Scale Separation Closure and Alfvén Wave Turbulence

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Abstract

Based on the concept of scale separation between coherent response function and incoherent source for renormalized turbulence theories, a closure scheme is proposed. A model problem dealing with shear-Alfvén wave turbulence is numerically solved; the solution explicitly shows expected turbulence features such as frequency shift from linear modes, band-broadening, and a power law dependence for the turbulence spectrum.

I. Introduction

It is generally believed that renormalized turbulence theories[1−6] present a most promising way to deal with fluid as well as plasma turbulence. The essence of these theories could be schematically expressed in a simple equation[2]

$$\epsilon_k \varphi_k = \tilde{\varphi}_k \tag{1a}$$

or equivalently

$$|\epsilon_k|^2 |\varphi_k|^2 \equiv |\epsilon_k|^2 I_k = |\tilde{\varphi}_k|^2 \tag{1b}$$

where $k = (\mathbf{k}, \omega)$ is the wave four vector, φ_k is the fluctuating quantity whose spectrum $I_k = |\varphi_k|^2$ we wish to obtain, ϵ_k is the renormalized propagator (response function, dielectric function in appropriate cases), and $\tilde{\varphi}_k$ is generally referred to as the incoherent source. Equation (1) is generally derived by splitting the nonlinear interaction into two parts: the coherent part which modifies (renormalizes) the linear propagator to yield the nonlinear response function ϵ_k, and the incoherent part $\tilde{\varphi}_k$ which accounts for the remainder of the nonlinear interaction. The form of Eq. (1) suggests that it is to be interpreted as a driven system with the incoherent source acting as an external driver. One must however remember that this analogy with an external source is essentially formal (barring some special cases where the turbulence may be externally driven) because the incoherent source comes purely from the interaction of those very waves whose spectrum is the object of concern.

The simplicity of Eq. (1) is quite superficial; both ϵ_k and $\tilde{\varphi}_k$ are functionals of the spectrum I_k (and, in general, functionals of higher order correlations). Equation (1), however, serves as a basis for constructing an unending chain of equations in which an nth order correlation is determined in terms of higher order correlations. This feature, is of course, common with the unnormalized theories.[7−8] One of the primary tasks of turbulence is to propose a suitable closure scheme to truncate this infinite set of equations. For example, the assumption of quasi-normality[7] has been often used to affect the closure in unrenormalized fluid theories. (These fluid theories suffer from overshoot problems, and other theoretical difficulties, and were thus abandoned.)

A major advance in fluid turbulence was the Direct Introduction Approximation[1] (DIA) initiated by Kraichman which led to an essentially renormalized theory that contains the lowest nonlinear correction to the linear response function. The treatment of the incoherent source (IS) terms, however, was still a problem. In the DIA application to plasma theory known as DIAC[5] (coherent approximation to DIA), the incoherent term is very conveniently set equal to zero. However, the validity of such an assumption is highly questionable, and we deal with this question a little later.

It is quite clear that a good understanding of turbulence theories is unlikely without a proper and deeper understanding of the incoherent source. At the very least, one must make reasonable assumptions (to be verified a posteriori) about the nature of the incoherent source, and develop a closure scheme based on these assumptions. We must mention that the treatment of the IS is a very active field of research, and several authors[9–10] have set up two-point correlation theories to handle the problem. This, however, is a subject of much complexity, and our main interest now is to propose a physically motivated simple closure scheme which treats the incoherent source properly.

In this paper we propose a closure scheme based on the concept of scale separation between the incoherent source $\tilde{\varphi}_{\mathbf{k},\omega}$ and the response function $\epsilon_{\mathbf{k},\omega}$. It is proposed that the fine structure of steady-state turbulence spectrum is controlled by the response function $\epsilon_{\mathbf{k},\omega}$, and that the incoherent source $\tilde{\varphi}_{\mathbf{k},\omega}$, though arbitrary in magnitude, is comparatively broad or slowly varying in $\mathbf{k}$ and ω. The primary motivation for this assumption comes from a variety of observations on the nature of turbulence in confined plasmas. In the introduction, we give only the qualitative arguments; leaving more technical and quantitative arguments to Sec. II. We also take a short detour to put our current arguments in perspective.

In the linear theory, the right-hand side of Eq. (1) goes to zero, and $\epsilon_{\mathbf{k},\omega}$ also becomes $\epsilon^{\text{linear}}_{\mathbf{k},\omega} \equiv \epsilon^{\ell}_{\mathbf{k},\omega}$, implying that nontrivial solutions exist only for $\epsilon^{\ell}_{\mathbf{k},\omega} = 0$, which is solved to obtain a point spectrum $\omega = \omega(\mathbf{k})$. Thus the entire information about the linear modes is contained in the linear dielectric response $\epsilon^{\ell}_{\mathbf{k},\omega}$. For finite amplitude modes, the right-hand side of Eq. (1), i.e., the incoherent source term is not zero and could acquire an arbitrary magnitude as the mode amplitudes build up. Setting $\tilde{\varphi}_{\mathbf{k},\omega} = 0$ under such conditions (as is done in DIAC) is at best a poor approximation. This approximation further leads to the so-called nonlinear dispersion relation, $\epsilon_{\mathbf{k},\omega} = 0$ which is then solved to obtain complex $\omega(\mathbf{k})$ giving the nonlinear growth or damping rates. There is still a unique complex ω for a given $\mathbf{k}$. This approach is simply an extension of linear concepts, and is probably an adequate description only in the transient state immediately following the linear state. By no means, we could safely extend this to describe fully developed turbulence where $\tilde{\varphi}_{\mathbf{k},\omega}$ is quite arbitrary, and the spectrum $I_{\mathbf{k},\omega}$ is to be determined for real ω and $\mathbf{k}$; there being no growth or damping of modes in the steady-state turbulence. Although $\epsilon_{\mathbf{k},\omega} = 0$ ceases to be a physically meaningful equation (because $\tilde{\varphi}_{k,\omega} \neq 0$ which leads to singular amplitudes at the zeros of $\epsilon_{k,\omega}$), there is still plenty of information contained in the dielectric response function $\epsilon_{\mathbf{k},\omega}$. In fact, this function contains the entire coherent response, linear as well as nonlinear, of the medium (plasma) to the development of large amplitude waves (which may happen due to an instability, for example). It is now pertinent to ask what determines the detailed nature of the wave spectrum $I_{\mathbf{k},\omega}$; is it the coherent nonlinear properties of the medium which sustains the waves, or is it the character of the nonlinear source which is obtained by interminable scattering of waves by one another. In several problems of physical interest, we are concerned with determining the properties of the medium (plasma) in the presence of turbulence, for example, the anomalous electron transport in a tokamak plasma which is supposed to be driven by low-frequency microinstabilities. We now argue that if the turbulence was to strongly affect the medium, i.e., to appreciably change $\epsilon_{\mathbf{k},\omega}$ from its linear value, then the turbulence (i.e., the turbulent spectrum) must also be strongly affected by $\epsilon_{\mathbf{k},\omega}$. Thus, in a steady state self-consistent interaction, the properties of $\epsilon_{\mathbf{k},\omega}$ and $I_{\mathbf{k},\omega}$ are strongly linked to each other. Under these conditions the nonlinear source, though arbitrary in magnitude, will not determine the detailed structure of $I_{k,\omega}$. We further note that in strong turbulence, the linear modes have broadened, and have constantly interacted with each other. These constant interactions when summed up to give the incoherent source are expected to be quite flat in ω and $\mathbf{k}$.

The preceding considerations do suggest that a non-perturbative approximation scheme based on a scale-separation hypothesis between $\epsilon_{\mathbf{k},\omega}$ and $\tilde{\varphi}_{\mathbf{k},\omega}$ (comparatively structureless in ω and $\mathbf{k}$) could make a good starting point for determining the spectrum of steady-state turbulence in situations where we expect the medium properties to be strongly affected by turbulence [see more discussion in Sec. II.]

The theory is obviously not applicable when $\tilde{\varphi}_{\mathbf{k},\omega}$ might have a sharper structure than $\epsilon_{\mathbf{k},\omega}$. It is physically quite difficult to imagine a situation when incoherent scattering would somehow manage to make a sharp structure in ω and $\mathbf{k}$, and thus determine the shape of $I_{\mathbf{k},\omega}$.

Finally, we must check the consistency of our approximations a posteriori, i.e., our results are indeed consistent with the assumption that $\epsilon_{\mathbf{k},\omega}$ has a sharper structure than $\tilde{\varphi}_{\mathbf{k},\omega}$.

The paper has two distinct parts; the first part deals with setting up the closure scheme and is discussed in Sec. II. The second part is devoted to a particular application; Alfvén wave turbulence, and is treated in Sec. III. A short discussion is given in Sec. IV.

II. Scale Separation and Closure

A. General Considerations and Closure

It is generally possible to describe (in Fourier space) a nonlinear plasma or fluid system by the equation[11]

$$\epsilon_k^\ell \varphi_k = \sum_{k'} V_{k,k'} \varphi_{k'} \varphi_{k-k'} \tag{2}$$

where $k = (\mathbf{k}, \omega)$, $V_{k,k'}$ are the nonlinear coupling coefficients, and ϵ_k^ℓ is the linear dielectric response function of the medium. The idea behind a renormalization theory is to separate the right-hand side nonlinear terms into two distinct parts; the coherent part proportional to φ_k which when added to ϵ_k^ℓ makes the nonlinear dielectric response function ϵ_k, and the remainder which forms the incoherent source (IS). By a straightforward manipulation of Eq. (2) one obtains the appropriate Eq. (1)

$$(\epsilon_k^\ell + C_k)\varphi_k \equiv \epsilon_k \varphi_k = \tilde{\varphi}_k \tag{3}$$

where $\tilde{\varphi}_k$ is the IS, and the lowest order nonlinear renormalizing correction

$$C_k \varphi_k = -2 \sum_{k'} \frac{V_{k,k'} V_{k-k',k}}{\epsilon_{k-k'}} |\varphi_{k'}|^2 \varphi_k. \tag{4}$$

The coherent part C_k depends (to this order) only on the fluctuating spectrum $|\varphi_{k'}|^2$. The incoherent part $\tilde{\varphi}_k$ is considerably more complicated because it involves lots of unknown correlations. But to proceeds further, one needs to understand the nature of $\tilde{\varphi}_k$.

The DIAC theory, obviously motivated by the desire to solve the problem, puts $\tilde{\varphi}_k = 0$. A possible physical argument could be that for highly dispersive media (as the plasmas are), the strong coupling condition in one dimension $\mathbf{k} = (0, 0, \kappa)$

$$\omega(\kappa_1) + \omega(\kappa_2) = \omega(\kappa_1 + \kappa_2) \tag{5}$$

is not likely to be satisfied, and therefore the interaction term $\varphi_{\kappa',\omega'}\varphi_{\kappa-\kappa',\omega-\omega'}$ should be negligible except for the coherent part. It turns out that this argument is not very convincing, and this procedure is likely to lead to conceptual problems in the description of strong turbulence. The assumption $\tilde{\varphi}_\kappa = 0$ automatically leads to $\epsilon_\kappa = 0$, which is very much like the linear eigenvalue problem. Although, the dielectric now has a nonlinear correction, the solution would still be a unique complex ω for a given κ. Thus, the waves grow or damp in the turbulent background, but the waves do not interact with each other. A definite correspondence between ω and κ really conflicts with the concept of mode-broadening, which should be one of the main features of a strong turbulence theory.

In fact, one finds that the incoherent wave-wave interaction is more responsible for transporting energy from one mode to another than the coherent interaction. In three dimensions, the three-wave coupling condition

$$\omega(\mathbf{k}_1) + \hat{\omega}(\mathbf{k}_2) = \omega(\mathbf{k}_1 + \mathbf{k}_2) \tag{6}$$

can always be satisfied no matter what kind of dispersion relation the wave might have. We furthermore notice that in one dimension, the four wave interaction condition $\omega(\kappa_1) + \omega(\kappa_2) + \omega(\kappa_3) = \omega(\kappa_1 + \kappa_2 + \kappa_3)$ could always be satisfied (as long as ω can be negative as well as positive) even though the three wave coupling condition may not. The contribution to IS from the four-wave coupling process is given by $\varphi_{\kappa'}\varphi_{\kappa''}\varphi_{\kappa-\kappa'-\kappa''}$ which is precisely the same order as the coherent part.

It should be stressed that the preceding discussion is valid only in the quasilinear regime. For turbulent situation, the waves have broadened bands implying that the dispersion curves are not lines but bands. This broadening allows a larger number of wave-interactions to contribute effectively to the IS. The measure of these wave-interactions should be much larger than the measure of wave-interactions contributing to the coherent part. Thus we can safely conclude that setting the IS equal to zero cannot be justified for a turbulent system.

We now try to motivate the scale-separation or the band separation argument to deal with the incoherent source. The IS terms could be written as

$$\tilde{\varphi}_k = \sum_{k'} V_{k,k'}\varphi_{k'}\varphi_{k-k'} - \text{coherent part} \simeq \sum_{k'} V_{k,k'}\varphi_{k'}\varphi_{k-k'}$$

because the coherent terms really form a very small subset of all the nonlinear terms. Since the IS is obtained by summing over the entire spectral range, we expect that the structure of the IS will be smeared out in k, particularly when the spectrum is already broadened by nonlinear effects. We expect further smearing out to come from the following process. A typical turbulence spectrum has a broadened spectrum of width Δk around some $k[k = (\mathbf{k}, \omega)]$. Since $\Delta\omega$ and $\Delta|k|$ are generally large [In drift wave turbulence $\Delta\omega \sim \omega$],[12] these neighboring waves of comparable amplitude will generate a strong and wide low k band. This low k band beating with the peak spectrum will spread the incoherent interaction over all k resulting in relative broadening.

Thus, we may expect that the incoherent source would be quite flat in the vicinity of the broadened linear peaks. This helps us to develop an approximation scheme to close the chain of equations contained in Eq. (1). In the zeroth approximation we treat the incoherent source IS as a constant whose value reflects the strength of the source.

Within the context of preceding discussion, we assume that the k dependence of $|\varphi_k|^2 = I_k$ is determined entirely by ϵ_k, and $\tilde{\varphi}_k$ can be treated as a constant source φ_0. Making use of this assumption in Eq. (1), and taking the modulus square of both sides, we obtain

$$|\epsilon_k^\ell + C_k|^2|\varphi_k|^2 \equiv |\epsilon_k|^2 I_k = |\varphi_0|^2 \equiv S_0 \tag{7}$$

which makes Eq. (4) to be $[\epsilon_{k-k'} = \epsilon_{k-k'}^\ell + C_{k-k'}]$

$$C_k = C_k[I_{k'}, C_{k'}] = -2\sum_{k'} \frac{V_{k,k'}V_{k-k',k}}{\epsilon_{k-k'}}|\varphi_{k'}|^2 = -2\sum \frac{V_{k,k'}V_{k-k',k}}{\epsilon_{k-k'}} I_{k'} \tag{8}$$

implying that Eqs. (7) and (8) are closed, and can be solved, in principle, for I_k and C_k. To obtain the spectrum of steady-state turbulence (I_k), we must find a way to solve Eq. (7) and (8) with the proviso that $k = (\mathbf{k}, \omega)$ must be real. Thus the scale-separation argument has led us to a relatively simple set of closed equations which describe steady-state turbulence. Notice that ϵ_k^ℓ and, $V_{k,k'}$ contain all the information about the medium and the wave motion around which the turbulence is built. We believe that the above formalism should be applicable to a broad range of plasma and fluid phenomena.

Equations (7) and (8), though comparatively simple are still too difficult to solve analytically for the spectrum I_k. We wait till Sec. III to numerically solve the problem of Alfvén wave turbulence. Now we take a digression to show that the problem of plasma heating by externally excited waves can also be tackled within the framework developed in this section.

B. Plasma Heating as a Special Case

In the initial stages, the wave heating[13] (externally driven) of plasmas can be viewed as a linear problem. But as the amplitude of the excited wave increases, nonlinear effects will be seen, and the generation and interaction of several modes could easily lead to a turbulent situation. The situation is, of course, quite similar to the turbulent system we have dealt with in this section, but with the difference, that now the source term must contain the effects of the external source in addition to the incoherent source. The basic equation describing the spectrum $I_k = |\varphi_k|^2$ in this case is

$$|\epsilon_k|^2 I_k = |\tilde{\varphi}_k^{\mathrm{ext}} + \tilde{\varphi}_k|^2 \tag{9}$$

where $\tilde{\varphi}_k^{\mathrm{ext}}$ represents the incoherent effects of the external driving source. The problem will reduce to the linear driven case if $\tilde{\varphi}_k = 0$, and $\epsilon_k \equiv \epsilon_k^\ell$. Clearly for large amplitude waves (high power heating), the characteristics of the waves in the plasma are bound to be quite different from its linear characteristics. Therefore to understand the mechanism as well as efficiency of high power heating, we need to study the fully nonlinear system schematically represented by Eq. (9). As outlined earlier, our attempt here is again to determine the structure of I_k, the range or band in which it is significantly different from zero.

We take this opportunity to state that our theory is incapable of providing much useful information in the transient stages when the system develops from a quasilinear state to a turbulent state. We concentrate only on the stationary or near-stationary turbulent state.

A very integral part of our approximation is the scale separation between the renormalized nonlinear dielectric ϵ_k and the source. We have given several arguments to show that the intrinsic incoherent source $\tilde{\varphi}_k$ is expected to satisfy these conditions. What about $\tilde{\varphi}_k^{\text{ext}}$? We believe that similar arguments are valid even for $\tilde{\varphi}_{\text{ext}}$, because the coherent aspects of the source (which may have strong peaks in the linear limit) will become a part of ϵ_k, and $\tilde{\varphi}_k^{\text{ext}}$ due to multiple incoherent interactions will be quite broad in $k(\mathbf{k}$ and $\omega)$.

We end this section by pointing out a very interesting and useful consequence of our assumption. For simplicity, we deal with a one-dimensional problem, i.e., $k = (\mathbf{k}, \omega) = [(0, 0, \kappa), \omega]$. Schematically, the essence of Eq. (7) or Eq. (9) is written as

$$I_{\kappa,\omega} = \frac{S_0}{|\epsilon_{\kappa,\omega}|^2} \tag{10}$$

where S_0 is independent of κ and ω, and is nonzero. We could now proceed formally and deduce that for a given κ, $I_{\kappa,\omega}$ will peak at that value of $\omega = \omega_0$ for which $|\epsilon_{\kappa,\omega}|^2$ is a minimum; i.e., the solution of

$$\left[\frac{d}{d\omega}|\epsilon_{\kappa,\omega}|^2\right]_{\omega=\omega_0} = 0 \tag{11}$$

for fixed k, determines the maximum value of the fluctuation spectrum. It must, of course, be borne in mind that the analysis is meaningful only for real κ and ω, and we must admit only the real roots of Eq. (11). Note that $|\epsilon_{\kappa,\omega}| = 0$ is also a root of Eq. (11), and is not admitted because of $I_{\kappa,\omega}$ going to infinity (S_0 is finite). Using Eq. (11), we can write Eq. (10) as

$$I_{\kappa,\omega} = \frac{\overline{S}_0}{\Delta^2 + (\omega - \omega_0)^2} \tag{12}$$

where

$$\Delta^2 = \left[\frac{1}{2}\frac{d^2}{d\omega^2}|\epsilon_{\kappa,\omega}|^2\right]^{-1}_{\omega=\omega_0} |\epsilon_{\kappa,\omega_0}|^2. \tag{13}$$

$$\overline{S}_0 = S_0 \cdot \left[\frac{1}{2} \frac{d^2}{d\omega^2} |\mathcal{E}_{k,\omega}|^2 \right]^{-1}_{\omega=\omega_0}$$

Thus, to leading order, we obtain a Lorentzian spectrum peaked at $\omega = \omega_0$, and with a width Δ. As expected, the general shape of the spectrum is independent of S_0, although the peak value is $(\overline{S}_0/\Delta^2)$. We must, however, point out that Eq. (12) is purely formal, because both ω_0 and Δ are functions of $\epsilon_{\kappa,\omega}$, which is itself a function of $I_{\kappa,\omega}$, the principal object of interest.

III. Alfvén Turbulence and Numerical Methods

In this section, we demonstrate that the theoretical methods developed in the last section can be applied to realistic turbulence problems. We choose a simplified model of Alfvén waves turbulence as a test case. We deal with a cylindrical plasma of radius r and axial length $2\pi R$ (to simulate a torus of radius R) within the framework of ideal magnetohydrodynamics (MHD). The basic renormalized equations are derived in Appendix A, and are ($\mathbf{k}$and ω dependence is explicitly displayed in this section)

$$|\epsilon_{\mathbf{k},\omega}|^2 |U_{\mathbf{k},\omega}|^2 \equiv |F_{\mathbf{k},\omega} + C_{\mathbf{k},\omega}|^2 |U_{\mathbf{k},\omega}|^2 = |\tilde{U}_{\mathbf{k},\omega}|^2, \tag{14}$$

$$C_{\mathbf{k},\omega} = \sum_{\mathbf{k}+\mathbf{k}'\neq 0, \omega-\omega'\neq 0} \frac{\omega(\omega-\omega')(k_\theta^2 - k_\theta'^2 - k_\theta k_\theta')}{F_{\mathbf{k}-\mathbf{k}',\omega-\omega'} + C_{\mathbf{k}-\mathbf{k}',\omega-\omega'}} \frac{|U_{\mathbf{k}',\omega'}|^2}{V_A^4} \tag{15a}$$

$$\equiv \sum_{\mathbf{k}'\neq 0, \omega'\neq 0} \frac{\omega\omega'\Big[-k_\theta^2 - k_\theta'^2 + 3k_\theta k_\theta' \Big]}{F_{\mathbf{k}',\omega'} + C_{\mathbf{k}',\omega'}} \frac{|U_{\mathbf{k}-\mathbf{k}',\omega-\omega'}|^2}{V_A^4} \tag{15b}$$

where Eq. (15b) is obtained from (15a) by simply letting $\mathbf{k}' \rightarrow \mathbf{k} - \mathbf{k}'$ and $\omega' \rightarrow \omega - \omega'$. In Eqs. (14) and (15), $U = U_\theta$ is the azimuthal component of the velocity field,

$$F_{\mathbf{k},\omega} = \frac{\omega^2}{v_A^2} - k_z^2, \tag{16}$$

and

$$k_z = n/R, \quad k_\theta = m/r \tag{17}$$

where m and n are respectively the azimuthal and axial wave numbers, $V_A = B_0/(4\pi\rho_0)^{1/2}$ is the Alfvén speed, ρ_0 is the plasma density and B_0 is the strong uniform axial magnetic field, $C_{\mathbf{k},\omega}$ is the coherent nonlinear response, and $\tilde{U}_{\mathbf{k},\omega}$ is the incoherent nonlinear source term.

Notice that Eqs. (14) and (15) are exactly the same form as Eqs. (7) and (8) implying that the closure is obtained simply by putting the incoherent term

$$|\tilde{U}_{\mathbf{k},\omega}|^2 = S_0 \tag{18}$$

where S_0 is independent of $\mathbf{k}$ and ω. The set of Eqs. (14) to (18) is numerically solved in Sec.. IIIB. In Sec. IIIA, we show how we could extract some useful information out of these equations for a special problem by using the method of Sec. IIB.

IIIA. Frequency Shift in Alfvén Wave Heating

In wave heating of plasmas, electromagnetic waves of a given description are generated in an antenna located near the edge of the plasma. These modes are converted to the appropriate natural oscillation modes of the plasma, and then can impart energy to the plasma (by resistivity or by Landau damping, etc.). Since the dominant modes in the plasma must, in general, correspond to the modes driven by the antenna, one has a prior knowledge of the wave numbers from a knowledge of the antenna structure. Now for a given set of wave numbers, the heating efficiency depends upon the frequency of the wave. In Alfvén wave heating, for example, $F = 0$, or $\omega^2 = k_z^2 V_A^2$ (the linear dispersion relation of the wave) is the condition for maximum efficiency in the linear regime. For high power or heating in the nonlinear regime, the optimum heating frequency must depend upon the wave energy I_k of the excited wave. This was essentially the content of Sec. IIB, where we derived a formula for ω_0, the frequency at which I_k peaks.

Let us consider an idealized Alfvén wave experiment where the antenna generates waves of $m = \pm 1$ and $n = \pm 1$ only. The best driving frequency for this case in the linear regime is obviously $\omega_L = \pm V_A/R$. However, we are interested in finding the optimum frequency ω_0 in the nonlinear regime. Let us assume that the steady state nonlinear stage is already set up, then ω_0 will obviously be the solution of Eq. (11). Because of the peculiar choice of the antenna, we could assume that the principal part of the excited spectrum is at $\omega = \pm\omega_0$. Thus the excited spectrum is

$$|U_{\mathbf{k},\omega}|^2 = |U_{m,n,\omega}|^2 = \begin{cases} U_0^2, & \text{if } m = \pm 1,\ n = \pm 1,\ \omega = \pm\omega_0 \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

Of course, the excited spectrum will have a width, which can be calculated from Eq. (13). However, as long as the width is small compared to $|\omega_0|$, the spectrum given in Eq. (14) will give approximately correct results. Making use of Eqs. (15b) and (19), it is clear that [we propose to calculate ω_0 for $m = 1$, $n = 1$]

$$C_{1,1,\omega} = \left(\frac{U_0}{V_A}\right)^2 \frac{1}{r^2 V_A^2} \left[\frac{\omega(\omega - \omega_0)}{F_{2,2,\omega-\omega_0} + C_{2,2,\omega-\omega_0}} + \frac{\omega(\omega + \omega_0)}{F_{2,2,\omega+\omega_0} + C_{2,2\omega+\omega_0}}\right] \tag{20}$$

because only $n' = n \pm 1$, $m' = \pm 1$, $m' = m \pm 1$, $\omega' = \omega \mp \omega_0$ can contribute to the sum in Eq. (15b). Notice that we are using the notation $C_{\mathbf{k},\omega} = C_{m,n,\omega}$. Equation (20) indicates that in order to calculate $C_{1,1,\omega}$, we need to calculate $C_{2,2,\omega-\omega_0}$, and of course to calculate $C_{2,2,\omega-\omega_0}$ we would need high $C_{m,n,\omega'}$'s. But to make analytical progress, we must make further approximations. In most cases of interest $U_0/V_A = \delta$ is a small quantity. We exploit this property $\delta << 1$ to propose the ordering $C_{m,n,\omega'} = \lambda_{mn}\delta/rR$, where $\lambda_{m,n}$ is a number we cannot determine by analytical theory. Notice that this ordering solves Eq. (20) provided the denominator $F_{2,2,\omega\mp\omega_0} + C_{2,2,\omega\mp\omega_0} \sim \delta/rR$. This will indeed turn out to be true. Within the context of the preceding discussion, we have [$\epsilon_{1,1,\omega} = F_{1,1,\omega} + C_{1,1,\omega}, \lambda_2 = \lambda_{2,2}$]

$$\epsilon_{1,1,\omega} = \frac{\omega^2}{V_A^2} - \frac{1}{R^2} + \frac{\delta^2}{r^2}\left[\frac{\omega(\omega - \omega_0)}{(\omega - \omega_0)^2 - 4(V_A^2/R^2) + \lambda_2\delta V_A^2/rR} + \frac{\omega(\omega + \omega_0)}{(\omega + \omega_0)^2 - 4(V_A^2/R^2) + \lambda_2\delta V_A^2/rR}\right]. \tag{21}$$

Since $\epsilon_{1,1,\omega}$ is real for this simple case; Eq. (11) is equivalent $d\epsilon_{1,1,\omega}/d\omega = 0$. Further, we have stipulated that the peaks are at $\omega = \pm\omega_0$.

Therefore, either of the equations $(d\epsilon_{1,1,\omega}/d\omega)_{\omega=\pm\omega_0}$ can be used to obtain $\omega_0(1,1) \equiv \omega_0$. Differentiating Eq. (21) and equating its value at ω_0 equal to zero, we obtain

$$\frac{2\omega_0}{V_A^2} + \frac{\delta^2}{r^2}\left[\frac{\omega_0}{-4\frac{V_A^2}{R^2} + \frac{\lambda_2\delta V_A^2}{rR}} + \frac{3\omega_0}{D} - \frac{2\omega_0^2\left[2\omega_0 + (\delta V_A^2/rR)\frac{\partial\lambda_2}{\partial\omega_0}\right]}{D^2}\right] = 0 \tag{22}$$

where

$$D = 4\left(\omega_0^2 - \frac{V_A^2}{R^2}\right) + \lambda_2 \frac{\delta V_A^2}{R}. \tag{23}$$

Since the assumed ordering requires $D \sim \delta$, the dominant balance in Eq. (22) comes from

$$\frac{2\omega_0}{V_A^2} - \frac{\delta^2}{r^2}\frac{4\omega_0^3}{D^2} = 0 \tag{24}$$

where both terms are of order unity. Equation (24) can be rewritten in the form

$$D \equiv 4\left(\omega_0^2 - \frac{V_A^2}{R^2}\right) + \lambda_2 \frac{\delta V_A^2}{rR} = \sqrt{2} \quad \delta\omega_0 \frac{V_A}{R} \tag{25}$$

which is approximately solved to obtain

$$\omega_0^2 = \frac{V_A^2}{R^2} + \frac{\delta V_A^2}{4r^2}\left[\sqrt{2} - \frac{\lambda_2 r}{R}\right]. \tag{26}$$

showing that the correction to the linear dispersion relation $(m = 1,\, n = 1)$

$$\omega_L^2 = V_A^2/R^2 \tag{27}$$

is of order $\delta = U_0/V_A$ rather than δ^2, which would be the quasilinear or the weak turbulence result. We must state that there is no way to determine the value of λ_2 by these simple analytical methods. However, we have already obtained a very interesting strong turbulence result that the frequency shift scales linearly with the perturbation. The spectrum width Δ can be readily obtained using Eq. (13) to be

$$\Delta = \lambda \frac{\delta V_A}{2\sqrt{2}r} \tag{28}$$

where λ is a number of order unity. We notice that the frequency shift $\omega_0 - \omega_L$ and the width of the spectrum, Δ, are both of order δ.

IIIB. Numerical Solution and Kolmogorov Spectrum

We begin this section with an investigation of the symmetry properties of $C_{\mathbf{k},\omega}$, the coherent modification of the response function. Reality of the wave field [the flow velocity $\mathbf{U}(\mathbf{x}, t)$ or the fluctuating magnetic field $\mathbf{b}(\mathbf{x}, t)$] imposes the universal symmetry

$$C_{\mathbf{k},\omega} = C^*_{-\mathbf{k},-\omega} \tag{29}$$

where $*$ stands for complex conjugation. Additional symmetries will, of course, depend upon the special properties of the system under consideration. For our problem, we could deduce further symmetries from Eqs. 15-17. Before doing this we must mention that the linear response function $F_{\mathbf{k},\omega}$ [Eq. (17)], in general, has an imaginary part (due to resistivity or Landau damping, for instance) proportional to ω.[14] This imaginary part is quite important to insure convergence and is retained in our numerical work, although we do not display it explicitly in Eq. (16). The existence of this imaginary part proportional to ω implies $F_{\mathbf{k},\omega} = F^*_{\mathbf{k},-\omega}$, which coupled with Eq. (15) leads to

$$C_{\mathbf{k},\omega} = C^*_{\mathbf{k},-\omega}. \tag{30}$$

Equations (29) and (30) trivially yield

$$C_{\mathbf{k},\omega} = C_{-\mathbf{k},\omega}. \tag{31}$$

The numerical procedure is initiated by choosing S_0 (the strength of the incoherent source) and an initial function $C^{(0)}_{\mathbf{k},\omega}$ which obeys the symmetry properties given in Eqs. (29)-(31). These values are used in Eq. (14) to compute $|U^{(1)}_{\mathbf{k},\omega}|^2 = I^{(1)}_{\mathbf{k},\omega}$. This form of $I^{(1)}_{\mathbf{k},\omega}$ along with $C^{(0)}_{\mathbf{k},\omega}$ are substituted in Eq. (15) to compute a new expression for $C_{\mathbf{k},\omega}$, which is used to begin the new iteration cycle to yield $I^{(2)}_{\mathbf{k},\omega}$ and $C^{(2)}_{\mathbf{k},\omega}$. The procedure is repeated till we obtain convergent results. We must point out that the calculation of $C^{(n)}_{\mathbf{k},\omega}$ using Eq. (15) is tantamount to doing a complicated multiple integral, and one can indeed run into convergence problems. We have been able to avoid convergence problems by 1) introducing a small imaginary part into the linear response function which effectively damps very high k_z modes, and 2) making the iteration procedure relatively slow, that is, the quantities $C_{\mathbf{k},\omega}$ are varied slowly from one iteration cycle to the other $[C^2 = (jC^0 + C^1)/j + 1]$, and 3) adding an extra positive number to the denominator and dismissing it cycle by cycle till it is zero.

These computing techniques seem a bit artificial, but they very effectively take care of oscillation problems which may be encountered in a normal procedure.

In our numerical calculations, we have further simplified the problem by concentrating on waves with $m = \pm 1$, thus effectively reducing the problem to a two-dimensional problem in $k_z(n/R)$ and ω.

In Fig. 1, we display the spectrum I_{k_z},ω as a function of k_z and ω on a two-dimensional plot. The spectrum is broad in both k_z and ω clearly indicating the fundamental difference between a turbulent and a linear spectrum.

After integrating the spectrum I_{k_z},ω over all ω, we obtain the spectrum I_{k_z}, and is displayed in Fig. 2 as a function of k_z. To make the k_z dependence more perspicuous, we have plotted in Fig. 3 a graph of $(-)\ell n I_{k_z}$ versus $\ell n k_z$ for the intermediate range (inertial range) of k_z. We notice that the curve is essentially a straight line implying that the Alfvén wave spectrum obeys a power law

$$I_{k_z} = C k_z^{-\alpha} \tag{32}$$

with the exponent $\alpha = 2$. This is similar to the Kolmogorov spectrum in hydrodynamical turbulence but with a different exponent; the exponent in hydrodynamical turbulence equals 1.67.

The form of the spectrum obtained is quite stable to the change of initial conditions and the magnitude of the damping term: We have varied these over a wide range, and our results remain the same.

Finally, we have gone back and estimated the $\omega - k_z$ dependence of the incoherent terms, and found that its structure is indeed much broader than the calculated I_{k_z},ω. Thus, the basis of the calculation, i.e., assuming that the structure of $\epsilon_{k,\omega}$ essentially determines the structure of I_{k_z},ω is verified a fortiori.

IV. Summary and Conclusions

We have proposed a physically motivated, non-perturbative closure scheme to deal with plasma or fluid turbulence. The scheme is based on a scale separation assumption; the structure (in $\mathbf{k},\omega$) of the turbulent spectrum is essentially determined by the structure of the nonlinear coherent response function $\epsilon_{\mathbf{k},\omega}$, and not by the structure of the incoherent source which is assumed to vary on a much larger scale in $\mathbf{k}$ and ω. This allows us to obtain a comparatively simple closed set of equations which can be, in principle, always solved.

After discussing a few simple analytically tractable applications, we have applied our formalism to a model problem: the shear-Alfvén turbulence. The problem is readily solved numerically to obtain the turbulent spectrum with the following features

(1) The calculated spectrum has explicit frequency shift from the linear theory as well as explicit band-broadening in $\mathbf{k}, \omega$ space,

(2) The frequency integrated spectrum obeys a power law in the intermediate k range, i.e., $I_k \sim k^{-\alpha}$, where α approaches the value 2 in the case we studied.

It is very encouraging that this closure scheme is capable of producing essential features associated with a turbulent spectrum. We hope to be able to compare our results with experimental studies of shear-Alfvén turbulence, when the experimental results become available. In the meantime, we are using our basic formalism to numerically solve 3-dimensional Navier Stokes turbulence. We believe that our methods should prove quite useful in studying a broad class of turbulent phenomena.

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Appendix A

Our starting point, the ideal cold magnetohydrodynamic (MHD) equations are

$$\frac{\partial \overline{\rho}}{\partial t} + \nabla \cdot (\overline{\rho}\mathbf{U}) = 0, \qquad (A-1)$$

$$\overline{\rho}\left[\frac{\partial}{\partial t} + \mathbf{U}\cdot\nabla\right]\mathbf{U} = \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B}, \qquad (A-2)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}) \qquad (A-3)$$

where $\overline{\rho}$ is the plasma density, $\mathbf{U}$ is the velocity and $\mathbf{B}$ is the magnetic field. We now separate the average (equilibrium) and the fluctuating quantities, i.e., $\overline{\rho} = \rho_0 + \rho$, $\mathbf{B} = \sqrt{4\pi}[B_0\hat{e}_z + \mathbf{b}]$, $\mathbf{U}$=$\mathbf{U}$, where $\sqrt{4\pi}B_0$ is the strong ambient axial magnetic field, ρ_0 is the average density, and ρ, $(4\pi)^{1/2}\mathbf{b}$ and $\mathbf{U}$ are respectively the fluctuating density, magnetic field, and velocity. Evolution of ρ, $\mathbf{b}$ and $\mathbf{U}$ is determined by

$$\frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot \mathbf{U} = -(\mathbf{U}\cdot\nabla)\rho - \rho\nabla\cdot\mathbf{U}, \qquad (A-4)$$

$$\rho_0\frac{\partial \mathbf{U}}{\partial t} - B_0\left(\frac{\partial \mathbf{b}}{\partial z} - \nabla b_z\right) = -\rho\frac{\partial \mathbf{U}}{\partial t} - \rho_0(\mathbf{U}\cdot\nabla)\mathbf{U} + (\mathbf{b}\cdot\nabla)\mathbf{b} - \frac{\nabla b^2}{2}, \qquad (A-5)$$

and

$$-\frac{\partial \mathbf{b}}{\partial t} + B_0\left[\frac{\partial \mathbf{U}}{\partial z} - \hat{e}_z\nabla\cdot\mathbf{U}\right] = -(\mathbf{b}\cdot\nabla)\mathbf{b} + \mathbf{b}(\nabla\cdot\mathbf{U}) + (\mathbf{U}\cdot\nabla)\mathbf{b} \qquad (A-6)$$

where we have put the nonlinear terms on the right-hand side of Eqs. (A-4) to (A-6). We now consider a cylindrical plasma of axial length $2\pi R$ (to possibly simulate a torus). Azimuthal and axial symmetry allow us to expand the fluctuating quantities as

$$f(r,\theta,z,t) = \sum_{m,n,\omega} f_{m,n,\omega}(r)e^{i\omega t + ik_\theta r\theta + ik_z z} \qquad (A-7)$$

where

$$k_z = \frac{n}{R} \quad , \quad k_\theta = \frac{m}{r} \qquad (A-8)$$

are the axial and azimuthal components of the wave-vector, and n and m are respectively the axial and azimuthal wave numbers. Making use of Eq. A-7, Eqs. (A-4)-(A-6) can be cast in the form

$$F_k\left[mu_{rk} + i\frac{\partial}{\partial r}ru_{\theta k}\right] = -\frac{\omega}{B_0^2}\frac{\partial}{\partial r}(rN_{\theta k}) - \frac{k_z}{B_0}\frac{\partial}{\partial r}(rM_{\theta k})$$
$$+\frac{im}{B_0^2}N_{rk} + \frac{ik_z m}{B_0}M_{rk} \qquad (A-9)$$

where

$$k = (\mathbf{k}, \omega) = (m, n, \omega) \equiv (k_\theta, k_z, \omega), \qquad (A-10)$$

$$F_k = \frac{\omega^2}{V_A^2} - k_z^2 \qquad (A-11)$$

is the linear dispersion function, and $N_{\theta k}(N_{rk})$, and $M_{\theta k}(M_{rk})$ are respectively the Fourier transforms of the nonlinear terms of $\theta(r)$ components of Eqs. (A-5) and (A-6). Although Eq. (A-9) can be subjected to the usual renormalization treatment, (and will be done at a later stage) we make a set of simplifying assumptions so that we can deal with a more perspicuous system and hence show the workings of our proposed closure scheme. Some of these assumptions are a bit artificial, but we make them for simplicity and clarity. Since we are dealing with shear Alfvén turbulence, we ignore the effects of compressibility. We also assume that $|U_\theta| > |U_r|$, so that the principal nonlinearity comes from $N_{\theta k}$. We neglect all other nonlinear terms. For this model problem, Eq. (A-9) reduces to $[k' = (\mathbf{k}', \omega)]$

$$F_k U_{\theta k} = \frac{i\omega}{B_0^2} \rho_0 \sum_{k'} k'_\theta U_{\theta(k-k')} U_{\theta(-k')} \qquad (A-12)$$

can be renormalized in a straightforward fashion to yield

$$\epsilon_k U_{\theta k} = (F_k + C_k) U_{\theta k} = \tilde{U}_{\theta,k}, \qquad (A-13)$$

and

$$C_k = \sum_{k'} \frac{\omega(\omega - \omega')(k_\theta^2 - k_\theta'^2 - k_\theta k_\theta')}{V_A^4 [F_{k-k'} + C_{k-k'}]} |U_{\theta k'}|^2 \qquad (A-14)$$

where $\tilde{U}_{\theta,k}$ is the nonlinear incoherent source, C_k is the coherent renormalizing modification of the linear response function F_k, and $(U_{\theta k})^2 = I_{\theta k}$ is the spectrum of turbulence.

Figure Captions

1. A plot of $I_{k_z,\omega}$ versus k_z and ω; the spectrum is broad in both k_z and ω.
2. A plot of frequency integrated spectrum $I_{k_z} = \int d\omega I_{k_z,\omega}$ versus k_z.
3. A plot of $-\ell n I_{k_z}$ versus $\ell n k_z$ in the intermediate range (initial range) of k_z; the straight line has a slope approximately equal to two.

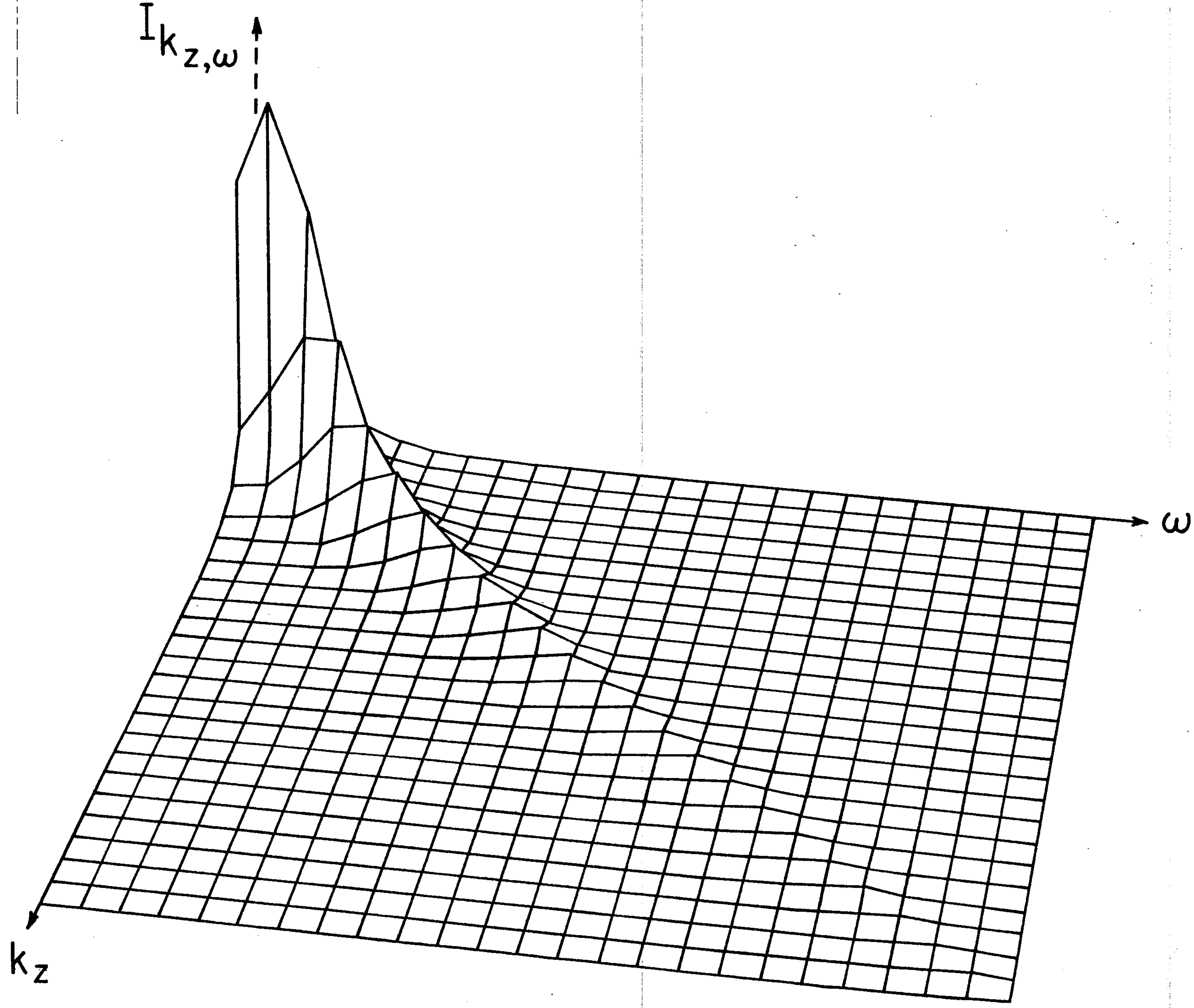

Fig. 1

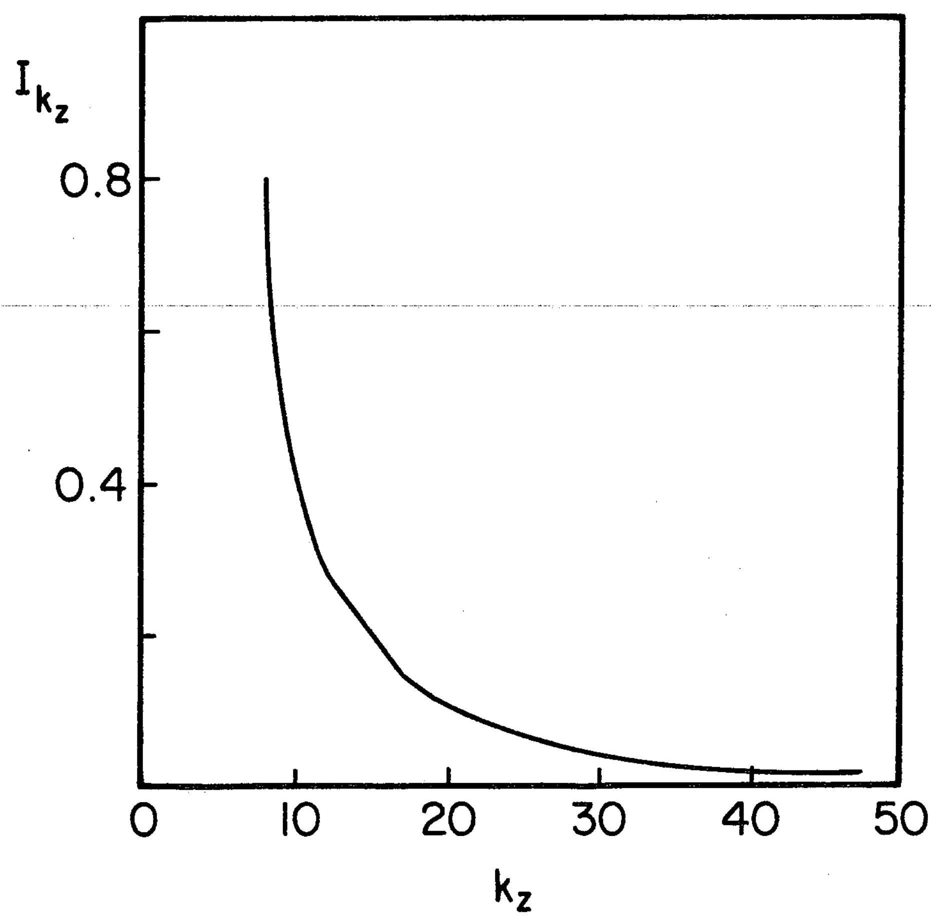

Fig. 2

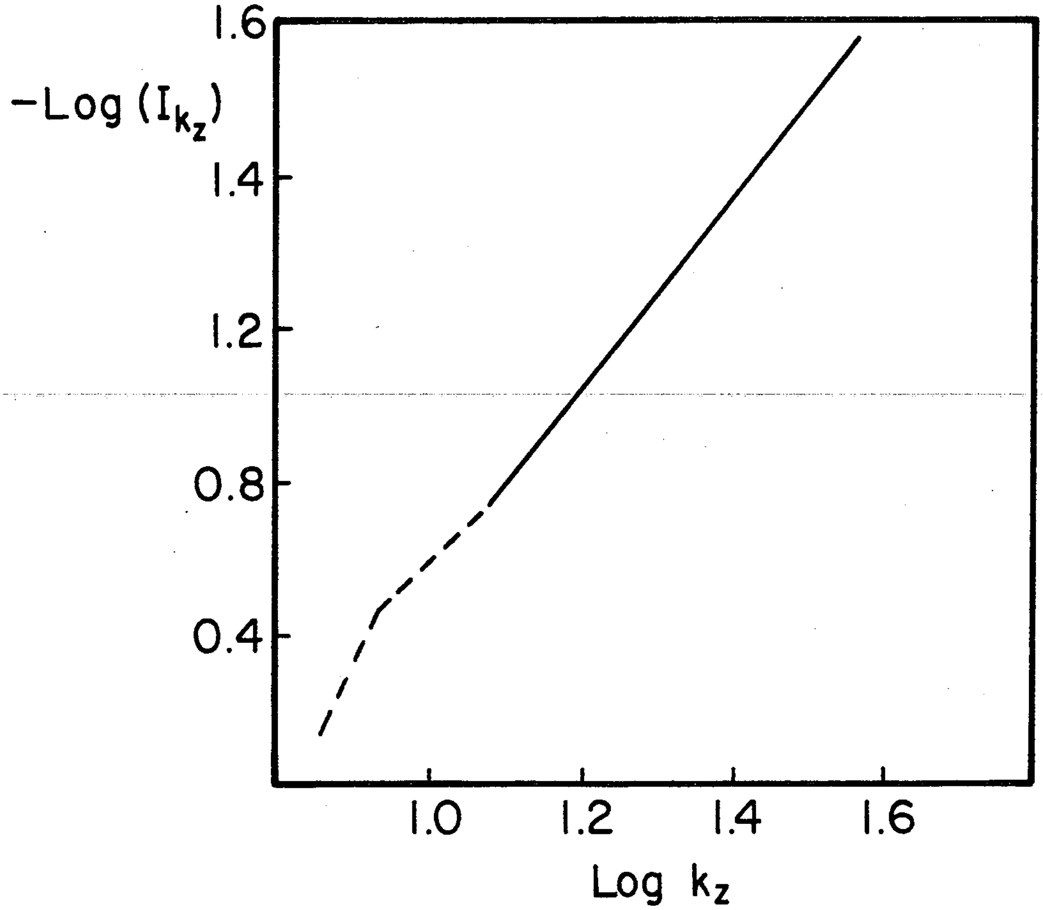

Fig. 3