^* \rangle &= g_{k_2} V_{k_2,-k_1} \left\langle \phi_{k_1}^{(0)} \phi_{k_1}^{*(0)} \right\rangle \left\langle h_k^{(0)} \phi_k^{*(0)} \right\rangle \\ &+ g_{k_2} V_{k_2,k} \left\langle \phi_k^{(0)} \phi_k^{*(0)} \right\rangle \left\langle \phi_{k_1}^{(0)} h_{k_1}^{*(0)} \right\rangle \\ &+ g_{k_2}\left(S_{k_2}+\xi_{k_2}\right)\left(\frac{v_{k_2 k}}{\epsilon_{k_2}}\left\langle \phi_{k_1}^{(0)} \phi_{k_1}^{*(0)} \right\rangle + \frac{v_{k_1 k}}{\epsilon_{k_1}}\left\langle \phi_{k_2}^{(0)} \phi_{k_2}^{*(0)} \right\rangle\right)\left\langle \phi_k^{(0)} \phi_k^{*(0)} \right\rangle \\ &+ g_{k_2}\left(S_{k_2}+\xi_{k_2}\right)\frac{v_{kk_1}^*}{\epsilon_k^*}\left\langle \phi_{k_1}^{(0)} \phi_{k_1}^{*(0)} \right\rangle \left\langle \phi_{k_2}^{(0)} \phi_{k_2}^{*(0)} \right\rangle, \end{aligned} \qquad (A8)$$

where we defined

$$\epsilon_k = 1 + \frac{4\pi N_0 e^2}{k^2}\left(\frac{1}{T_i}+\frac{1}{T_e}\right) - \frac{4\pi N_0 e}{k^2}\int d\mathbf{v} g_k \left(S_k + \xi_k\right) J_0(\gamma) \qquad (A9)$$

and

$$v_{kk_1} = \frac{4\pi N_0 e}{k^2}\int d\mathbf{v} J_0(\gamma) g_k \left[V_{kk_1} g_{k_2}\left(S_{k_2}+\xi_{k_2}\right) + (1 \leftrightarrow 2)\right]. \qquad (A10)$$

Substituting Eq. (A8) into Eq. (A6), we obtain, dropping the superscript (0),

$$\begin{aligned} P_k = g_k\left(S_k+\xi_k\right) I_k + g_k \Bigg[&\sum_{k_1} V_{kk_1} g_{k_2} V_{k_2,-k_1} I_{k_1} P_k + \sum_{k_1} V_{kk_1} g_{k_2} v_{k_2 k} P_{-k_1} I_k \\ &+ \sum_{k_1} V_{kk_1} g_{k_2}\left(S_{k_2}+\xi_{k_2}\right)\left(\frac{v_{k_2 k}}{\epsilon_{k_2}} I_{k_1} + \frac{v_{k_1 k}}{\epsilon_{k_1}} I_{k_2}\right) I_k \\ &+ \sum_{k_1} V_{kk_1} g_{k_2}\left(S_{k_2}+\xi_{k_2}\right)\frac{v_{kk_1}^*}{\epsilon_k^*} I_{k_1} I_{k_2} + d_k P_k - \xi_k I_k \Bigg]. \end{aligned} \qquad (A11)$$

The resonance-broadening d_k and the vertex renormalization function ξ_k are defined so as to eliminate the terms proportional to P_k and I_k, respectively, so we obtain

$$d_k = -\sum_{k_1} V_{kk_1} g_{k_2} V_{k_2,-k_1} I_{k_1} \tag{A12}$$

and

$$\xi_k = \sum_{k_1} V_{kk_1} g_{k_2} V_{k_2 k} P_{-k_1} + \sum_{k_1} V_{kk_1} g_{k_2} \left(S_{k_2} + \xi_{k_2}\right) \left(\frac{v_{k_2 k}}{\epsilon_{k_2}} I_{k_1} + \frac{v_{k_1 k}}{\epsilon_{k_1}} I_{k_2} \right). \tag{A13}$$

Equation (A11) is then written

$$P_k = g_k \left(S_k + \xi_k\right) I_k + g_k \sum_{k_1} V_{kk_1} g_{k_2} \left(S_{k_2} + \xi_{k_2}\right) v^*_{kk_1} \frac{I_{k_1} I_{k_2}}{\epsilon^*_k}$$

and substituting this equation into Eq. (A7) gives

$$\epsilon_k I_k = \frac{1}{2} \sum_{k_1} \frac{v_{kk_1} v^*_{kk_1}}{\epsilon^*_k} I_{k_1} I_{k_2}. \tag{A14}$$

Appendix B: Renormalization of the Fluid Model Equation

In this appendix, we renormalize the model fluid equation developed by Horton, Choi and Tang[5] which retain only the convective nonlinearity in the ion pressure balance equation and is applicable to the toroidal regime.

The model equations are (Eq. (17) and (18) of Ref. 5)

$$\left(1+k^2\right)\frac{d\phi_k}{dt} = iku_k\phi_k + 2ik\epsilon_n P_k \tag{B1}$$

$$\frac{dP_k}{dt} = -ik(1+\eta)\phi_k + \sum_{k_1}\left(\mathbf{k}_1\times\mathbf{k}_2\right)_{\|}\phi_{k_1}P_{k_2}. \tag{B2}$$

We regard the linear equation of (B1) as "Poisson-like equation" and the nonlinear equation (B2) as "Vlasov-like equation." Following the procedures of Sec. II and Appendix A, we obtain the renormalized set of equations. They are

$$\epsilon_k I_k = \frac{1}{2}\sum_{k_1} v_{kk_1}v^*_{kk_1}I_{k_1}I_{k_2} \tag{B3}$$

with

$$\epsilon_k = \left(-2\epsilon_n k\right)^{-1}\left[\omega\left(1+k^2\right) - ku_k + 2\epsilon_n k g_k S_k + 2\epsilon_n k\sum_{k_1}\frac{v_{kk_2}v_{k_2k}}{\epsilon_{k_2}}I_{k_1}\right], \tag{B4}$$

$$g_k = \left(\omega + d_k\right)^{-1}, \tag{B5}$$

$$S_k = k(1+\eta) + b_k, \tag{B6}$$

$$V_{kk_1} = i\left(\mathbf{k}_1\times\mathbf{k}_2\right)_{\|}, \tag{B7}$$

$$d_k = -\sum_{k_1} V_{kk_1}g_{k_2}V_{k_2,-k_1}I_{k_1}, \tag{B8}$$

$$b_k = \sum_{k_1} V_{kk_1}g_{k_2}V_{k_2k}g_{-k_1}S_{-k_1}I_{k_1}, \tag{B9}$$

and

$$v_{kk_1} = g_k\left(V_{kk_1}g_{k_2}S_{k_2} + V_{kk_2}g_{k_1}S_{k_1}\right). \tag{B10}$$

With the assumption of $\frac{d_k}{\omega}, \frac{b_k}{\omega} < 1$, the nonlinear dispersion relation becomes

$$\epsilon_k = D_k^\ell(\omega) + X_k(\omega) + Y_k(\omega) = 0, \qquad (B11)$$

with

$$D_k^\ell(\omega) = \left(1 + k^2\right)\omega^2 - k u_k \omega + k^2 \gamma_0^2, \qquad (B12)$$

$$X_k(\omega) = k\gamma_0^2 \sum_{k_1} (\mathbf{k}_1 \times \mathbf{k})_\parallel^2 \frac{I_{k_1}}{\omega_2} \left(\frac{k}{\omega} - \frac{k_1}{\omega_1}\right), \qquad (B13)$$

and

$$Y_k(\omega) = \gamma_0^4 \sum_{k_1} (\mathbf{k}_1 \times \mathbf{k})_\parallel^2 \frac{I_{k_1}}{D_{k_2}^\ell} k k_2 \left(\frac{k_2}{\omega_2} - \frac{k_1}{\omega_1}\right)\left(\frac{k_1}{\omega_1} - \frac{k}{\omega}\right). \qquad (B14)$$

Treating X_k and Y_k as perturbations, we calculate the frequency shift

$$\Delta\omega = -\left[X_k(\omega) + Y_k(\omega)\right]_{\omega=\omega_0} \Big/ \left.\frac{\partial D_k^\ell(\omega)}{\partial \omega}\right|_{\omega=\omega_0}$$

which gives $\mathrm{Im}(\Delta\omega)$ as

$$\mathrm{Im}(\Delta\omega) = -\frac{1}{2\gamma_0} \sum_{\mathbf{k}_1 > 0} (\mathbf{k}_1 \times \mathbf{k})_\parallel^2 I_{k_1} \frac{2}{k + k_1} - \frac{1}{2\gamma_0} \sum_{\mathbf{k}_1 > 0} (\mathbf{k}_1 \times \mathbf{k})_\parallel^2 I_{k_1} \frac{1}{k_1} \frac{k - k_1}{k + k_1}. \qquad (B15)$$

The result of Eq. (B15) is exactly the same ones as the first two terms in the parentheses of Eq. (34) in Sec. IV of the nonlinear growth rate. These contributions are from the fluid nonlinearity of the $\mathbf{E} \times \mathbf{B}$ convection of the pressure fluctuations as evidenced in Eq. (B2).

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