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A PARADIGM FOR JOINED HAMILTONIAN AND DISSIPATIVE SYSTEMS

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Abstract

A paradigm for describing dynamical systems that have both Hamiltonian and dissipative parts is presented. Features of generalized Hamiltonian systems and metric systems are combined to produce what are called metriplectic systems. The phase space for metriplectic systems is equipped with a bracket operator that has an antisymmetric Poisson bracket part and a symmetric dissipative part. Flows are obtained by means of this bracket together with a quantity called the generalized free energy, which is composed of an energy and a generalized entropy. The generalized entropy is some function of the Casimir invariants of the Poisson bracket. Two examples are considered: (1) a relaxing free rigid body and (2) a plasma collision operator that can be tailored so that the equilibrium state is an arbitrary monotonic function of the energy.

Prologue

It is with pleasure that I submit this somewhat preliminary work in honor of M. D. Kruskal. I consider this to be an outgrowth of my first work in this area, Ref. [1], which was influenced in a principal way by the famous six papers on the KdV equation by Kruskal and collaborators (in particular Ref. [2]). I recall with fondness Martin's support, encouragement and unflagging questioning.

I. Introduction

This paper is about a formalism for describing systems with dissipation. Physicists, particularly those that work in areas such as fluid mechanics, plasma physics or kinetic theory, view dynamical equations in a split manner: terms that cause dissipation are distinguished from those that don't. The formalism presented here concentrates on this split. The nondissipative dynamics is described as a generalized Hamiltonian system while the dissipative dynamics is described in terms of a metric. A bracket that incorporates both is defined on the phase space; thus we obtain a construct that has Riemannian as well as symplectic components. We define a generalized free energy and use it with this bracket to represent equations with dissipation in a manner analogous to the way Hamiltonian systems are represented in Poisson bracket form. This bracket representation has built into it conservation of dynamical constraints, such as energy, as well as guaranteed entropy production. Here the aim is to treat classical systems that relax to a time-independent equilibrium state, but the possibility of treating other systems, e.g. with more exotic attracting sets, exists.

The remainder of the paper is organized into five sections. In Sect. II we discuss generalized Hamiltonian mechanics; i.e. where the Poisson bracket is generalized. This is basically a review of material with a long history that includes work motivated by Lie, Dirac, and others. We recommend Ref. [3] (and references therein) for a geometrical mathematical slant and Ref. [4] for a coordinate approach. A very readable exposition is given in Ref. [5]. In Section III systems that are strictly dissipative are treated. The type of systems discussed have, in a natural way, Liapunov functions that guarantee asymptotic stability. This is because there is an associated metric. References [6]-[9] are useful for background material. In Section IV we put together Sections II and III and define metriplectic systems that have both Hamiltonian and dissipative parts. In the final two sections we treat two examples, one finite dimensional and one infinite dimensional. The former is a relaxing free rigid body while the latter is a plasma kinetic example. References [10]-[13] contain previous work concerning dissipative formalisms.

II. Generalized Hamiltonian Mechanics

Usually Hamilton's equations of motion are written in terms of the coordinates q_k and conjugate momenta p_k as

$$q_k=[q_k,H]\,,\quad \dot{p}_k=[p_k,H] \qquad\qquad k=1,2,\ldots,N \tag{1}$$

where H is the Hamiltonian and the Poisson bracket of two functions, f and g, of the p's and q's is given by

$$[f,g]=\sum_{k=1}^{N} \frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_k}\,. \tag{2}$$

If we relabel by defining z^i, $i=1,2,\ldots,2N$, by $z^i=q_i$ for $i=1,2,\ldots,N$ and $z^i=p_i$ for $i=N+1, N+2,\ldots,2N$ then Hamilton's equations and the Poisson bracket become

$$\dot{z}^i=J_c^{ij}\frac{\partial H}{\partial z^j}=[z^i,H] \tag{3}$$

and

$$[f,g]=\frac{\partial f}{\partial z^i}J_c^{ij}\frac{\partial g}{\partial z^j} \tag{4}$$

where we sum repeated indicies. The quantity J_c^{ij} is a contravariant tensor that is called the cosymplectic form; it is given by

$$(J_c)=\begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix} \tag{5}$$

where I_N is the NxN unit matrix. The cosymplectic form is so named because it is the dual or inverse of the symplectic two-form that is the essential structure for a geometrical description.

The form of Eq. (3) affords a convenient way to view the split between kinematics and dynamics. Here the kinematics is embodied in the definition of the Poisson bracket or equivalently the (J_c^{ij}), which determines the structure of the phase space. The dynamics is of course supplied by the choice of H. For usual Hamiltonian systems phase space is a symplectic manifold; i.e. a differentiable manifold with a closed nondegenerate two-form, the so called symplectic two-form. In this work we emphasize the cosymplectic form rather than the symplectic form. This point of view facilitates our goal of generalizing the kinematics, as well as the dynamics.

Conventional phase space has built into it a Lie algebra structure as determined by (J_c^{ij}). This Lie algebra is that associated with the group of canonical transformations; i.e. changes of coordinates that preserve the form of (J_c^{ij}). Specifically the Lie algebra of phase space is composed of real valued functions defined on phase space with a product given by Eq. (4). This product has the properties of antisymmetry and bilinearity. In addition it satisfies the Jacobi identity:

$$[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = 0 \quad . \tag{6}$$

Also it is a derivation

$$[fg,h] = f[g,h] + [f,h]g \quad . \tag{7}$$

All of these properties are a consequence of Eqs. (4) and (5). They also survive an arbitrary change of coordinates, but $(J_c^{ij}) \rightarrow (J^{ij})$ where (J^{ij}) need not have the form of Eq. (5) and may obtain dependence upon the phase space coordinates. Conversely, if we have a bracket of the form of Eq. (4) with (J_c^{ij}) replaced by some (J^{ij}) that retains the properties of antisymmetry and the Jacobi identity, then Darboux's theorem tells us that if the determinant of (J^{ij}) is not zero, we can construct a local canonical coordinate system in which the bracket obtains the form of Eq. (4). If we relax the requirement that $\det(J^{ij}) \neq 0$ then in the vicinity of a point where the rank of (J^{ij}) is 2M we can construct a coordinate system such that $(J^{ij}) \rightarrow (J_c'^{ij})$ where $(J_c'^{ij})$ has the following form:

$$(J_c'^{ij}) = \left[\begin{array}{cc|c} 0 & I_M & 0 \\ -I_M & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \tag{8}$$

A system of equations is Hamiltonian in a generalized sense if it can be written in the form

$$\dot{z}^i = J^{ij} \frac{\partial H}{\partial z^j} \qquad i = 1,2,\ldots,L \tag{9}$$

where the only requirements on (J^{ij}) are that the bracket it defines,

$$[f,g] = \frac{\partial f}{\partial z^i} J^{ij} \frac{\partial g}{\partial z^j} \quad , \tag{10}$$

must be antisymmetric and satisfy the Jacobi identity. When (J^{ij}) is not of the form of Eq. (5) the bracket given by Eq. (10) is called noncanonical. We note that L, the dimension of (J^{ij}), need not be even; thus cases where $\det(J^{ij})=0$ are acceptable. In fact, in these cases the phase space has an interesting structure. If the rank of (J^{ij}) is equal to 2M in the vicinity of some point in phase space then there exists L-2M null eigenvectors for (J^{ij}). The possibility exists that one of these null eigenvectors can be written as the gradient of some phase space function C; i.e.

$$J^{ij} \frac{\partial C}{\partial z^j} = 0 \quad . \tag{11}$$

This turns out to be true. Moreover, it can be shown that the null space is spanned by such gradients: $\partial C^{(\alpha)}/\partial z^i$, $\alpha = 1,2,\ldots,L-2M$. The quantities $C^{(\alpha)}$ are called Casimirs. They are constants of motion that are built into the phase space since given any Hamiltonian, H, the following holds:

$$\dot{C}^{(\alpha)} = [C^{(\alpha)},H] = 0 \qquad \alpha = 1,2,\ldots,L-2M. \tag{12}$$

Thus trajectories are confined to lie in surfaces defined by the constancy of the $C^{(\alpha)}$'s. These surfaces have dimension 2M and are imbedded in the whole phase space of dimension L. They are actually symplectic manifolds. A picturesque phraseology that has emerged is to say when (J^{ij}) is degenerate, phase space foliates into symplectic leaves. In Fig. [1] we depict this foliation for Euler's equations (c.f. Section V) where the leaves are concentric spheres.

III. Metric Systems

Prior to defining a metric system, let us briefly recall some definitions of stability for a dynamical system

$$\dot{z}^i = F^i(z) \qquad i=1,2,\ldots N \tag{13}$$

where $z \equiv (z^1, z^2, \ldots z^N)$. A phase space point z_e is clearly an equilibrium point for Eq. (13) if $F^i(z_e)=0$ for $i=1,2,\ldots N$. Such an equilibrium point is stable if for every neighborhood M of z_e there is a neighborhood N of z_e such that if initially $z(0)$ is in M, then the solution will remain in N for all time. Here we are interested in asymptotic stability. A system is asymptotically stable if in addition $\lim_{t\to\infty} z(t)=z_e$. These types of stability are indicated pictorially in Figs. (2a) and (2b) respectively.

One way of ascertaining stability or asymptotic stability is by means of Liapunov's method. If one can find a nice funtion $L(z)$ defined on phase space points such that in a neighborhood M of z_e, $L(z_e)=0$ and $L(z)<0$ for $z \neq z_e$; if moreover $\dot{L} \geqslant 0$ for $z \neq z_e$ and $\dot{L}(z_e)=0$, then z_e is a stable equilibrium point. If the function $\dot{L}(z)$ is definite, i.e. $\dot{L}(z)>0$ for $z \neq z_e$ and $\dot{L}(z_e)=0$, then we have asymptotic stability.

We define a metric system as follows:

$$\dot{z}^i = g^{ij}\frac{\partial S}{\partial z^j} = (z^i, S) \qquad i,j=1,2\ldots N \tag{14}$$

where $S(z)$ is some phase space function that has the natural physical interpretation as entropy. The bracket, (,), of Eq. (14) is defined on two phase space functions f and g by

$$(f,g) = \frac{\partial f}{\partial z^i} g^{ij} \frac{\partial g}{\partial z^j} \quad . \tag{15}$$

This bracket is clearly linear in each of its arguments; we further require that it be symmetric $(f,g)=(g,f)$, which in turn requires $g^{ij}=g^{ji}$. In general g^{ij} may depend upon z. (A consistent terminology would be to call g^{ij} a cometric form.)

The above definition of a metric system is the natural starting place for building asymptotic stability into a phase space. If we add the requirement that g^{ij} must have positive definite eigenvalues over the whole of the phase space of interest, then isolated maxima of S are asymptotically stable equilibrium points. To see this, first observe that since we have assumed positive definite eigenvalues the quantity g^{ij} has no null eigenvectors and thus equilibria correspond to extremal points of S. If such a point, z_e, is an isolated maximum, then one can add a constant to S such that $S(z_e)=0$ and $S(z_e)<0$ for some deleted neighborhood of z_e. Evidently,

$$\dot{S}=\frac{\partial S}{\partial z^i}\dot{z}^i=\frac{\partial S}{\partial z^i}g^{ij}\frac{\partial S}{\partial z^j}\geqslant 0 \quad , \tag{16}$$

where the equality is achieved only at the point $z=z_e$. Thus S serves as a strong Liapunov function; i.e. one that guarantees asymptotic stability.

This is a felicitous state of affairs since one need only examine the extremal points of S and determine which are maxima. Asymptotic stability of these equilibria is built into the phase space by virtue of the assumptions we have made on g^{ij}. (Similarly a statement about instability can be made concerning equilibria that are minima of S.)

When the phase space for metric systems is equipped with a nondegenerate g^{ij} many questions of Riemannian geometry arise. We briefly indicate a few ideas along this line.

Consider the usual expression for the length of a segment of a curve

$$s(t_0,t_1;z) = \int_{t_0}^{t_1} \sqrt{g_{ij}\frac{dz^i}{dt}\frac{dz^j}{dt}}\,dt \tag{17}$$

where s can be considered to be a functional of an arbitrary curve z(t). If we seek an extremal of s by letting $z \rightarrow z + \delta z$ subject to $\delta z(t_0)=\delta z(t_1)=0$ we obtain the well known equation for geodesic curves. The form of Eq. (17) is nice since arc length is a geometrical quantity and thus invariant under reparametrization of the curve. At the expense of losing this property, but for ease in manipulation we seek extremal curves for the integral over the incremental length squared

$$s^{(2)}(t_0,t_1;z) = \int_{t^0}^{t_1} g_{ij} \frac{dz^i}{dt} \frac{dz^j}{dt}\, dt \quad . \tag{18}$$

Extremals of Eqs. (17) and (18) are the same. Upon variation of $s^{(2)}$ we obtain

$$\ddot{z}^{\ell} + \Gamma^{\ell}_{ij} \dot{z}^i \dot{z}^j = 0 \tag{19}$$

where as usual

$$\Gamma^{\ell}_{ij} = \frac{1}{2} g^{\ell k}\left(\frac{\partial g_{ik}}{\partial z^j} + \frac{\partial g_{jk}}{\partial z^i} - \frac{\partial g_{ij}}{\partial z^k}\right) \quad . \tag{20}$$

Let us now constrain our curve to be a solution trajectory of a metric system; i.e. where $\dot{z}^i$ is determined by Eq. (14). Inserting Eq. (14) into Eq. (18) yields the following:

$$s^{(2)}(t_0,t_1;z) = \int_{t_0}^{t_1} (S,S)\, dt \tag{21}$$

$$= \int_{t_0}^{t_1} \dot{S}\, dt \quad = \quad S(t_1) - S(t_0) \quad = \quad \Delta S \quad .$$

Thus we have established a connection between $s^{(2)}$, which is related to the length of a trajectory, and ΔS, the change in entropy after a time interval $\Delta t = t_1 - t_0$; hence, extremizing the change in entropy over a time interval is equivalent to the variational principle for geodesic, but with the constraint, Eq. (14). It is natural to ask the following: What is required of S or g^{ij} such that solutions of Eq. (14) correspond to geodesic; i.e. what is necessary for the compatibility of Eq. (14) and (19)? This issue can be explored by taking a time derivative of Eq. (14) and comparing the result with Eq.(18). These equations are seen to be redundant if $\ddot{S} = 0$, or

$$\ddot{S} = \dot{z}^{\ell}\frac{\partial \dot{S}}{\partial z^{\ell}} = \dot{z}^{\ell}\frac{\partial}{\partial z^{\ell}}(S,S) = 0 \quad , \tag{22}$$

which can be viewed as a constraint involving both g^{ij} and S. This constraint is equivalent to the constancy of $\dot{S}$; i.e. for some constant K

$$K = \frac{1}{2}(S,S) = \frac{1}{2}g^{ij}\frac{\partial S}{\partial z^i}\frac{\partial S}{\partial z^j} \quad . \tag{23}$$

Observe that Eq. (23) is just the Hamilton-Jacobi equation for geodesic motion. The coordinates z^i span the configuration space while the conjugate momenta are $p_i = \partial S/\partial z^i$. The entropy in Eq. (23) takes the role of the generating function that integrates the geodesic flow. If g^{ij} is specified then we have compatibility if the S that generates the dynamics in configuration space, via Eq. (14), is also the generating function for integrating the 2N dimensional geodesic flow. Alternately we can specify S and view Eq. (23) as placing a constraint on g^{ij}.

Consider now another feature of the phase space for metric systems. If we have a nondegenerate g^{ij} then the phase space has associated with it a Riemannian curvature tensor, $R^i{}_{jkl}$, defined in the usual way. This can serve as a means for classifying dissipation; e.g. the corresponding phase space may be flat, have constant curvature etc. If we remove the requirement that g^{ij} be nondegenerate then the notion of Riemannian curvature can still survive. For example if we require that the rank of g^{ij} be fixed then we can have Riemannian or metric leaves embedded in our larger space, in analogy with the degenerate case discussed on Sect. II. When this happens g^{ij} will have null eigenvectiors, which play a part in the metriplectic construct that is treated in the next section.

IV. Metriplectic Systems

We now turn to the task of combining the features of the systems discussed in Sections II and III in order to obtain a framework for describing a class of dissipative dynamical systems. To begin with we build into the phase space certain of these features. The picture we have in mind is a phase space that has embedded symplectic leaves as well as metric leaves. This is achieved by defining the metriplectic phase space as a differentiable manifold that is equipped with a cosymplectic form and a metric form. Thus there is a bilinear bracket operation on functions of the dynamical variables given by

$$\{f,g\} = [f,g] + (f,g) = \frac{\partial f}{\partial z^i} J^{ij} \frac{\partial g}{\partial z^j} + \frac{\partial f}{\partial z^i} g^{ij} \frac{\partial g}{\partial z^j} \quad . \tag{24}$$

On this primitive level we only require that J^{ij} be antisymmetric and endow [f,g] with the Jacobi identity while g^{ij} must be symmmetric. For the present purposes we add the requirements that g^{ij} have fixed rank and that its nonzero eigenvalues be positive.

Fundamentally it is desirable for dissipative systems to conserve dynamical constraints such as energy and momentum while producing entropy. For this reason one can build into the phase space the preservation of such constants of motion. This is done by requiring the phase space gradients of these quantities to be null eigenvectors of g^{ij}. Since we are going to use Eq. (24) to determine the arena for the dynamics, it will be apparent that this confines a trajectory to a surface defined by these dynamical constants.

A metriplectic flow is defined in terms of Eq. (24) and a quantity F, called the generalized free energy, as follows:

$$\dot{z}^i = \{z^i, F\} = J^{ij} \frac{\partial F}{\partial z^j} + g^{ij} \frac{\partial F}{\partial z^j} \quad . \tag{25}$$

The first term of Eq. (25) generates flow within symplectic leaves while the second term can generate flow out of symplectic leaves along the dynamical constraint surface. Figure (1) depicts this for the example discussed in Section V.

The generalized free energy is defined by

$$F = H - S \quad , \tag{26}$$

where H is the Hamiltonian or energy for the nondissipative portion (i.e. $g^{ij}=0$) and the quantity S, called a generalized entropy, is an arbitrary function of the Casimirs of J^{ij}. (One could place a constant T in front of S without loss of generality.) If we accept for the moment that S is an entropy-like object then this definition can be motivated by analogy with equilibrium thermodynamics. Recall that in the energy formulation of thermodynamics the equilibrium state is obtained by extremizing the energy at constant entropy. The reason that F is an appealing choice for generating the dynamics is that here the critical points of F correspond to dynamical equilibria since in view of Eq. (25) $\partial F/\partial z^i = 0 \Rightarrow \dot{z}^i = 0$.

We give two arguments for suggesting why Casimirs are entropy-like objects. In the first place we argue from experience. For cases where the physics is understood, such as in fluid mechanics and kinetic theory, the entropies have been identified as a Casimirs (c.f. Section VI) of the Hamiltonian portion of these systems. Secondly, Casimirs are known to be constants that arise from a relabelling symmetry of some underlying description.[14-16] They occur when there is a process of phase space reduction that results in a loss of information. For example the Casimir $S[f]=\int s(f)dz$ of Section VI for the Vlasov-Poisson system arises from the fact that there is a redundancy of continuum particle states that yield the same phase space density f. One can label the particle states by initial conditions, but functions of the initial conditions can serve as well.

In closing this section we point out some interesting additional properties that a metriplectic system may possess. For example, it may turn out that the metric leaves are flat or of constant curvature. It is clearly possible for symplectic and metric leaves to be orthoogonal. The notion of a metriplectic two form $m = J_{ij}dz^i \wedge dz^j + g_{ij}dz^i \otimes dz^j$ emerges. Discussion of these ideas and others may be considered in a future work.

V. Relaxing Free Rigid Body

In order to illustrate the formalism outlined in the previous section we treat an example. We begin by considering the motion of a rigid body with fixed center of mass under no torques. The motion of such a free rigid body is governed by Euler's equations

$$\begin{aligned} \dot{\omega}_1 &= \omega_2\omega_3(I_2 - I_3) \\ \dot{\omega}_2 &= \omega_3\omega_1(I_3 - I_1) \\ \dot{\omega}_3 &= \omega_1\omega_2(I_1 - I_2) \quad . \end{aligned} \tag{27}$$

Here we have done some scaling, but the dynamical variables ω_i, $i = 1,2,3$, are related to the three principal axis components of the angular velocity, while the constants I_i, $i = 1,2,3$, are related to the three principal moments of inertia.

This system conserves the following expressions for rotational kinetic energy and squared magnitude of the angular momentum:

$$H = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) \tag{28a}$$

$$l^2 = \omega_1^2 + \omega_2^2 + \omega_3^2 \quad . \tag{28b}$$

The quantity H can be used to cast Eqs. (27) into Hamiltonian form in terms of a noncanonical Poisson Bracket[4] that involves the three dynamical variables, ω_i. The matrix (J^{ij}) introduced in Sect. III has a null eigenvector that is given by $\partial l^2/\omega_i$; i.e. l^2 is a Casimir. The noncanonical Poisson bracket is

$$[f,g] = \frac{\partial f}{\partial \omega_i}\, \omega_k \epsilon_{ijk} \frac{\partial g}{\partial \omega_j} \qquad i,j,k = 1,2,3 \tag{29}$$

where ϵ_{ijk} is the Levi-Civita symbol. Evidently Eqs. (27) are equivalent to

$$\dot{\omega}_i = [\omega_i, H] \qquad i = 1,2,3 \tag{30}$$

and we have for an arbitrary function $S(l^2)$, $[S,f]=0$ for all f.

So far we have endowed the phase space, which has coordinates ω_i, with a cosymplectic form. Let us now add to this a metric component. In this case a dynamical constraint manifold corresponds to a surface of constant energy, i.e. an ellipsiod. We wish to construct a (g^{ij}) that has $\partial H/\omega_i$ as a null eigenvector, while possessing two nonzero eigenvalues of the same sign. This is convieniently done by defining the bracket (,) in terms of a projection matrix; i.e.

$$(f,h) = -\lambda\left[\left(\frac{\partial H}{\partial \omega_i}\frac{\partial H}{\partial \omega_j} - \delta_{ij}\frac{\partial H}{\partial \omega_\ell}\frac{\partial H}{\partial \omega_\ell}\right)\frac{\partial f}{\partial \omega_i}\frac{\partial h}{\partial \omega_j}\right] \quad . \tag{31}$$

For now we take λ to be constant, but it could depend upon ω. Explicitly the (g^{ij}) is given by

$$(g^{ij}) = \lambda\begin{bmatrix} I_2^2\omega_2^2 + I_3^2\omega_3^2 & -I_1I_2\omega_1\omega_2 & -I_1I_3\omega_1\omega_3 \\ -I_1I_2\omega_1\omega_2 & I_1^2\omega_1^2 + I_3^2\omega_3^2 & -I_2I_3\omega_1\omega_3 \\ -I_1I_3\omega_1\omega_3 & -I_2I_3\omega_1\omega_3 & I_1^2\omega_1^2 + I_2^2\omega_2^2 \end{bmatrix}. \tag{32}$$

We are now in a position to display a class of metriplectic flows for the rigid body; i.e.

$$\dot{\omega}_i = \{\omega_i,F\} = [\omega_i,F] + (\omega_i,F) \tag{33}$$

$$= J^{ij}\frac{\partial H}{\partial \omega_j} + g^{ij}\frac{\partial S}{\partial \omega_j} \qquad i=1,2,3 \quad ,$$

where $F=H-S$, H is given by Eq. (28a) and S is an arbitrary function of l^2. For the case $i=1$ we have

$$\dot{\omega}_1 = \omega_2\omega_3(I_2-I_3) + 2\lambda S'(l^2)\omega_1[I_2(I_2-I_1)\omega_2^2 + I_3(I_3-I_1)\omega_3^2]. \tag{34}$$

The other two equations are obtained upon cyclic permutation of the indices. By design this system conserves energy but produces the generalized entropy $S(l^2)$ if $\lambda > 0$, which could be chosen to correspond to angular momentum.

It is well known that equilibria of Euler's equations composed of pure rotation about either of the principal axes corresponding to the largest and smallest principal moments of inertia are stable. If we suppose that $I_1 < I_2 < I_3$, then stability of an equilibrium defined by $\omega_1 = \omega_2 = 0$ and $\omega_3 = \omega_0$ can be shown by means of the following Liapunov function (see e.g. Ref. [7]):

$$F = \frac{1}{2}[(I_1 - I_3)\omega_1^2 + (I_2 - I_3)\omega_2^2] - \frac{\epsilon}{4}[\omega_1^2 + \omega_2^2 + \omega_3^2 - \omega_0^2]^2 , \qquad (35)$$

where ϵ is an arbitrary positive constant. If we recognize that F is a generalized free energy, i.e. $F = H - \hat{S}(l^2)$ where $\hat{S}(l^2) = I_3 l^2/2 + \epsilon(l^2 - \omega_0^2)^2/4$, then it is clear that Euler's equations conserve F. Also it is evident that the phase space point $\omega_e = (0,0,\omega_0)$ is an isolated maximum for F.

If we append to Euler's equations the terms arising from the metric, as shown in Eq. (34) and its two companion equations with $S' = \hat{S}' = I_3/2 + \epsilon(l^2 - \omega_0^2)/2$, then we have a system for which the equilibrium $\omega_e = (0,0,\omega_0)$ is asymptotically stable. For example, if initially $\omega_1 = \omega_0/2$, $\omega_2 = \omega_0/2$ and $\omega_3 = \omega_0/2$, and the constants ϵ and λ are arranged properly then the system will relax to ω_e. This relaxation must take place along the dynamical constraint surface. The metriplectic phase space, with a trajectory corresponding to this motion, is depicted in Fig. (1).

VI. Extension to Field Theory - A Plasma Physics Example

For field theories the infinite dimensional generalization of the structure discussed in Sect. III is required. Thus if $\psi^i(x,t)$, $i=1,2,\ldots M$, are the field variables, then we are interested in systems represented by

$$\frac{\partial\psi^i}{\partial t} = \{\psi^i, F\} = [\psi^i, F] + (\psi^i, F) \quad , \tag{36}$$

where F is now a generalized free energy functional and the two brackets on the right are suitably generalized.

Consider first the Hamiltonian portion of the field theory. The associated Poisson bracket has the form

$$[A,B] = \int \frac{\delta A}{\delta\psi^i} O^{ij} \frac{\delta B}{\delta\psi^j}\, d\tau \tag{37}$$

where $d\tau$ is an x-space volume element, A and B are functionals of the ψ^i and $\delta A/\delta\psi^i$, the functional derivative, is defined by

$$\frac{d}{d\epsilon} A[\psi^i + \epsilon\delta\psi]\bigg|_{\epsilon=0} = \int \frac{\delta A}{\delta\psi^i}\,\delta\psi\, d\tau \quad . \tag{38}$$

The quantity O^{ij} is a cosymplectic operator; i.e. an operator that is anti-self-adjoint (so that $[A,B]=-[B,A]$) and also satisfies a requirement that insures the Jacobi identity for $[A,B]$ (see e.g. Ref. [17]). Although these properties are required, the O^{ij} need not have any specific form. Usually quantum field theories are written in terms of canonical variables, in which case the cosymplectic operator has the same form as that of Eq. (5), but this is not the case for fields that describe continuous media in Eulerian form. For those fields $O^{ij}=\psi^k C_k{}^{ij}$ where the quantities $C_k{}^{ij}$ are structure operators for some Lie algebra. Unlike the canonical case this O^{ij} is linear in the field variables; also the $C_k{}^{ij}$ may involve differential operators.

Similarly we generalize the symmetric bracket as follows:

$$(A,B) = \int \frac{\delta A}{\delta \psi^i} G^{ij} \frac{\delta B}{\delta \psi^j}\, d\tau \tag{39}$$

where G^{ij} is an operator analogous to the g^{ij}. We require G^{ij} to be self-adjoint and also to possess the necessary null space for the preservation of dynamical constraints.

As an example we consider the Vlasov-Poisson equation with the addition of a collision term; i.e.

$$\frac{\partial f}{\partial t}(z,t) = -\vec{v} \cdot \frac{\partial f}{\partial \vec{x}} + \frac{\partial \phi}{\partial \vec{x}}(\vec{x};f) \cdot \frac{\partial f}{\partial \vec{v}} = \left.\frac{\partial f}{\partial t}\right|_c \tag{40}$$

where $f(z,t)$ is the phase space density for a species of particles and $z = (\vec{x}, \vec{v})$ denotes a point of the particle phase space. The potential $\phi(\vec{x};f) = \int V(\vec{x}, \vec{x}\,')f(z')dz'$, where V is the single particle potential (assumed spatially invariant). For the moment we leave $\partial f/\partial t|_c$ unspecified; we will determine a general form for it by constructing a symmetric bracket with desired properties.

The Hamiltonian structure for the Vlasov-Poisson equation was introduced in Ref. [18], which we reproduce for completeness. The Hamiltonian functional in this case is the total energy

$$H[f] = \int T(z)f(z)\,dz + \frac{1}{2}\iint V(z,z')f(z)f(z')\,dz\,dz' \tag{41}$$

where $T = v^2/2$ is the particle kinetic energy (we have set the mass and charge to unity). The noncanonical Poisson bracket is the following:

$$[A,B] = \int f(z')[\delta A/\delta f(z'), \delta B/\delta f(z')]_c\, dz' \tag{42}$$

where $[f,g]_c = \partial f/\partial \vec{x} \cdot \partial g/\partial \vec{v} - \partial f/\partial \vec{v} \cdot \partial g/\partial \vec{x}$. Thus in this case the cosymplectic operator is $O = -[f, \cdot\,]_c$. Using $\delta H/\delta f = T + \phi \equiv H_p$, where H_p is the particle energy, we see that the following is equivalent to the Vlasov-Poisson equation:

$$\frac{\partial f}{\partial t} = [f, H] = -[f, H_F] \qquad . \tag{43}$$

We construct a symmetric bracket so as to generalize the form of the collision terms obtained by Landau and by Lenard and Balescu. Since this form is that of a Fokker-Planck equation we know that G^{ij} must be a second order differential operator in $\vec{v}$. Also since realistic collision operators are global in velocity space, we are led to the following form for (A,B):

$$(A,B) = -\iint \left[\frac{\partial}{\partial v_i}\frac{\delta A}{\delta f(z)} - \frac{\partial}{\partial v_i{}'}\frac{\delta A}{\delta f(z')}\right]$$

$$\times \left[\frac{\partial}{\partial v_j}\frac{\delta B}{\delta f(z)} - \frac{\partial}{\partial v_j{}'}\frac{\delta B}{\delta f(z')}\right] T_{ij}(z,z')dzdz' \quad . \tag{44}$$

We wish to restrict T_{ij} so the mass, momentum and energy generate null eigenvectors. This is achieved by letting $T_{ij} = w_{ij}(z,z')M(f(z))M(f(z'))/2$ where M is an arbitrary function of f and w_{ij} has the following symmetries: (i) $w_{ij}(z,z') = w_{ji}(z,z')$, (ii) $w_{ij}(z,z') = w_{ij}(z',z)$ and (iii) $(v_i - v_i{}')w_{ij} = 0$. The three properties of the w_{ij} are the same as those possessed by the tensor that appears in both the Landau and the Lenard-Balescu collision operators. It can in fact be either of these objects; in the former case it is given by

$$w_{ij}{}^{(L)} = (1/g)(\delta_{ij} - g_i g_j/g^2)\delta(\vec{x} - \vec{x}') \tag{45}$$

where L is a constant, δ_{ij} is the Kronecker delta, $\delta(\vec{x} - \vec{x}')$ is the Dirac delta and $g_i = v_i - v_i{}'$.

Let us now see what sort of collision term arises from the above assumptions. The generalized free energy in this case is $F = H - S$ where $S[f] = \int s(f)dz$ and s(f) is an arbitrary function of f. Inserting F into the symmetric bracket yields

$$\left.\frac{\partial f}{\partial t}\right|_c = (f,F) = \frac{\partial}{\partial v_i}\int w_{ij}\left[M(f(v))\frac{\partial f(v')}{\partial v_j'} - M(f(v'))\frac{\partial f(v)}{\partial v_j}\right]dv', \tag{46}$$

a prototypical collision operator. Although by construction we know that Eq.(46) will not destroy conservation of mass etc., verification of other improtant properties such as maintainance of the positivity of f by the dymamics and relaxation to a stationary state can be shown by essentially the same means used in Ref. [19].

The relaxation property requires a compatibility condition on the functions s(f) and M(f). For an H-theorem the following is required

$$M\frac{\partial^2 s}{\partial f^2} = 1 \qquad . \tag{47}$$

Thus we have a collision operator that is mated to a particular entropy functional. The state to which the system relaxes is determined by $\delta F/\delta f = 0$. This yields

$$H_p - s'(f) = 0 \qquad . \tag{48}$$

For a consistent solution s' must be monotonic; the relaxed state is given by

$$f_e = (s')^{-1}(H_p) \qquad , \tag{49}$$

and thus we have a framework that can be tailored to produce a system that relaxes to any equilibrium state where f_e is a monotonic function of the particle energy. Some special cases are of interest: (1) if we pick $M=f$ then compatibility requires $s=-kf\ln f$. The collision operator in this case is of the usual Landau form. (2) If we pick M to be quadratic in f, i.e. $M=f(1-f)$, then the compatible entropy is $s=-k(f\ln f+(1-f)\ln(1-f))$ and we have a system that relaxes to the Fermi-Dirac distribution. The collision operator obtained in this case is of the form of that given in Ref. [20],where a kinetic theory for arguments of Lynden-Bell[21] is discussed.

In closing we point out that a connection between the fluctuation spectrum of the Vlasov equation and our general collision operator can be made. If we make the following Bogoliubov-type assumption:

$$\langle \delta f \, \delta f \rangle_{k,\omega} = \delta(\vec{v} - \vec{v}\,')\delta(\omega - \vec{k} \cdot \vec{v})M(f), \tag{50}$$

then, paralleling the arguments of Ref. [20], a collision operator of the form of Eq. (46) is obtained.

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Figure Captions

Fig. 1. Depiction of the metriplectic phase space for the relaxing free rigid body. Symplectic leaves are concentric spheres while constant energy surfaces are ellipsoids.

Fig. 2. (a) Stability. (b) Asymptotic stability.

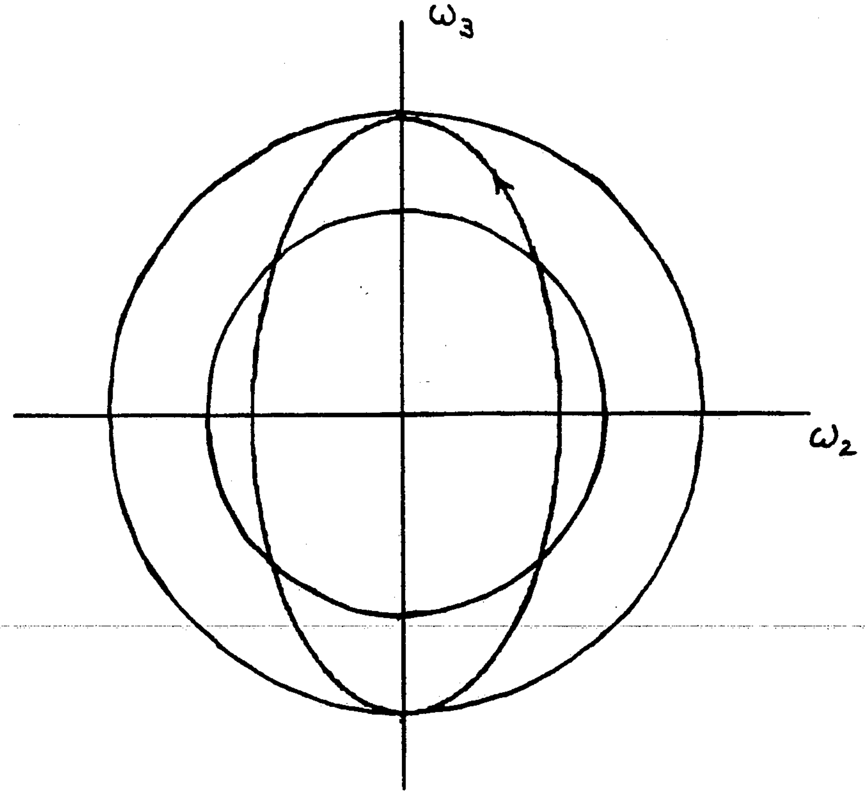
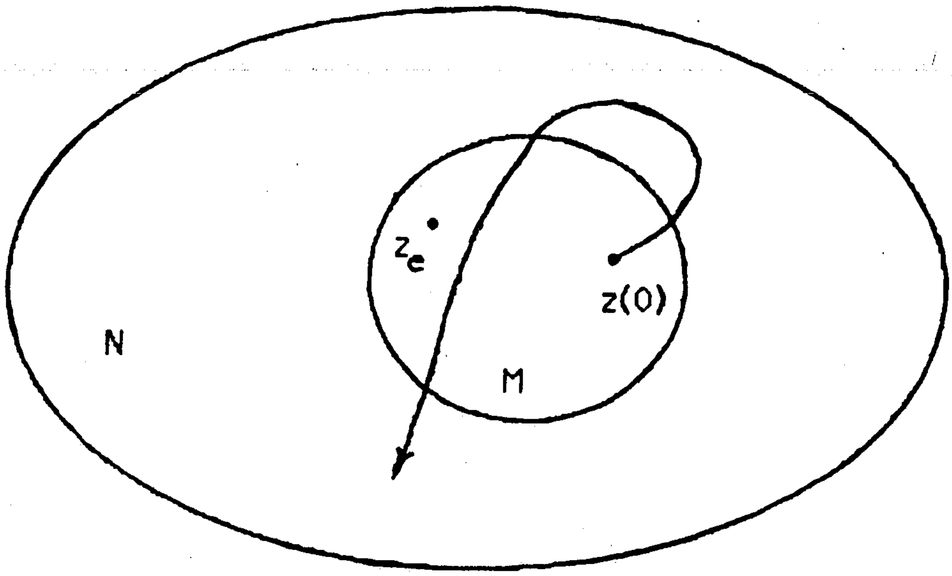
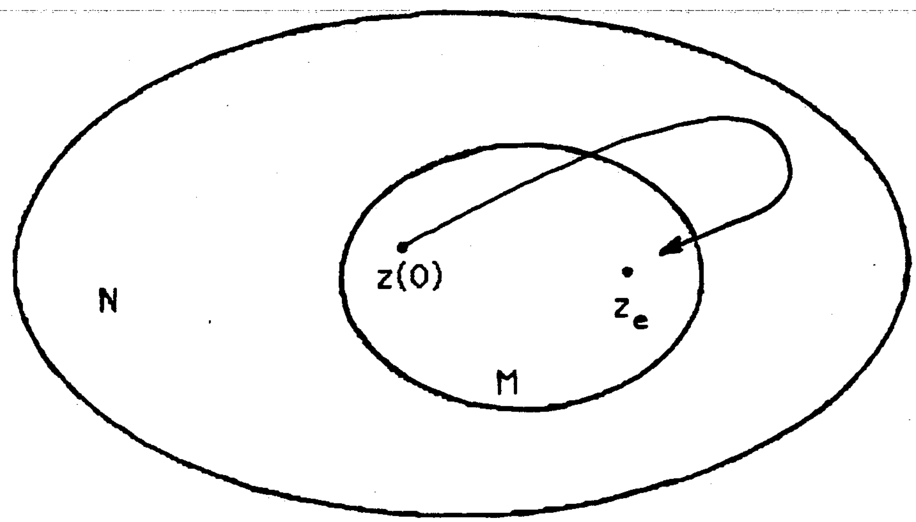


FIGURE 1



2a.



2b.

FIGURE 2