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Local Conservation Laws for the Maxwell-Vlasov and Collisionless Kinetic Guiding Center Theories

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Abstract

Using one of the author's recent variational formulation of the Maxwell-Vlasov and guiding center theories [Ref. 1], the energy-momentum and angular momentum tensors for such theories are derived and the corresponding local conservation laws are proven. The energy-momentum tensor is shown to be symmetric in its spatial components while the angular momentum density is naturally antisymmetric.

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I. Introduction

In this work we use a recent variational formulation for the Maxwell-Vlasov equations and related theories (Ref. 1, henceforth called I) to obtain local conservation laws. This formulation differs from previous such theories. In particular it differs from the well-known Low Lagrangian formulation[2] in that it is not only an Eulerian description for the electromagnetic fields, but also treats the particles or guiding centers in an Eulerian manner. This facilitates application. The formulation of I differs from another Eulerian variational principle given in Refs. 3, which is based upon a noncanonical Hamiltonian description. In Refs. 3, the particles are treated in an Eulerian manner by decomposing the phase space density into "Clebsch-like" potentials, while in I the Eulerian description of the particles is based upon a Hamilton-Jacobi theory, which plays the role of a tool; all final expressions are expressible in normal terms.

The problem of obtaining conservation laws for guiding center theories can be characterized as follows: given the solutions of the collisionless kinetic guiding center equations; i.e. the guiding center phase space density $f_{g\nu}(\underline{x}, v_{\|}, \mu, t)$ for particles of species ν, one can immediately write down expressions for the guiding center charge and current densities

$$\rho_g = \sum_\nu \int f_{g\nu} dv_{\|} B d\mu$$

$$\underline{j}_g = \sum_\nu e_\nu \int \left(\underline{v}_{D_\nu} + \underline{v}_{\|}\right) f_{g\nu} dv_{\|} B d\mu, \tag{1}$$

where the $\underline{v}_{D_\nu}$ are the drift velocities for particles of species ν. The real charge and current densities differ, however, from these expressions by certain polarization and magnetization contributions

$$\rho = \rho_g + \nabla \cdot \underline{P}, \quad \underline{j} = \underline{j}_g - \frac{\partial \underline{P}}{\partial t} + \nabla \times \underline{M}. \tag{2}$$

The problem is then to find the correct expressions for the electric polarization $\underline{P}$ and the magnetization $\underline{M}$ such that all the necessary conservation laws hold. These quantities were obtained in I where the new method generalized previous results by also including the polarization drift. This difficult problem resisted solution until now. Also in I, expressions for the energy and energy flux densities were determined . Here the full energy-momentum tensor and the angular momentum tensor are obtained via Noether's theorem. The appropriate symmetries of these tensors are shown and the local conservation laws are proven. These results are of importance for applications; e.g. knowledge of the total angular momentum combined with the energy can lead to an "energy" principle for linear stability analysis.

The tensors are obtained in the usual way by first determining the general expression for an arbitrary variation of the total Lagrangian density in normal position space $\underline{x}$. We note, however, that there is a slight complication because the "particle" part of the Lagrangian is primarily defined on an extended space $\underline{y} = (\underline{y}_1, \underline{y}_2)$ where $\underline{y}_1$ is identical to $\underline{x}$ and $\underline{y}_2$ is an additional coordinate that is needed in order to describe guiding centers. By means of translational invariance in $\underline{x}$-space and time, and rotational invariance in $\underline{x}$-space the canonical tensors are obtained. These tensors are not gauge invariant, but each can be split into a divergence free gauge invariant part and a divergence free non-gauge invariant part. The gauge invariant energy-momentum tensor turns out to be symmetric in its spatial components. This is shown to follow from its relationship to the gauge invariant part of the angular momentum tensor. All of these expressions are also applicable to relativistic theories.

Two applications are considered: first we treat the Maxwell-Vlasov equations and show that the gauge invariant parts of our tensors reduce to the usual well-known expressions. Following this we treat the Maxwell-kinetic guiding center theory based on Littlejohn's guiding center equations of motion.[4]

II. Variation of the Lagrangian Density

In this section we review the variational principle of I, establish some convenient notation and obtain an expression for $\delta\mathcal{L}$ that is used in Sec. III.

Consider variation of the following action:

$$A = \int \mathcal{L} dt d^3x, \tag{3}$$

where

$$\mathcal{L} = \mathcal{L}_M + \sum_\beta \int d^{n_2}y_2 \mathcal{L}_\beta \tag{4}$$

$$\mathcal{L}_M = \frac{1}{8\pi}(E^2 - B^2) \tag{5}$$

$$\underline{E} = -\frac{\partial\Phi}{\partial x} - \frac{1}{c}\frac{\partial \underline{A}}{\partial t}, \quad \underline{B} = \nabla \times \underline{A} \tag{6}$$

and

$$\mathcal{L}_\beta = -\phi_\nu\left(\frac{\partial S_\nu}{\partial t} + e_\nu\Phi + \hat{H}_\nu\left(\frac{\partial S_\nu}{\partial \underline{x}} - \frac{e_\nu}{c}\underline{A}, \frac{\partial S}{\partial \underline{y}_2}, \underline{E}, \underline{B}\right)\right). \tag{7}$$

The Hamiltonian for the "particles" of species ν is given by

$$H_\nu = e_\nu \Phi + \hat{H}_\nu(\underline{p} - \frac{e_\nu}{c}\underline{A}, \underline{P}_2, \underline{E}, \underline{B}), \tag{8}$$

where in general $\hat{H}_\nu$ could also depend upon derivatives of $\underline{E}$ and $\underline{B}$, but this is unnecessary for our present applications. Here $\underline{p} \equiv \underline{P}_1$ is canonically conjugate to $\underline{x} \equiv \underline{y}_1$ and $\underline{P}_2$ is canonically conjugate to the variable $\underline{y}_2$ that is needed for guiding center theory. In the formulation presented in I and also here $\hat{H}_\nu$ does not depend upon $\underline{y}_2$ but does depend upon $\underline{P}_2$. However, if desired the meaning of $\underline{y}_2$ and $\underline{P}_2$ can be interchanged. The quantity S_ν is the Hamilton-Jacobi function and ϕ_ν is a density function. Both are functions of $\underline{y} = (\underline{y}_1, \underline{y}_2), t$ and a set of constants of integration $\underline{\alpha} = (\underline{\alpha}_1, \underline{\alpha}_2)$ that are needed for complete solutions S_ν of the Hamiltonian-Jacobi equations $\partial S_\nu/\partial t + H_\nu = 0$. The label β is used as a shorthand for ν and $\underline{\alpha}$. Also $\sum_\beta$ is used to mean $\sum_\nu \int d^{n_1+n_2}\alpha$, where n_1 is the dimension of the vector space $\underline{y}_1 = \underline{x}$ (i.e. $n_1 = 3$) and n_2 is the dimension of the vector space $\underline{y}_2$.

The quantities to be varied in Eq. (3) are $\Phi, \underline{A}, \phi_\nu$, and S_ν. Their variations must vanish at certain times t_1, t_2 and on certain surfaces in $\underline{y}$-space so that surface terms can be neglected upon partial integration.

In order to simplify subsequent calculations we introduce the following notation:

$$\psi^1_\beta = \phi_\nu, \quad \psi^2_\beta = S_\nu, \quad (\Phi_\nu) = (-\Phi, \underline{A})$$

$$(z^\mu_a) = (ct, \underline{x}), \quad (z^\mu) = (ct, \underline{y}) \tag{9}$$

where $ct = z^0 = z^0_a = -z_{a0}$ and

$$\hat{\mathcal{L}}_\beta = \mathcal{L}_\beta + g_\beta(\underline{y}_2)\mathcal{L}_M \tag{10}$$

where $g_\beta(y_2)$ is an artifice that is used to align the "particle" and field regions of integration. It is arbitrary except for the requirement that

$$\sum_\beta \int d^{n_2} y_2 g_\beta(y_2) = 1. \tag{11}$$

For any quantity Q

$$\frac{\partial Q}{\partial z^\mu} = Q_{,\mu}. \tag{12}$$

If Q is independent of $\underline{y}_2$ then also

$$\frac{\partial Q}{\partial z_a^\mu} = Q_{,\mu}. \tag{13}$$

In addition, we use the summation convention. (Sometimes indices occur twice when summation is not intended, but this will be obvious from the context). Finally, the fields $\underline{E}$ and $\underline{B}$ are given by the electromagnetic field tensor

$$F_{\mu\lambda} = \Phi_{\mu,\lambda} - \Phi_{\lambda,\mu} \qquad \mu, \lambda = 0, 1, 2, 3. \tag{14}$$

Thus

$$\begin{aligned} E_i &= F_{0i} \qquad i = 1, 2, 3 \\ B_i &= -\frac{1}{2}\epsilon_{ikl}F^{kl} \qquad i = 1, 2, 3 \end{aligned} \tag{15}$$

where $\epsilon_{ikl} = \underline{e}_i \cdot (\underline{e}_k \times \underline{e}_l)$ for $i, k, l = 1, 2, 3$ and the $\underline{e}_i$ are constant orthonormal unit vectors in $\underline{x}$-space.

Given the above notation we can write the Lagrangian density as

$$\mathcal{L} = \sum_\beta \int d^{n_2}y_2 \hat{\mathcal{L}}_\beta, \tag{16}$$

using Eqs. (10) and (4). Its variation is given by

$$\begin{aligned} \delta\mathcal{L} = \sum_\beta \int d^{n_2}y_2 \Bigg[& \delta\psi_\beta^i \left(\frac{\partial \hat{\mathcal{L}}_\beta}{\partial \psi_\beta^i} - \frac{\partial}{\partial z^\mu} \frac{\partial \hat{\mathcal{L}}_\beta}{\partial \psi_{\beta,\mu}^i} \right) \\ & + \delta\Phi_\lambda \left(\frac{\partial \hat{\mathcal{L}}_\beta}{\partial \Phi_\lambda} - \frac{\partial}{\partial z_a^\mu} \frac{\partial \hat{\mathcal{L}}_\beta}{\partial \Phi_{\lambda,\mu}} \right) + \frac{\partial}{\partial z^\mu} \left(\delta\psi_\beta^i \frac{\partial \hat{\mathcal{L}}_\beta}{\partial \psi_{\beta,\mu}^i} \right) \\ & + \frac{\partial}{\partial z_a^\mu} \left(\delta\Phi_\lambda \frac{\partial \hat{\mathcal{L}}_\beta}{\partial \Phi_{\lambda,\mu}} \right) + \delta z^\mu \left(\frac{\partial \hat{\mathcal{L}}_\beta}{\partial z^\mu} \right)_{explicit} \Bigg]. \end{aligned} \tag{17}$$

Here the last term arises due to explicit dependence of $\hat{\mathcal{L}}_\beta$ upon z^μ. From Eq. (17) we see that variation of the action given by Eq. (3) yields the following Euler-Lagrange equations:

$$\begin{aligned} \frac{\partial \hat{\mathcal{L}}_\beta}{\partial \psi_\beta^i} - \frac{\partial}{\partial z^\mu} \frac{\partial \hat{\mathcal{L}}_\beta}{\partial \psi_{\beta,\mu}^i} &= 0 \\ \frac{\partial \hat{\mathcal{L}}_\beta}{\partial \Phi_\lambda} - \frac{\partial}{\partial z_a^\mu} \frac{\partial \hat{\mathcal{L}}_\beta}{\partial \Phi_{\lambda,\mu}} &= 0. \end{aligned} \tag{18}$$

Equations (18) are equivalent to the Maxwell-Vlasov or related kinetic equations, depending upon the choice for $\hat{H}_\nu$. These equations contain the expressions for the electric polarization $\underline{P}$ and the magnetization $\underline{M}$. This is shown explicitly in I. Now using Eqs. (18), we obtain from Eq. (17) the nontrivial relation

$$\delta\mathcal{L} = \sum_\beta \int d^{n_2}y_2 \left[\frac{\partial}{\partial z^\mu}\left(\delta\psi^i_\beta \frac{\partial\hat{\mathcal{L}}_\beta}{\partial\psi^i_{\beta,\mu}}\right) + \frac{\partial}{\partial z^\mu_a}\left(\delta\Phi_\lambda \frac{\partial\hat{\mathcal{L}}_\beta}{\partial\Phi_{\lambda,\mu}}\right)\right]. \tag{19}$$

In Eq. (19) we have neglected the explicit part since it is zero for the applications we present here. The first term of Eq. (19) is easily integrated over $\underline{y}_2$. Thus if we neglect $\underline{y}_2$-space boundary terms $\partial/\partial z^\mu$ can be replaced by $\partial/\partial z^\mu_a$. (In the guiding center case $\underline{y}_2$ has only one component, $v_\parallel$. As $|v_\parallel| \to \infty$ or $|y_2| \to \infty$ there is no surface contribution because any physical f_g must vanish in this limit.) Our final expression is

$$\delta\mathcal{L} = \frac{\partial}{\partial z^\mu_a} \sum_\beta \int d^{n_2}y_2 \left[\delta\psi^i_\beta \frac{\partial\hat{\mathcal{L}}_\beta}{\partial\psi^i_{\beta,\mu}} + \delta\Phi_\lambda \frac{\partial\hat{\mathcal{L}}_\beta}{\partial\Phi_{\lambda,\mu}}\right]. \tag{20}$$

III. The Energy-Momentum Tensor

We first construct the canonical energy-momentum tensor. This is obtained from Eq. (20) by considering a variation of the entire physical system through an infinitesimal distance ϵ^μ in the space $\underline{z}_a$. A scalar function $F(\underline{z}_a)$ is thus transformed according to

$$F(\underline{z}_a) = F'(\underline{z}_a + \underline{\epsilon}), \tag{21}$$

and the function F' differs infinitesimally from F by the following:

$$\delta F(\underline{z}_a) = F'(\underline{z}_a) - F(\underline{z}_a) = -\epsilon^\mu \frac{\partial}{\partial z^\mu_a} F(\underline{z}_a). \tag{22}$$

Applying this formula to the quantities $\mathcal{L}, \psi^i_\beta$ and Φ_λ for arbitrary ϵ^μ yields the local conservation equations

$$\frac{\partial\Theta^\mu_\rho}{\partial z^\mu_a} = 0, \tag{23}$$

where

$$\Theta^\mu_\rho = \sum_\beta \int d^{n_2}y_2 \left[\frac{\partial\psi^i_\beta}{\partial z^\rho_a}\frac{\partial\hat{\mathcal{L}}_\beta}{\partial\psi^i_{\beta,\mu}} + \Phi_{\lambda,\rho}\frac{\partial\hat{\mathcal{L}}_\beta}{\partial\Phi_{\lambda,\mu}}\right] - \mathcal{L}\delta^\mu_\rho \tag{24}$$

is the canonical tensor. This result was obtained by other means in I. We note that Θ^{μ}_{ρ} has no clear physical significance since it is not gauge invariant. However, we show how this tensor can be split into the sum of a gauge invariant part and a non-gauge invariant part, each of which is independently conserved. In I this was done only for the Θ^{μ}_{0} components.

Observe that $\hat{\mathcal{L}}_{\beta}$ does not depend upon $\psi^{1}_{\beta,\mu}$, thus we only have the $i=2$ component $\psi^{2}_{\beta,\rho} = S_{\nu,\beta}$. Furthermore,

$$\frac{\partial \hat{\mathcal{L}}_{\beta}}{\partial S_{\nu,\rho}} = -\phi_{\nu}(V^{\rho}_{\nu}) \tag{25}$$

where $(V^{\rho}_{\nu}) = (c, \underline{V}_{\nu})$. The quantity $\underline{V}_{\nu}$ is defined by $V_{\nu i} = \partial \hat{H}_{\nu}/\partial(\partial S_{\nu}/\partial y_{1i})$ and it corresponds to the velocity of particles of species ν in $\underline{x}$-space. Observe, $\partial \hat{\mathcal{L}}_{\beta}/\partial S_{\nu,\rho}$ is gauge invariant. Similarly,

$$\frac{\partial \hat{\mathcal{L}}}{\partial \Phi_{\lambda,\nu}} = \frac{1}{2}\frac{\partial \hat{\mathcal{L}}_{\beta}}{\partial F_{\sigma\rho}}\frac{\partial F_{\sigma\rho}}{\partial \Phi_{\lambda,\mu}} = \frac{\partial \hat{\mathcal{L}}_{\beta}}{\partial F_{\lambda\mu}} \tag{26}$$

is also gauge invariant. With this insight we let $\Theta^{\mu}_{\rho} = T^{\mu}_{\rho} + N^{\mu}_{\rho}$, where

$$T^{\mu}_{\rho} = \sum_{\beta}\int d^{n_2}y_2\left[\left(\frac{\partial \psi^{2}_{\beta}}{\partial z^{\rho}_{a}} - \frac{e_{\nu}}{c}\Phi_{\rho}\right)\frac{\partial \hat{\mathcal{L}}_{\beta}}{\partial \psi^{2}_{\beta,\mu}} + (\Phi_{\lambda,\rho} - \Phi_{\rho,\lambda})\frac{\partial \hat{\mathcal{L}}_{\beta}}{\partial \Phi_{\lambda,\mu}}\right] - \mathcal{L}\delta^{\mu}_{\rho} \tag{27}$$

and

$$N^{\mu}_{\rho} = \sum_{\beta}\int d^{n_2}y_2\left[\frac{e_{\nu}}{c}\Phi_{\rho}\frac{\partial \hat{\mathcal{L}}_{\beta}}{\partial \psi^{2}_{\beta,\mu}} + \Phi_{\rho,\lambda}\frac{\partial \hat{\mathcal{L}}_{\beta}}{\partial \Phi_{\lambda,\mu}}\right]. \tag{28}$$

Taking the divergence of Eq. (28) with respect to z^{μ}_{a} yields

$$\begin{aligned}\frac{\partial N^{\mu}_{\rho}}{\partial z^{\mu}_{a}} = &\sum_{\beta}\int d^{n_2}y_2\left[\frac{e_{\nu}}{c}\Phi_{\rho,\mu}\frac{\partial \hat{\mathcal{L}}_{\beta}}{\partial S_{\nu,\mu}} + \frac{e_{\nu}}{c}\Phi_{\rho}\frac{\partial}{\partial z^{\mu}_{a}}\frac{\partial \hat{\mathcal{L}}_{\beta}}{\partial S_{\nu,\mu}}\right.\\ &\left. + \Phi_{\rho,\lambda\mu}\frac{\partial \hat{\mathcal{L}}_{\beta}}{\partial \Phi_{\lambda,\mu}} + \Phi_{\rho,\lambda}\frac{\partial}{\partial z^{\mu}_{a}}\frac{\partial \hat{\mathcal{L}}_{\beta}}{\partial \Phi_{\lambda,\mu}}\right].\end{aligned} \tag{29}$$

Using Eqs. (18), (10) and (7) we have

$$\frac{\partial}{\partial z^{\mu}_{a}}\frac{\partial \hat{\mathcal{L}}_{\beta}}{\partial \Phi_{\lambda,\mu}} = \frac{\partial \hat{\mathcal{L}}_{\beta}}{\partial \Phi_{\lambda}} = -\frac{e_{\nu}}{c}\frac{\partial \hat{\mathcal{L}}_{\beta}}{\partial S_{\nu,\lambda}} \quad . \tag{30}$$

Thus the first and last terms of Eq. (29) cancel. Again from Eq. (18) for the case $i=2$, and the recognition that $\hat{\mathcal{L}}_{\beta}$ does not depend upon S_{ν} we find that the second term vanishes since

$$\sum_{\beta}\int d^{n_2}y_2\frac{e_{\nu}}{c}\Phi_{\rho}\frac{\partial}{\partial z^{\mu}_{a}}\frac{\partial \mathcal{L}}{\partial S_{\nu,\mu}} = -\sum_{\beta}\int d^{n_2}y_2\frac{e_{\nu}}{c}\Phi_{\rho}\frac{\partial}{\partial y^{j}_{2}}\frac{\partial \hat{\mathcal{L}}_{\beta}}{\partial(\partial S_{\nu}/\partial y^{j}_{2})} = 0. \tag{31}$$

The index j in Eq. (31) is summed up to n_2. The third term of Eq. (9) vanishes by virtue of the fact that $\Phi_{\rho,\lambda\mu}$ is symmetric in λ and μ while $\partial\hat{\mathcal{L}}_\beta/\partial\Phi_{\lambda,\mu}$ is antisymmetric (c.f. Eq. (26)). Thus the physical energy-momentum tensor is given by T^μ_ρ, which can be rewritten as

$$T^\mu_\rho = \sum_\beta \int d^{n_2}y_2 \left[(S_{\nu,\rho} - \frac{e_\nu}{c}\Phi_\rho) \frac{\partial\hat{\mathcal{L}}_\beta}{\partial S_{\nu,\mu}} + F_{\lambda\rho}\frac{\partial\hat{\mathcal{L}}_\beta}{\partial F_{\lambda\mu}} \right] - \mathcal{L}\delta^\mu_\rho. \tag{32}$$

In the next section we will ascertain the symmetry of T^μ_ρ by means of the angular momentum tensor.

IV. The Angular Momentum Tensor

As usual the angular momentum tensor is obtained from infinitesimal rotational invariance. If we rotate our system in $\underline{x}$-space according to

$$\delta x_i = \epsilon_{ik}x_k, \qquad \epsilon_{ik} = -\epsilon_{ki}, \qquad i = 1,2,3 \tag{33}$$

then the variation of scalars is still given by Eq. (22) but a 3-vector such as Φ_ℓ will change according to

$$\delta\Phi_\ell = -\epsilon_{ik}x_k\Phi_{\ell,i} + \epsilon_{\ell k}\Phi_k, \quad \ell = 1,2,3. \tag{34}$$

Using Eq. (34) and the following variations:

$$\begin{aligned} \delta\mathcal{L} = -\delta x_i\frac{\partial\mathcal{L}}{\partial x_i} &= -\epsilon_{ik}x_k\frac{\partial\mathcal{L}}{\partial x_i} = -\epsilon_{ik}\frac{\partial}{\partial x_i}(x_k\mathcal{L}) \\ \delta\psi^2_\beta &= -\epsilon_{ik}x_k\psi^2_{\beta,i} \\ \delta\Phi_0 &= -\epsilon_{ik}x_k\Phi_{0,i}, \end{aligned} \tag{35}$$

together with Eq. (20) yields

$$\begin{aligned} \epsilon_{ik}\frac{\partial}{\partial x_i}(x^k\mathcal{L}) = \frac{\partial}{\partial z^\mu_a}\sum_\beta \int d^{n_2}y_2\epsilon_{ik}x_k\left[\psi^2_{\beta,i}\frac{\partial\hat{\mathcal{L}}_\beta}{\partial\psi^2_{\beta,\mu}} + \Phi_{\lambda,i}\frac{\partial\hat{\mathcal{L}}_\beta}{\partial\Phi_{\lambda,\mu}}\right] \\ - \frac{\partial}{\partial z^\mu_a}\sum_\beta \int d^{n_2}y_2\epsilon_{ik}\Phi_k\frac{\partial\hat{\mathcal{L}}_\beta}{\partial\Phi_{i,\mu}}. \end{aligned} \tag{36}$$

Since ϵ_{ik} is antisymmetric but otherwise arbitrary, upon making use of Eq. (24), Eq. (36) leads to

$$\frac{\partial}{\partial z^\mu_a}\left[x^k\Theta^\mu_i - x^i\Theta^\nu_k - \sum_\beta \int d^{n_2}y_2\left(\Phi_k\frac{\partial\hat{\mathcal{L}}_\beta}{\partial\Phi_{i,\mu}} - \Phi_i\frac{\partial\hat{\mathcal{L}}_\beta}{\partial\Phi_{k,\mu}}\right)\right] = 0. \tag{37}$$

Inserting $\Theta_i^\mu = T_i^\mu + N_i^\mu$ into Eq. (37) and using $\partial N_i^\mu / \partial z_a^\mu = 0$ yields

$$\frac{\partial}{\partial z_a^\mu}\left[x^k T_i^\mu - x^i T_k^\mu\right] + \frac{\partial}{\partial z_a^\mu} Q_{ki}^\mu = 0 \tag{38}$$

where

$$\frac{\partial Q_{ki}^\mu}{\partial z_a^\mu} = \sum_\beta \int d^{n_2} y_2 \left[\frac{e_\nu}{c}\Phi_i \frac{\partial \hat{\mathcal{L}}_\beta}{\partial \psi_{\beta,k}^2} - \frac{e_\nu}{c}\Phi_k \frac{\partial \hat{\mathcal{L}}_\beta}{\partial \psi_{\beta,i}^2} - \Phi_k \frac{\partial \hat{\mathcal{L}}_\beta}{\partial \Phi_i} + \Phi_i \frac{\partial \hat{\mathcal{L}}_\beta}{\partial \Phi_k}\right]. \tag{39}$$

In Eq. (38) terms have cancelled due to the antisymmetry of $\partial \hat{\mathcal{L}}_\beta / \partial \Phi_{i,\mu}$; also the last two terms were obtained using Eq. (18). Finally, from Eq. (29) we see that the right hand side of Eq. (38) vanishes. Thus

$$\frac{\partial}{\partial z_a^\mu}\left(x^k T_i^\mu - x^i T_k^\mu\right) = T_i^k - T_k^i = 0 \tag{40}$$

and we have established that the energy-momentum tensor is symmetric in its spatial components. The form of the angular momentum tensor is simply given by

$$M_{ki}^\mu = x^k T_i^\mu - x^i T_k^\mu. \tag{41}$$

V. Explicit Form of the Energy-Momentum Tensor

Consider now the first term of Eq. (32). Using Eq. (25) and the following:

$$\phi_\nu = w_\nu \hat{f}_\nu\left(\underline{\alpha}_1, \underline{\alpha}_2, \frac{\partial S_\nu}{\partial \underline{\alpha}_1}, \frac{\partial S_\nu}{\partial \underline{\alpha}_2}\right),$$

$$w_\nu = \left\| \frac{\partial^2 S_\nu}{\partial \alpha_i \partial y^k} \right\|, \tag{42}$$

where w_ν is the Van Vleck determinant and $\hat{f}_\nu$ is a general constant of motion closely related to the distribution function (see I for details), we obtain

$$\begin{aligned}
&\sum_\beta \int d^{n_2} y_2 \left(S_{\nu,\rho} - \frac{e_\nu}{c}\Phi_\rho\right) \frac{\partial \hat{\mathcal{L}}_\beta}{\partial S_{\nu,\mu}} \\
&= -\sum_\nu \int d^{n_1+n_2}\alpha d^{n_2} y_2 \left(S_{\nu,\rho} - \frac{e_\nu}{c}\Phi_\rho\right) V_\nu^\mu w_\nu \hat{f}_\nu.
\end{aligned} \tag{43}$$

The quantities $S_{\nu,\rho} - \frac{e_\nu}{c}\Phi_\rho$ are given by

$$S_{\nu,0} - \frac{e_\nu}{c}\Phi_0 = \frac{1}{c}(\frac{\partial S_\nu}{\partial t} + e_\nu \Phi) = -\frac{1}{c}\hat{H}_\nu, \qquad \rho = 0$$

$$S_{\nu,\rho} - \frac{e_\nu}{c}\Phi_\rho = p_\rho - \frac{e_\nu}{c}A_\rho, \qquad \rho = 1,2,3. \tag{44}$$

The second term of Eq. (32), with the aid of Eq. (16), can be rewritten as

$$\sum_\beta \int d^{n_2}y_2 F_{\lambda\rho}\frac{\partial \hat{\mathcal{L}}_\beta}{\partial F_{\lambda\mu}} = F_{\lambda\rho}\frac{\partial \mathcal{L}}{\partial F_{\lambda\mu}} = F_{0\rho}\frac{\partial \mathcal{L}}{\partial F_{0\mu}} + F_{\ell\rho}\frac{\partial \mathcal{L}}{\partial F_{\ell\mu}}$$
$$= E_\rho \frac{\partial \mathcal{L}}{\partial E_\mu} + F_{\ell\rho}\frac{\partial \mathcal{L}}{\partial F_{\ell\mu}}. \tag{45}$$

The last term of Eq. (45) can be evaluated using

$$F_{\ell\rho} = \epsilon_{i\rho\ell}B^i = (\underline{B} \times \underline{e}_\rho)\cdot \underline{e}_\ell \qquad \rho, \ell = 1,2,3. \tag{46}$$

Thus for $\ell, \mu = 1,2,3$

$$\frac{\partial \mathcal{L}}{\partial F_{\ell\mu}} = \left(\frac{\partial \mathcal{L}}{\partial \underline{B}} \times \underline{e}_\mu\right)\cdot \underline{e}_\ell \tag{47}$$

and the following relations for $F_{\ell\rho}\partial\mathcal{L}/\partial F_{\ell\mu}$ hold:

$$F_{\ell\rho}\frac{\partial \mathcal{L}}{\partial F_{\ell\mu}} = \underline{B}\cdot\frac{\partial \mathcal{L}}{\partial \underline{B}}\delta_{\rho\mu} - B_\mu \frac{\partial \mathcal{L}}{\partial B_\rho} \qquad \rho,\mu = 1,2,3 \tag{48}$$

$$F_{\ell 0}\frac{\partial \mathcal{L}}{\partial F_{\ell\mu}} = \left(\frac{\partial \mathcal{L}}{\partial \underline{B}} \times \underline{E}\right)_\mu \qquad \rho = 0; \mu = 1,2,3 \tag{49}$$

$$F_{\ell\rho}\frac{\partial \mathcal{L}}{\partial F_{\ell 0}} = \left(\underline{B} \times \frac{\partial \mathcal{L}}{\partial \underline{E}}\right)_\rho, \qquad \rho = 1,2,3; \mu = 0 \tag{50}$$

$$F_{\ell 0}\frac{\partial \mathcal{L}}{\partial F_{\ell 0}} = \underline{E}\cdot\frac{\partial \mathcal{L}}{\partial \underline{E}} \qquad \rho = \mu = 0 \quad . \tag{51}$$

Evaluating Eqs. (48)-(51) for the electromagnetic field portion of $\mathcal{L}$, $\mathcal{L}_M = \frac{1}{8\pi}(E^2 - B^2)$, yields

$$\frac{1}{4\pi}(E_\rho E_\mu + B_\rho B_\mu) - \frac{B^2}{4\pi}\delta_{\rho\mu} \qquad \rho,\mu = 1,2,3 \tag{52}$$

$$\frac{1}{4\pi}(\underline{E} \times \underline{B})_\mu \qquad \rho = 0; \mu = 1,2,3 \tag{53}$$

$$-\frac{1}{4\pi}(\underline{E} \times \underline{B})_\rho \qquad \rho = 1,2,3; \mu = 0 \tag{54}$$

$$\frac{1}{4\pi}E^2 \qquad \rho = \mu = 0 \quad . \tag{55}$$

Evaluating Eqs. (48)-(51) for the "particle" portion of $\mathcal{L}, \sum_\beta \int d^{n_2}y_2\mathcal{L}_\beta$, yields the following for Eq. (45):

$$-\sum_\nu \int d^{n_1+n_2}\alpha d^{n_2}y_2 w_\nu \hat{f}_\nu \left[E_\rho \frac{\partial \hat{H}_\nu}{\partial E_\mu} - B_\mu \frac{\partial \hat{H}_\nu}{\partial B_\rho} + \underline{B}\cdot\frac{\partial \hat{H}}{\partial \underline{B}}\delta_{\mu\nu}\right] \qquad \rho,\mu = 1,2,3 \tag{56}$$

$$\sum_\nu \int d^{n_1+n_2}\alpha d^{n_2}y_2 w_\nu \hat{f}_\nu \left(\underline{E}\times\frac{\partial \hat{H}_\nu}{\partial \underline{B}}\right)_\mu \qquad \rho = 0; \mu = 1,2,3 \tag{57}$$

$$-\sum_\nu \int d^{n_1+n_2}\alpha d^{n_2}y_2 w_\nu \hat{f}_\nu \left(\underline{B}\times\frac{\partial \hat{H}_\nu}{\partial \underline{E}}\right)_\rho \qquad \rho = 1,2,3; \mu = 0 \tag{58}$$

$$-\sum_\nu \int d^{n_1+n_2}\alpha d^{n_2}y_2 w_\nu \hat{f}_\nu \underline{E}\cdot\frac{\partial \hat{H}_\nu}{\partial \underline{E}} \qquad \rho = \mu = 0. \tag{59}$$

Here we have used $\phi_\nu = w_\nu \hat{f}_\nu$.

It remains to determine the last term of Eq. (32),

$$\mathcal{L}\delta^\mu_\rho = \left[\frac{1}{8\pi}(E^2 - B^2) - \sum_\nu \int d^{n_1+n_2}\alpha d^{n_2}y_2 w_\nu \hat{f}_\nu \left(\frac{\partial S_\nu}{\partial t} - e_\nu\Phi_0 + \hat{H}_\nu\right)\right]\delta^\mu_\rho. \tag{60}$$

As a consequence of the Euler-Lagrange equations, Eq. (18) with $i = 1$, the last term of Eq. (60) vanishes and we are left with

$$\mathcal{L}\delta^\mu_\rho = \frac{1}{8\pi}(E^2 - B^2)\delta^\mu_\rho. \tag{61}$$

In I it was shown that one can replace

$$\int d^{n_1}\alpha_1 d^{n_2}y_2 w_\nu \hat{f}_\nu \ldots \quad \text{by} \quad \int d^3p f_\nu \ldots \tag{62}$$

where d^3p is a $\underline{p}$-space volume element and $f_\nu(\underline{x},\underline{p},t;\underline{\alpha}_2)$ is the distribution function for "particles" of species ν defined on normal phase, $\underline{x},\underline{p}$. The quantity $\underline{\alpha}_2 = \underline{P}_2$ is the constant value of $\underline{P}_2$ corresponding to the independence of H_ν of $\underline{y}_2$. This constant is to be chosen so that the correct equations of motion for the "particles" is obtained. This means that f_ν is proportional to a δ-function in $\underline{\alpha}_2$. In the $\underline{p}$-representation, $S_{\nu,p}$ is given by Eq. (44). We can now write down the explicit form of the energy-momentum tensor T^μ_ρ:

$$T^0_0 = \sum_\nu \int d^3p d^{n_2}\alpha_2 f_\nu \left(\hat{H}_\nu - \underline{E}\cdot\frac{\partial \hat{H}_\nu}{\partial \underline{E}}\right) + \frac{1}{8\pi}(E^2 + B^2) \tag{63}$$

$$cT_0^\mu = \sum_\nu \int d^3p d^{n_2}\alpha_2 f_\nu \left(\hat{H}_\nu V_\nu^\mu + c\left(\underline{E} \times \frac{\partial \hat{H}_\nu}{\partial \underline{B}} \right)_\mu \right) + \frac{c}{4\pi} (\underline{E} \times \underline{B})_\mu \quad \mu = 1,2,3 \tag{64}$$

$$T_\rho^0 = -\sum_\nu \int d^3p d^{n_2}\alpha_2 f_\nu \left[c\left(p_\rho - \frac{e_\nu}{c} A_\rho \right) + \left(\underline{B} \times \frac{\partial \hat{H}_\nu}{\partial \underline{E}} \right)_\rho \right] - \frac{1}{4\pi} (E \times B)_\rho \quad \rho = 1,2,3 \tag{65}$$

$$\begin{aligned} T_\rho^\mu = &-\sum_\nu \int d^3p d^{n_2}\alpha_2 f_\nu \left[\left(p_\rho - \frac{e_\nu}{c} A_\rho \right) V_\nu^\mu + E_\rho \frac{\partial \hat{H}_\nu}{\partial E_\mu} - B_\mu \frac{\partial \hat{H}_\nu}{\partial B_\rho} + \underline{B} \cdot \frac{\partial \hat{H}_\nu}{\partial \underline{B}} \delta_\rho^\mu \right] \\ &+ \frac{1}{4\pi} \left(E_\rho E_\mu + B_\rho B_\mu \right) - \frac{1}{8\pi} (B^2 + E^2) \delta_\rho^\mu \qquad \mu, \rho = 1,2,3 \quad . \end{aligned} \tag{66}$$

Equations (64) and (65) agree with the expressions for the energy and energy flux densities of I, while Eqs. (66) and (67) are new.

VI. The Maxwell-Vlasov Theory

For the Maxwell-Vlasov equations we restrict our general results by setting $\underline{y} = \underline{x}, \underline{\alpha} = \underline{\alpha}_1, \int d^n\alpha_2 \ldots \to 1 \ldots$ and $\partial \hat{H}_\nu / \partial \underline{E} = \partial \hat{H}_\nu / \partial \underline{B} = 0$. In this way Eqs. (63)-(66) yield the well-known expressions

$$T_0^0 = \sum_\nu \int d^3p f_\nu \hat{H}_\nu + \frac{1}{8\pi} (E^2 + B^2)$$

$$cT_0^\mu = \sum_\nu \int d^3p f_\nu \hat{H}_\nu V_\nu^\mu + \frac{c}{4\pi} (\underline{E} \times \underline{B})_\mu, \quad \mu = 1,2,3$$

$$T_\rho^0 = -\sum_\nu \int d^3p f_\nu c (\underline{p} - \frac{e_\nu}{c} \underline{A})_\rho - \frac{1}{4\pi} (\underline{E} \times \underline{B})_\rho, \quad \rho = 1,2,3$$

$$\begin{aligned} T_\rho^\mu = &-\sum_\nu \int d^3p f_\nu (\underline{p} - \frac{e_\nu}{c} \underline{A})_\rho V_\nu^\mu + \frac{1}{4\pi} (E_\rho E_\mu + B_\rho B_\mu) \\ &- \frac{1}{8\pi} (E^2 + B^2) \delta_\rho^\mu, \qquad \rho, \mu = 1,2,3. \end{aligned} \tag{67}$$

VII. Kinetic Guiding Center – Maxwell Theory

Using Wimmel's variational formulation[5] for Littlejohn's[4] guiding center equations of motion that include polarization drifts, one has as shown in I

$$\frac{\partial \hat{H}_\nu}{\partial \underline{E}} = \frac{m_\nu c}{B^2}(\underline{v}_{D_\nu} - \underline{v}_E) \times \underline{B}$$

$$\begin{aligned}\frac{\partial \hat{H}_\nu}{\partial \underline{B}} = &- \frac{m_\nu v_\|}{B}(\underline{v}_{D_\nu} - \underline{v}_E) - \frac{m_\nu c}{B^2}(\underline{v}_{D_\nu} - \underline{v}_E) \times \underline{E} \\ &+ \frac{2m_\nu}{B}\underline{v}_E \cdot (\underline{v}_{D_\nu} - \underline{v}_E)\underline{b} + \mu \underline{b} \end{aligned} \tag{68}$$

$$\underline{p} - \frac{e_\nu}{c}\underline{A} = m_\nu(v_\|\underline{b} + \underline{v}_E)$$

$$\underline{V}_\nu = v_\|\underline{b} + \underline{v}_{D_\nu}$$

$$\hat{H}_\nu = \frac{m_\nu}{2}(v_\|^2 + v_E^2) + \mu B, \quad \underline{b} = \underline{B}/B.$$

In these expressions the quantity $\underline{\alpha}_2$ has already been assigned its appropriate value. Also, one can replace

$$\int d^3p d^{n_2}\alpha_2 f_\nu \ldots \quad \text{by} \quad \int \hat{B}_{\nu\|} d\mu dv_\| f_{g\nu} \ldots \tag{69}$$

where

$$\hat{B}_{\nu\|} = \underline{b} \cdot \nabla \times \left[\frac{m_\nu c}{e_\nu}(v_\|\underline{b} + \underline{v}_E)\right] + B, \tag{70}$$

and $f_{g\nu}(\underline{x}, v_\|, \mu, t)$ is a solution of the kinetic guiding center equation

$$\frac{\partial f_{g\nu}}{\partial t} + (\underline{v}_{D_\nu} + v_\|\underline{b}) \cdot \frac{\partial f_{g\nu}}{\partial \underline{x}} + \dot{v}_\| \frac{\partial f_{g\nu}}{\partial v_\|} = 0. \tag{71}$$

Expressions for $\dot{v}_\|$ and $\underline{v}_{D_\nu}$ can, for instance, be found in I. The normalization of $f_{g\nu}$ is such that

$$\int \hat{B}_{\nu\|} d\mu dv_\| f_{g\nu} = n_{g\nu}(\underline{x}, t), \tag{72}$$

where $n_{g\nu}$ is the guiding center density of particles of species ν. We can now write the energy-momentum tensor:

$$\begin{aligned} T_0^0 = &\sum_\nu \int \hat{B}_{\nu\|} d\mu dv_\| f_{g\nu} \left[\frac{m_\nu}{2}(v_\|^2 + v_{D_\nu}^2) - (\underline{v}_{D_\nu} - \underline{v}_E)^2 + \mu B\right] \\ &+ \frac{1}{8\pi}(E^2 + B^2) \end{aligned} \tag{73}$$

$$cT_o^\mu = \sum_\nu \int \hat{B}_{\nu\|} d\mu dv_\| f_{g\nu} \Bigg[\left(\frac{m_\nu}{2}(v_\|^2 + v_E^2) + \mu B \right) \left(\underline{v}_{D_\nu} + v_\| \underline{b} \right)_\mu$$

$$+ \left(\mu B + 2m_\nu \underline{v}_E \cdot (\underline{v}_{D_\nu} - \underline{v}_E) \right) v_{E\mu} - \frac{m_\nu c^2}{B^2} \left[\underline{E} \times \left((\underline{v}_{D_\nu} - \underline{v}_E) \times \underline{E} \right) \right]_\mu$$

$$- \frac{m_\nu v_\| c}{B} \left(\underline{E} \times (\underline{v}_{D_\nu} - \underline{v}_E) \right)_\mu \Bigg] + \frac{c}{4\pi} (\underline{E} \times \underline{B})_\mu \qquad \mu = 1,2,3 \tag{74}$$

$$T_\rho^0 = -\sum_\nu \int \hat{B}_{\nu\|} d\mu dv_\| f_{g\nu} c m_\nu (v_\| \underline{b} + \underline{v}_{D_\nu})_\rho - \frac{1}{4\pi} (\underline{E} \times \underline{B})_\rho \qquad \rho = 1,2,3 \tag{75}$$

$$T_\rho^\mu = -\sum_\nu \int \hat{B}_{\nu\|} d\mu dv_\| f_{g\nu} m_\nu \Bigg[(v_\| \underline{b} + \underline{v}_{D_\nu})_\rho (v_\| \underline{b} + \underline{v}_{D_\nu})_\mu - v_{D_{\nu\rho}} v_{D_{\nu\mu}}$$

$$+ v_{E\rho} v_{D_{\nu\mu}} + \frac{c}{B^2} \left(E_\rho (\underline{v}_{D_\nu} \times \underline{B})_\mu + B_\mu (\underline{v}_{D_\nu} \times \underline{E})_\rho \right) + \frac{c^2}{B^2} (E_\rho E_\mu - E^2 b_\rho b_\mu)$$

$$+ \left(2\underline{v}_E \cdot (\underline{v}_{D_\nu} - \underline{v}_E) + \frac{1}{m_\nu} \mu B \right) (\delta_\rho^\mu - b_\rho b_\mu) - \underline{v}_E \cdot (\underline{v}_{D_\nu} - \underline{v}_E) \delta_\rho^\mu \Bigg]$$

$$+ \frac{1}{4\pi} (E_\rho E_\mu + B_\rho B_\mu) - \frac{1}{8\pi} (E^2 + B^2) \delta_\rho^\mu \qquad \mu, \rho = 1,2,3. \tag{76}$$

It was shown in Sec. IV that the expression given by Eq. (77) is symmetric in ρ and μ. This property is not apparent because it is hidden in the special forms for the drift velocities. For the case where $\underline{v}_{D_\nu} = \underline{v}_E$, the symmetry can easily be shown directly. Also for $\underline{E} = 0$ the symmetry is obvious.

VIII. Summary

By means of a variational formulation for the Maxwell-Vlasov and related theories, which describes both the fields and the "particles" in the Eulerian picture, the canonical energy-momentum and angular momentum tensors were obtained. This was done by applying the field theoretic method that relates invariance properties and conservation laws. The tensors found this way were not gauge invariant. Splitting the canonical energy-momentum tensor into a divergence free gauge invariant part T_λ^μ and a divergence free non-gauge invariant part had the consequence that the angular momentum tensor likewise split up in a natural way into a divergence free gauge invariant part $M_{ki}^\mu (k, i = 1,2,3, \mu = 1,2,3,4)$ and a divergence free non-gauge invariant part. M_{ki}^μ is related to T_i^μ by $M_{ki}^\mu = x^k T_i^\mu - x^i T_k^\mu$.

From this the relation $T_i^k = T_k^i$ follows immediately. This is a result that appears to be not easily proven explicitly. T_ρ^μ was expressed in normal terms and explicit expressions were given for the Maxwell-Vlasov theory and a Maxwell-kinetic guiding center theory based on Littlejohn's guiding center equations.

Finally we comment about two examples to which the formalism presented here is applicable. By appropriate choices of $\hat{H}_\nu$ the energy-momentum and angular momentum tensors for relativistic systems, or systems for which the "particle" hamiltonian depends upon derivatives of $\underline{E}$ and $\underline{B}$, are easily obtained. The former case is immediate by making use of Eqs. (63)-(66), while the latter requires a slight generalization. The relativistic Maxwell-Vlasov equation is an example of the former case. It can be formulated in a covariant manner by rewriting the "particle" portion of the Lagrangian density, Eq. (7), with a particular choice of $\hat{H}_\nu$. The electromagnetic field portion of course requires no alteration. For a single species the appropriate "particle" form is

$$\phi\left(\frac{\partial S}{\partial t} + e\Phi + c\sqrt{\left(\frac{\partial S}{\partial \underline{x}} - \frac{e}{c}\underline{A}\right)^2 + m_0^2c^2}\right)$$
$$= \psi\left[\left(\frac{\partial S}{\partial x^\mu} - \frac{e}{c}\Phi_\mu\right)\left(\frac{\partial S}{\partial x_\mu} - \frac{e}{c}\Phi^\mu\right) + m_o c^2\right] \quad .$$

The equality here follows from the substitution

$$\phi = \psi\left[-\frac{\partial S}{\partial t} - e\Phi + c\sqrt{\left(\frac{\partial S}{\partial \underline{x}} - \frac{e}{c}\underline{A}\right)^2 + m_0^2c^2}\right]$$

which introduces the quantity ψ that is (like S) a Lorentz scalar invariant. Upon variation we see that ψ satisfies a covariant continuity equation

$$\frac{\partial}{\partial x^\mu}(\psi u^\mu) = 0$$

where

$$u^\mu = \frac{1}{m_0}\left(\frac{\partial S}{\partial x_\mu} - \frac{e}{c}\Phi^\mu\right)$$

is the velocity 4-vector.

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