DOE/ET-53088-175

IFSR #175

VARIATIONAL QUADRATIC FORM FOR LOW FREQUENCY ELECTROMAGNETIC PERTURBATIONS; (I) FORMALISM

H.L. Berk
Institute for Fusion Studies
The University of Texas at Austin
Austin, Texas 78712

and

B.G. Lane
Plasma Fusion Center
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

June 1985

Variational Quadratic Form for Low Frequency Electromagnetic Perturbations; (I) Formalism

H. L. Berk

Institute for Fusion Studies

The University of Texas at Austin

Austin, Texas 78712

B. G. Lane

Plasma Fusion Center

Massachusetts Institute of Technology

Cambridge, Massachusetts 02139

Abstract

A variational formalism is obtained in the limit of large perpendicular wavenumber which simultaneously includes electrostatic and electromagnetic perturbations, finite Larmor radius corrections, equilibrium plasma rotation and arbitrary particle bounce effects. A tractable final expression is obtained and kinetic integrals are evaluated in special limits. The more accurate non-eikonal expression is obtained from the asymptotic matching of the eikonal form to a more restrictive non-eikonal quadratic forms derived elsewhere.

I. Introduction

In this work we derive the variational form governing the collisionless linearized mode equations for low frequency perturbations in magnetic configurations with frequencies much less than the ion cyclotron frequency. We take into account pressure, equilibrium flow, finite Larmor-radius, arbitrary bounce effects and describe simultaneously conventional MHD and electrostatic effects. The equations are derived in the eikonal limit starting from the results of a formalism developed by Wong.[1] By comparing our result with previously derived forms in the literature,[2,3,4,5,6,7,8] we can use asymptotic matching techniques to determine the non-eikonal form of the variational expression.

The use of a variational expression to determine the governing equations of a system is a compact way to derive the appropriate normal mode equations and to achieve insight on how to determine approximate solutions. The method has been used for obtaining the kinetic energy principle,[2,3] the normal mode equations in rotation problems,[4] finite Larmor radius effects in systems with pressure anisotropy,[5] and the response to very low frequency systems.[6,7,8,9]

The present work reduces the exact eikonal variational form which is difficult to analyze, to a transparent structure for describing combined MHD and electrostatic perturbation problems with arbitrary ordered bounce frequency effects and leading order finite Larmor radius corrections. This reduced variational form in the eikonal limit is valid over a larger parameter regime than any of the individual forms given in Refs. 2-9. In particular, electrostatic perturbations are described by the inclusion of an additional term to the MHD-like variational expression. We also obtain nonlocal kinetic forms that can be explicitly evaluated in several different bounce frequency regimes. As an example, our final form explicitly shows that the description of electrostatic trapped particle modes in tandem mirrors requires that the transit time through the end-cell be short compared to a wave period, while the bounce-time through the central cell can be arbitrary. Further, the variational form allows for an evaluation of the resonance contributions, which lead to dissipation over a wide range of bounce frequencies for test functions of interest in trapped particle mode analysis.

This is a two-part investigation. The present paper is concerned with the derivation of the governing variational form and the evaluation of kinetic integrals of interest for electrostatic trapped particle modes in tandem mirrors. A forthcoming paper will apply this variational form to several aspects of trapped particle modes in tandem mirrors.

In Sec. II we show how the quadratic form developed by Wong can be reduced to a form useful for analysis. In Sec. III we evaluate the kinetic integrals in several bounce frequency regimes for a standard tandem mirror model. By matching our expression the results at various other works, we show in Sec. IV how a non-eikonal variational form can be obtained for a symmetric tandem mirror. Applications of this form will be discussed in the forthcoming paper. In Appendix A we show some details of our algebraic manipulations. In Appendix B we present an alternate, and somewhat more general form of the eikonal variational expression which may be particularly useful for flute modes and for describing systems with very hot particles.

II. High Mode Number Analysis

We begin with the variational quadratic form appropriate for high mode number, electromagnetic perturbations developed by Wong:[1].

$$
\begin{aligned}
W = \frac{1}{4\pi}\int_{-\infty}^{+\infty}\frac{ds}{B}\left[\frac{k_\perp^2 c^2}{\omega^2}\left(\frac{\partial\chi_1}{\partial s}\right)^2 + B_\parallel^2\right] + \int_{-\infty}^{+\infty}\frac{ds}{B}\sum_j\int d^3v\Bigg\{ q_j^2\phi_1^2\frac{\partial F_j}{\partial\epsilon} \\
+ \left\{ q_j^2\phi_1^2 + \frac{q_j^2 v_\parallel^2}{\omega^2}\left(\frac{\partial\chi_1}{\partial s}\right)^2(1-J_0^2) - \left[q_j\phi_1 J_0 + \frac{q_j B_\parallel}{k_\perp c}\left(\frac{2\mu B}{m_j}\right)^{1/2} J_1\right]^2\right\}\frac{1}{B}\frac{\partial F_j}{\partial\mu} \\
- \left(\frac{\partial F_j}{\partial\epsilon} + \frac{\ell c}{\omega q_j}\frac{\partial F_j}{\partial\alpha}\right)\Bigg\{ 2q_j\chi_1 J_0\left[q_j\phi_1 J_0 + \frac{q_j B_\parallel}{k_\perp c}\left(\frac{2\mu B}{m_j}\right)^{1/2} J_1\right] \\
- q_j^2\chi_1^2\left(\frac{\omega-\omega_{d_j}}{\omega}\right)J_0^2\Bigg\}\Bigg\} + D_0
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
D_0 = i\sum_j\frac{2\pi}{m_j^2}\int d\epsilon d\mu\left(\omega\frac{\partial F_j}{\partial\epsilon} + \frac{\ell c}{q_j}\frac{\partial F_j}{\partial\alpha}\right) \\
\times\int_{t^-}^{t^+} dt\int_{-\infty}^{+\infty} dt' \quad \exp\left[i\Omega(t,t')\mathrm{sg}(t-t')\right]A_j(t)A_j(t')
\end{aligned}
\tag{2}
$$

$$A_j(t) = q_j\phi_1 J_0 + \frac{q_j B_{\|}}{k_\perp c}\left(\frac{2\mu B}{m_j}\right)^{1/2} J_1 - q_j\chi_1\left(\frac{\omega - \omega_{dj}}{\omega}\right) J_0 \tag{3}$$

$$t(\epsilon, \mu, q_j, s) = \int^s \frac{ds'}{|v_{\|}(s')|} \qquad \Omega_j(t, t') = \int_{t'}^{t} dt''(\omega - \omega_{dj})$$

$$\mathrm{sg}(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}. \tag{4}$$

We list below the definitions of the various quantities appearing in Eqs. (1)-(4):

j refers to species

$m_j \equiv$ mass, $q_j \equiv$ charge, $c \equiv$ velocity of light

$\mathbf{B} \equiv \nabla\alpha \times \nabla\theta \equiv$ equilibrium magnetic field

$\Phi = \Phi(\alpha, s) \equiv$ equilibrium electrostatic potential

$\alpha \equiv$ magnetic flux, $s \equiv$ distance along field line

$\epsilon \equiv \frac{1}{2}m_j v^2 + q_j\Phi \equiv$ particle energy

$\mu \equiv \frac{1}{2}m_j v_\perp^2/B \equiv$ particle magnetic moment

$v_{\|} = \pm(2/m_j)^{1/2}\left[\epsilon - \mu B(s) - q_j\Phi(s)\right]^{1/2}$

$S(\alpha, \theta) \equiv$ eikonal

$\mathbf{k} = \frac{\partial S}{\partial \alpha}\nabla\alpha + \frac{\partial S}{\partial \theta}\nabla\theta \equiv \lambda\nabla\alpha + \ell\nabla\theta$

$k_\perp^2 = \mathbf{k}\cdot\mathbf{k}$, $\mathbf{b} = \mathbf{B}/B$, $B = |\mathbf{B}|$

$\omega_{dj} \equiv \mathbf{k}\cdot\mathbf{b}\times\left[m_j v_{\|}^2(\mathbf{b}\cdot\nabla)\mathbf{b} + \mu\nabla B + q_j\nabla\Phi\right]/m_j\omega_{cj}$

$\omega_{cj} \equiv q_j B/m_j c \equiv$ cyclotron frequency

$F_j = F_j(\epsilon, \mu, \alpha) \equiv$ equilibrium distribution function

$\mathbf{B}_1 = \nabla \times \mathbf{A}_1 = \mathbf{b}B_{\|} + i\mathbf{k}\times\mathbf{b}A_{\|} + \mathcal{O}(1/kr) \equiv$ perturbed magnetic field

$\mathbf{E}_1 = -\nabla\phi_1 + i\omega A_{\|}\mathbf{b}/c \equiv$ perturbed electric field

$A_{\|} \equiv (-ic/\omega)\partial\chi_1/\partial s$

$J_0(x) \equiv$ Bessel function of zeroth order

$J_1(x) \equiv$ Bessel function of first order

$$x = k_\perp(2\mu B/m_j)^{1/2}/\omega_{cj}. \tag{5}$$

We shall denote $s^{\pm}(\epsilon, \mu, q_j)$ as the turning points of a particle (i.e., where $v_{\|} = 0$) of energy ϵ, magnetic moment μ and charge q_j and $t^- = t(s^-)$ and $t^+ = t(s^+)$.

Note that, in the original Vlasov equation, the dependence of the perturbations on χ_1 is only through $\partial\chi_1/\partial s$. The functional form of Eq. (1) appears to depend on χ_1 which involves an arbitrary integration constant. To eliminate this artificial dependence we will manipulate the variational form so that only explicit $\partial\chi_1/\partial s$ dependence appears.

To proceed, we note the following identities. Given the function $a\big(s(t)\big)$ and $b\big(s(t)\big)$ that are single-valued functions of s, and the definitions,

$$I_1(a,b) = \int_{t^-}^{t^+} dt a(t) \int_{-\infty}^{\infty} dt' b(t') \exp\left[i\Omega(t,t')\mathrm{sg}(t-t')\right] \tag{6}$$

$$I_2(a, db/dt) = \int_{t^-}^{t^+} dt a(t) \int_{-\infty}^{\infty} dt' \frac{db(t')}{dt'} \mathrm{sg}(t'-t) \exp\left[i\Omega(t,t')\mathrm{sg}(t-t')\right], \tag{7}$$

it then follows that

$$I_1(a,b) = I_1(b,a) \tag{8}$$

$$I_2\left(a, \frac{db}{dt'}\right) = -I_2\left(\frac{db}{dt}, a\right). \tag{9}$$

One can also show that

$$I_1\left(\frac{da}{dt}, \frac{db}{dt'}\right) = I_1\left(\frac{db}{dt}, \frac{da}{dt'}\right). \tag{10}$$

(Note, Eq. (10) is different from Eq. (8) as $da(t)/dt \equiv v_{\|}(\partial a/\partial s)$ is not a single valued function of s, but changes sign according to the sign of $v_{\|}$.)

We now introduce the transformation $(1-\omega_E/\omega)\psi = \phi_1$, with $\omega_E = \ell c(\partial\Phi/\partial\alpha)$ and use the identity

$$i \exp\big(i\Omega_j(t,t')\mathrm{sg}(t-t')\big)\left(1-\frac{\omega_{d_j}}{\omega}\right)A_j(t) = \frac{\mathrm{sg(t-t')}}{\omega}\left\{\frac{d}{dt}\left[\exp\big(i\Omega_j(t,t')\mathrm{sg}(t-t')\big)A_j(t)\right]\right.$$

$$\left. - \exp\big(i\Omega_j(t,t')\mathrm{sg}(t-t')\big)\frac{d}{dt}A_j(t)\right\}. \tag{11}$$

Using identities (8)-(10) and integrating the resulting explicit total derivatives, we find

$$D_0 = \int \frac{ds}{B} \sum_j \int d^3v \left(\omega \frac{\partial F_j}{\partial \epsilon} + \frac{\ell c}{q_j} \frac{\partial F_j}{\partial \alpha} \right) \left[q_j (\psi - \chi_1) \left(1 - \frac{\omega_{d_j}}{\omega} \right) J_0 + \frac{2 q_j B_\parallel}{k_\perp c} \left(\frac{2\mu B}{m_j} \right)^{1/2} J_1 \right.$$
$$\left. + 2 q_j \psi \left(\frac{\omega_{B_j} + \omega_{\kappa_j}}{\omega} \right) J_0 \right] q_j (\psi - \chi_1) J_0 + D_1 \tag{12}$$

$$D_1 = 2\pi i \sum_j \frac{1}{m_j^2} \int d\epsilon d\mu \left(\omega \frac{\partial F_j}{\partial \epsilon} + \frac{\ell c}{q_j} \frac{\partial F_j}{\partial \alpha} \right)$$
$$\times \int_{t^-}^{t^+} dt \int_{-\infty}^{+\infty} dt' \left[-i \operatorname{sg}(t - t') \frac{q_j}{\omega} \frac{d}{dt'} \left(\psi(t') J_0 - \chi_1(t') J_0 \right) + \frac{q_j B_\parallel(t')}{k_\perp c} \left(\frac{2\mu B}{m_j} \right)^{1/2} J_1 \right.$$
$$\left. + q_j \left(\frac{\omega_{B_j} + \omega_{\kappa_j}}{\omega} \right) J_0 \psi(t') \right]$$
$$\left[i \operatorname{sg}(t - t') \frac{q_j}{\omega} \frac{d}{dt} \left(\psi(t) J_0 - \chi_1(t) J_0 \right) + \frac{q_j B_\parallel(t)}{k_\perp c} \left(\frac{2\mu B}{m_j} \right)^{1/2} J_1 \right.$$
$$\left. + q_j \left(\frac{\omega_{B_j} + \omega_{\kappa_j}}{\omega} \right) J_0 \psi \right] \exp\left[i\Omega(t, t') \operatorname{sg}(t - t') \right] \tag{13}$$

where

$$\omega_{B_j} \equiv \frac{\mathbf{k} \times \mathbf{b} \cdot \mu \nabla B}{m_j \omega_{c_j}}$$
$$\omega_{\kappa_j} \equiv \frac{v_\parallel^2 \mathbf{k} \times \mathbf{b} \cdot \vec{\kappa}}{\omega_{c_j}}$$
$$\vec{\kappa} \equiv \mathbf{b} \cdot \nabla \mathbf{b} \equiv \kappa_\alpha \nabla \alpha + \kappa_\theta \nabla \theta. \tag{14}$$

Thus D_0 contains local terms and kinetic terms which depend on the orbit integral of the particle through the potential structure. We combine these local terms with the remaining local terms of Eq. (1) to obtain the following variational form:

$$W = W_{\text{local}} + W_{\text{kin}}$$

$$W_{\text{local}} = \frac{1}{4\pi} \int_{-\infty}^{+\infty} \frac{ds}{B} \left[\frac{k_\perp^2 c^2}{\omega^2} \left(\frac{\partial \chi_1}{\partial s} \right)^2 + B_\parallel^2 \right]$$
$$+ \int_{-\infty}^{+\infty} \frac{ds}{B} \sum_j \frac{4\pi}{m_j^2} \int \frac{d\epsilon d\mu}{|v_\parallel|} \left\{ q_j^2 \psi^2 \left(1 - \frac{\omega_E}{\omega} \right)^2 \frac{\partial F_j}{\partial \epsilon} \right.$$

$$
\begin{aligned}
&+\left[q_j^2\psi^2\left(1-\frac{\omega_E}{\omega}\right)^2+q_j^2\frac{v_\parallel^2}{\omega^2}\left(\frac{\partial\chi_1}{\partial s}\right)^2(1-J_0^2)\right.\\
&\left.-\left(q_j\psi J_0\left(1-\frac{\omega_E}{\omega}\right)+\frac{q_jB_\parallel}{k_\perp c}\left(\frac{2\mu B}{m_j}\right)^{1/2}J_1\right)^2\right]\frac{1}{B}\frac{\partial F_j}{\partial\mu}\\
&-\left(\frac{\partial F_j}{\partial\epsilon}+\frac{\ell c}{q_j\omega}\frac{\partial F_j}{\partial\alpha}\right)\left[q_j^2\psi^2\left(1-\frac{\omega_E}{\omega}\right)J_0^2\right.\\
&\left.\left.+\frac{2q_jB_\parallel}{k_\perp c}\left(\frac{2\mu B}{m_j}\right)^{1/2}q_j\psi J_1J_0+q_j^2\psi^2\left(\frac{\omega_{B_j}+\omega_{\kappa_j}}{\omega}\right)J_0^2\right]\right\}. \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
W_{\text{kin}}&=i\sum_j\frac{2\pi}{m_j^2}\int d\epsilon d\mu\left(\omega\frac{\partial F_j}{\partial\epsilon}+\frac{\ell c}{q_j}\frac{\partial F_j}{\partial\alpha}\right)\\
&\int_{t^-}^{t^+}dt\int_{-\infty}^{+\infty}dt'\exp\left[i\Omega(t,t')\mathrm{sg}(t-t')\right]\\
&\times\left[-i\frac{\mathrm{sg}(t-t')}{\omega}q_j\frac{d}{dt'}(\psi(t')-\chi_1(t'))J_0+\frac{q_jB_\parallel(t')}{k_\perp c}\left(\frac{2\mu B}{m_j}\right)^{1/2}J_1+q_j\frac{(\omega_{B_j}+\omega_{\kappa_j})}{\omega}J_0\psi(t')\right]\\
&\times\left[i\frac{\mathrm{sg}(t-t')}{\omega}q_j\frac{d}{dt}(\psi(t)-\chi_1(t))J_0+\frac{q_jB_\parallel(t)}{k_\perp c}\left(\frac{2\mu B}{m_j}\right)^{1/2}J_1+q_j\frac{(\omega_{B_j}+\omega_{\kappa_j})}{\omega}J_0\psi(t)\right].
\end{aligned}
$$

(16)

It is convenient to introduce new field variables in order to facilitate comparison to familiar MHD results. We therefore write

$$
\begin{aligned}
Q_L&=B_\parallel+\left[\frac{\partial B}{\partial\alpha}-\frac{\lambda}{\ell}\frac{\partial B}{\partial\theta}\right]\phi\\
\phi&=\frac{\ell c\psi}{\omega}\\
\chi&=\frac{\ell c\chi_1}{\omega}. \qquad (17)
\end{aligned}
$$

The quantity Q_L is the modulus of the perturbed magnetic field in a frame moving with the total $\mathbf{E}\times\mathbf{B}$ velocity and $\phi=\vec{\xi}\cdot\nabla\alpha$ where $\vec{\xi}$ is the plasma displacement. In W_{local} we expand the Bessel functions to third order in the Larmor radius, while in W_{kin} the Bessel functions are expanded only to first order. With these approximations for the finite

Larmor radius terms, we find after considerable algebra (some details of the manipulations are presented in Appendix A) the following forms for W_{local} and W_{kin}

$$
\begin{aligned}
W_{\text{local}} = \int_{-\infty}^{+\infty} \frac{ds}{B} \Bigg\{ & \frac{1}{4\pi} \frac{\sigma k_\perp^2}{\ell^2} \left(\frac{\partial \chi}{\partial s} \right)^2 + \frac{\tau}{4\pi} \left[Q_L - \frac{\sigma}{\tau} B \phi \left(\kappa_\alpha - \frac{\lambda}{\ell} \kappa_\theta \right) \right]^2 \\
& - \frac{\phi^2 \sigma}{\tau} \left[\frac{\partial}{\partial \alpha} P_\perp(\alpha, B, \Phi) + \frac{\partial \Phi}{\partial \alpha} \frac{\partial}{\partial \Phi} P_\perp(\alpha, B, \Phi) \right] \left(\kappa_\alpha - \frac{\lambda}{\ell} \kappa_\theta \right) \\
& - \phi^2 \frac{\partial P_\parallel}{\partial \alpha}(\alpha, B, \Phi) \left(\kappa_\alpha - \frac{\lambda}{\ell} \kappa_\theta \right) \\
& - \sum_j \frac{m_j k_\perp^2 \phi^2}{B^2 \ell^2} \Bigg[n_j (\omega - \omega_E)^2 - (\omega - \omega_E) \frac{\ell c}{q_j B} \frac{D}{D\alpha} (P_{\perp_j} B) \\
& \qquad + \frac{\ell^2 c^2}{2 q_j^2} \frac{1}{B} \frac{\partial B}{\partial \alpha} \frac{D L_j}{D \alpha} \Bigg] \Bigg\}
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
W_{\text{kin}} = i \sum_j & \frac{2\pi}{(m_j \ell c)^2} \int d\epsilon d\mu \left(\omega \frac{\partial F_j}{\partial \epsilon} + \frac{\ell c}{q_j} \frac{\partial F_j}{\partial \alpha} \right) \\
& \cdot \int_{t^-}^{t^+} dt \int_{-\infty}^{+\infty} dt' \Bigg[i \operatorname{sg}(t' - t) q_j \frac{d}{dt'} (\phi(t') - \chi(t')) + \mu \ell c Q_L(t') \\
& + m_j v_\parallel^2(t') \ell c \left(\kappa_\alpha(t') - \frac{\lambda}{\ell} \kappa_\theta(t') \right) \phi(t') \Bigg] \exp\left[i \operatorname{sg}(t - t') \Omega(t, t') \right] \\
& \Bigg[i \operatorname{sg}(t - t') q_j \frac{d}{dt} (\phi(t) - \chi(t)) + \mu \ell c Q_L(t) \\
& + m_j v_\parallel^2(t) \ell c \left(\kappa_\alpha(t) - \frac{\lambda}{\ell} \kappa_\theta(t) \right) \phi(t) \Bigg]
\end{aligned}
\tag{19}
$$

where $n_j \equiv$ density of species j

$$\left[P_{\perp_j}(\alpha, B, \Phi), P_{\parallel_j}(\alpha, B, \Phi) \right] \equiv \text{ perpendicular and parallel pressure of species } j$$

$$P_{\perp,\parallel}(\alpha, B, \Phi) = \sum_j P_{\perp,\parallel_j}(\alpha, B, \Phi)$$

$$L_j(\alpha, B, \Phi) = \frac{4\pi}{m_j^{3/2}} \int d\epsilon d\mu \frac{\mu^2 B^3 F_j(\epsilon, \mu, \alpha)}{[2(\epsilon - \mu B - q\Phi)]^{1/2}}$$

$$\sigma \equiv 1 + \frac{4\pi (P_\perp - P_\parallel)}{B^2}$$

$$\tau = 1 + \frac{4\pi}{B}\frac{\partial P_\perp}{\partial B}(\alpha, B, \Phi)$$

$$\frac{DQ(\alpha, B, \Phi)}{D\alpha} = \frac{\partial Q}{\partial \alpha} + \frac{\partial B}{\partial \alpha}\frac{\partial Q}{\partial B} + \frac{\partial \Phi}{\partial \alpha}\frac{\partial Q}{\partial \Phi} \equiv \frac{\nabla Q \cdot \nabla \alpha}{|\nabla \alpha|^2}.$$

In Eqs. (18) and (19) we have neglected finite Larmor radius terms multiplying Q_L as in most applications $r^2 Q_L/\phi \simeq \mathcal{O}(\kappa r)$. In Appendix B alternative forms to Eq. (19) are presented and finite Larmor radius terms multiplying Q_L are retained. These FLR terms can be important when dealing with hot particle dynamics, but generally can be ignored for more conventional MHD and trapped particle mode problems.

Equations (18) and (19) are the generalization of kinetic quadratic forms existing in the literature[2–8] in that they treat the effects of (a) parallel electric field perturbations (more precisely, finite $\partial(\phi - \chi)/\partial s$ simultaneously with the usual MHD terms, (b) finite bounce frequency, (c) MHD kinetic effects when $\omega \simeq \omega_*^i \equiv (\ell c/q_i)(\partial P_{\perp i}/\partial \alpha)$ with arbitrary bounce frequency. It reproduces the kinetic variational form for high bounce frequency.[8] The finite Larmor radius structure is the same as derived elsewhere in the long-thin and $(\partial\phi/\partial s) = (\partial\chi/\partial s)$ limits.

We note that now only $(\partial\chi/\partial s)$ enters into the variational form. It enters in the perturbed magnetic energy $\left(B_\perp^2 \propto (\partial\chi/\partial s)^2\right)$ and in the parameter closely associated with the perturbed parallel electric field, $\partial(\phi - \chi)/\partial s$. This is consistent with the fact that the original Vlasov equation depended only on $(\partial\chi/\partial s) \propto A_\parallel$. The remaining terms in the quadratic form are proportional to ϕ or Q_L. The coalescence of MHD modes with electrostatic modes in the flute limit is evident since in both cases the term $\partial(\phi - \chi)/\partial s = (\partial\chi/\partial s) = 0$ and the quadratic form is a functional of the two variables ϕ and Q_L. We also note that the Euler-Lagrange equations for $(\partial\chi/\partial s)$ are distinctly different from the equations for ϕ.

III. Evaluation of Kinetic Integrals

Here we consider several models and limits to evaluate W_1. It is convenient to assume that either

(a) $q_j d(\phi - \chi)/dt << \ell\mu c Q_L + m_j v_\parallel^2 \ell c(\kappa_\alpha - \lambda\kappa_\theta/\ell)\phi$ or

(b) $q_j d(\phi - \chi)/dt >> \ell \mu c Q_L + m_j v_{\|}^2 \ell c(\kappa_\alpha - \lambda\kappa_\theta/\ell)\phi$.

In Appendix B additional forms are given when these approximations are not employed.

First we briefly discuss case (a), a regime where electrostatic effects are neglected. This regime has recently been studied in Ref. 10, and the reader is referred to that work for a more detailed study.

To proceed, we observe that the integrand of Eq. (19) is a periodic function of period $t^+ - t^-$, and if we break the t integrand into periodic segments, and sum the resulting series, we find that Eq. (19), can be written as

$$W_{\text{kin}}^{(a)} = \sum_j \frac{4\pi}{m_j^2} \int d\epsilon d\mu \left(\omega \frac{\partial F_j}{\partial \epsilon} + \frac{\ell c}{q_j} \frac{\partial F_j}{\partial \alpha} \right)$$
$$\int_{t^-}^{t^+} dt \int_{t_-}^{t^+} dt' K_c(t,t') \left[\mu Q_L(t') + m_j v_{\|}^2(t') \left(\kappa_\alpha(t') - \frac{\lambda}{\ell} \kappa_\theta(t') \right) \phi(t') \right]$$
$$\cdot \left[\mu Q_L(t) + m_j v_{\|}^2(t) \left(\kappa_\alpha(t) - \frac{\lambda}{\ell} \kappa_\theta(t) \right) \phi(t) \right] \tag{20}$$

where

$$K_c(t,t') = - \frac{\cos[\Omega(t^<, t^-)] \cos[\Omega(t^+, t^>)]}{\sin[\Omega(t^+, t^-)]} \tag{21a}$$

$$= -\sin[\Omega(t^+, t^<)] \cos[\Omega(t^+, t^>)] - \cos[\Omega(t', t^+)] \cos[\Omega(t^+, t)] \cot[\Omega(t^+, t^-)] \tag{21b}$$

where $t^> = \max(t', t)$, $t^< = \min(t', t)$. We note that the first form of the propagator $K_c(t,t')$ in Eq. (21a) is the form obtained by Rutherford and Frieman.[11] From the second form of the propagator, which follows straightforwardly from the first using trigonometric identities, we can establish that for real ω, the imaginary contribution from a given region of phase space is signed. To demonstrate this we write

$$\text{Im} \cot[\Omega(t^+, t^-)] = -\pi \sum_{n=-\infty}^{\infty} \delta[n\pi - \Omega(t^+, t^-)] \tag{22}$$

and we find

$$\text{Im}\, W_{\text{kin}}^{(a)} = \sum_{n,j} \frac{4\pi^2}{m_j} \int d\epsilon d\mu \left(\omega \frac{\partial F_j}{\partial \epsilon} + \frac{\ell c}{q_j} \frac{\partial F_j}{\partial \alpha} \right)$$
$$\cdot \delta[n\pi - \Omega(t^+, t^-)] \left\{ \int_{t^-}^{t^+} dt \cos[\Omega(t^+, t)] \left[\mu Q_L(t) + m_j v_{\|}^2(t) \left(\kappa_\alpha - \frac{\lambda}{\ell} \kappa_\theta \right) \phi(t) \right] \right\}^2 . \tag{23}$$

For case (b), we neglect the Q_L and κ terms, but keep the $\phi - \chi$ terms. We then again break the t integral into periodic components and find that Eq. (19) can be written as,

$$W_{\rm kin}^{(b)} = \sum_j \frac{4\pi q_j^2}{\ell^2 m_j^2 c^2} \int d\epsilon d\mu \left(\omega \frac{\partial F_j}{\partial \epsilon} + \frac{\ell c}{q_j} \frac{\partial F_j}{\partial \alpha} \right)$$
$$\int_{t^-}^{t^+} dt' \int_{t^-}^{t^+} dt \ K_s(t,t') \frac{d}{dt'}[\phi(t') - \chi(t')] \frac{d}{dt}[\phi(t) - \chi(t)] \tag{24}$$

where

$$K_s(t',t) = \frac{\sin[\Omega(t^<, t^-)]\sin[\Omega(t^+, t^>)]}{\sin[\Omega(t^+, t^-)]} \tag{25a}$$

$$= \cos[\Omega(t^+, t^<)]\sin[\Omega(t^+, t^>)]$$
$$- \cot[\Omega(t^+, t^-)]\sin[\Omega(t^+, t')]\sin[\Omega(t^+, t)]. \tag{25b}$$

Equation (24) is the primary form of the kinetic term we will be using for collisionless trapped particles, and we shall evaluate this form in more detail for test functions perturbations in which ϕ and χ are piecewise constant in different regions of a tandem mirror. In these specific evaluations it is convenient to assume one has an azimuthally symmetric tandem about the mid-plane so that $K_s(t,t') = K_s(-t,-t')$, where t is measured from the mid-plane. From symmetry it follows that eigenfunctions can be chosen to be either symmetric or anti-symmetric about the mid-plane. Using the symmetry of the eigenfunction, one can combine contributions from t and $-t$, and then manipulate Eq. (24) so that $K_s(t,t')$ can be replaced by

$$K(t,t') = \cos[\Omega(t_b, t^<)]\sin[\Omega(t_b, t^>)] - \mathrm{trg}[\Omega(t_b, 0)]\sin[\Omega(t_b, t)]\sin[\Omega(t_b, t')] \tag{26}$$

where $t = \int_0^s ds/|v_\parallel|$, $s = 0$ is the midplane, $t_b = t^+ = -t^-$ and

$$\mathrm{trg} x = \begin{cases} \cot \mathrm{x} & \text{for symmetric eigenfunction;} \\ -\tan \mathrm{x}, & \text{for antisymmetric eigenfunction.} \end{cases}$$

We also note that for a tandem mirror with quadrupole anchors that if we neglect κ_θ in $W_{\rm local}$ and W_1 that the use of mid-plane symmetry is exactly satisfied. From this observation it follows that our symmetric model is rigorously justified for test functions that have zero amplitude in the anchor region and if

$$\int_0^t \omega_\kappa dt << \Omega(t,0),$$

which is generally true in the long thin approximation.

We now evaluate W_1 for a three-cell configuration shown in Fig. 1 with reflection symmetry assumed about the midplane for symmetric trial functions ϕ and χ which are chosen to be piecewise constant; i.e. $\phi = \phi_c$, $\chi = \chi_c$ in the central cell and $\phi = \phi_a$, $\chi = \chi_a$ in the anchor region. We choose $Q_L = 0 + \mathcal{O}(\kappa r)$ to eliminate the large stabilizing τQ_L^2 term in Eq. (18). Then to $\mathcal{O}(\kappa^2 r^2)$, we find,

$$W_{\rm kin}^{(b)} = \sum_j \frac{8\pi q_j^2}{\ell^2 c^2 m_j^2} \Delta\phi^2 \int d\epsilon d\mu \left(\omega \frac{\partial F_j}{\partial \epsilon} + \frac{\ell c}{q_j} \frac{\partial F_j}{\partial \alpha} \right)$$
$$\left[\cos[\Omega(t_b, t(s_0))] \sin[\Omega(t_b, t(s_0))] - \mathrm{trg}[\Omega(t_b, 0)] \sin^2[\Omega(t_b, t(s_0))] \right] \tag{27}$$

where $\Delta\phi = (\phi_c - \chi_c) - (\phi_a - \chi_a)$, s_0 is the position of the interface between the central cell and plug, and $d[\phi(s) - \chi(s)]/ds = -\mathrm{sg}(s)\Delta\phi\delta(s - s_0)$, which is the conventional test function for trapped particle modes generalized so that $\chi \neq 0$. For definiteness, we have taken the eigenfunction symmetric about the midplane so that $\mathrm{trg}x = \cot x$. We further assume that the distribution function in the central cell is primarily Maxwellian, so that

$$F_0(\epsilon, \mu, \alpha) = \frac{n(\alpha)}{[2\pi T_j(\alpha)/m_j]^{3/2}} \exp[-\epsilon/T_j(\alpha)]. \tag{28}$$

The kinetic integral can then be rewritten as,

$$W_{\rm kin}^{(b)} = -\sum_j \frac{8\pi q_j^2 \Delta\phi^2}{\ell^2 c^2 m_j^2 T_j} \int d\epsilon d\mu (\omega - \omega_{Ec} - \hat{\omega}_{jc}^*) F_j$$
$$[\cos[\Omega(t_b, t(s_0))] \sin[\Omega(t_b, t(s_0))] - \cot\Omega(t_b, 0) \sin^2[\Omega(t_b, t(s_0))]] \tag{29}$$

where $\omega_{Ec} = \ell c \partial\Phi_c/\partial\alpha \equiv E \times B$ rotation frequency in central cell.

$$\hat{\omega}^*_{jc} = \frac{c\ell T_j}{q_j n_c}\frac{Dn_{cj}}{D\alpha}\left[1 + \eta_{jc}\left(\frac{\epsilon - q_j\Phi_c}{T_j} - \frac{3}{2}\right)\right]$$

$$\eta_{jc} = \frac{\partial T_j}{T_j \partial\alpha}\Big/\frac{Dn_{cj}}{n_{cj}D\alpha},$$

$n_{cj} \equiv$ density of species j in central cell. If, as in Fig. 1, the electric field drift frequency has a single central cell value ω_{Ec} and a single anchor cell value ω_{Ea}, we have that,

$$\begin{aligned} \Omega(t_b, 0) &= \omega\tau_b - \omega_{Ec}\tau_c - \omega_{Ea}(\tau_b - \tau_c) \\ \Omega\big(t_b, t(s)\big) &= (\omega - \omega_{Ea})(\tau_b - \tau_c) \end{aligned} \tag{30}$$

where

$$\tau_c = \left(\frac{m_j}{2}\right)^{1/2} \frac{L_c}{[\epsilon - \mu B_c - q_j\Phi_c]^{1/2}} \tag{31a}$$

$$\tau_b - \tau_c = \left(\frac{m_j}{2}\right)^{1/2} \frac{L'_a}{[\epsilon - \mu B_p - q_j\Phi_p]^{1/2}} \tag{31b}$$

We also use L to denote the overall half length of the tandem, L_c the central cell half length, L_a the anchor cell length, and L_Φ the length between the mirror peak and anchor potential peak. $L'_a = L_a$ for electrons and $L'_a = L_\Phi$ for ions. We also assume $L_\Phi << L_a$. In Eq. (30) we have neglected the curvature drift, as a result of the long thin approximation, and ω_B, which is valid at low beta if $\omega_{jc} \approx \hat{\omega}^*_{jc} \approx \omega_E$.

There are three regimes where one can reduce Eq. (29) still further,

(a) $\Omega(t_b, 0) << 1$,

(b) $\Omega\big(t_b, 0\big) >> 1 >> \Omega\big(t_b, t(L_c)\big)$

(c) $\Omega\big(t_b, t(L_c)\big) >> 1$.

Region (a) is the high bounce frequency region for which in most of the phase space we have

$$\cos[\Omega\big((t_b, t(L_c)\big)] \doteq 1, \quad \sin[\Omega\big((t_b, t(L_c)\big)] \doteq \Omega[(t_b), t(L_c)]$$

$$\cot[\Omega(t_b, 0)] \doteq \frac{1}{\Omega(t_b, 0)}. \tag{32}$$

However, there is a small class of particles with a low parallel velocity in the central cell for which $\Omega(t_b, 0)$ is a finite number so that there is an imaginary part of the $\cot x$ function as given in Eq. (22). It may be important to calculate this dissipative contribution.

First, we substitute Eq. (32) into Eq. (29), neglecting the imaginary contribution. We find

$$W_{\rm kin}^{(b)} = -\sum_j \frac{8\pi\Delta\phi^2}{T_j m_j^2 c^2}\frac{q_j^2}{\ell^2}\int d\epsilon d\mu(\omega - \omega_{Ec} - \hat{\omega}_{jc}^*)F_j \frac{\Omega(t(L_c),0)\Omega(t_b,t(L_c))}{\Omega(t_b,0)}. \tag{33}$$

For the specific electric fields of our model we have,

$$W_{\rm kin}^{(b)} = -\sum_j \frac{8\pi\Delta\phi^2}{T_j m_j^2 c^2}\frac{q_j^2}{\ell^2}\int d\epsilon d\mu F_j(\omega - \omega_{Ec} - \hat{\omega}_{jc}^*)L_a' L_c \cdot \frac{(\omega-\omega_{Ec})(\omega-\omega_{Ea})}{\left[|v_{\|a}|(\omega-\omega_{Ec})L_c + |v_{\|c}|(\omega-\omega_{Ea})L_a'\right]} \tag{34}$$

where

$$|v_{\|a}| = (2/m_j)^{1/2}[\epsilon - \mu B_a - q_j\Phi_a]^{1/2}$$

$$|v_{\|c}| = (2/m_j)^{1/2}[\epsilon - \mu B_c - q_j\Phi_c]^{1/2}.$$

The remaining integrals are somewhat complicated to perform. Two models are tractable. In the first, developed by Byers and Cohen,[12] one takes $B_a = B_c$, $\Phi_a = \Phi_c$ and the response then is,

$$W_{\rm kin}^{(b)} = -\sum_j \frac{(\Delta\phi)^2 q_j^2}{\ell^2 T_j c^2} N_{\rm pass}^{(j)} \frac{(\omega-\omega_{Ec}-\omega_{jc}^*)(\omega-\omega_{Ec})(\omega-\omega_{Ea})}{(\omega-\omega_{Ec}) + L_a'(\omega-\omega_{Ea})/L_c} \tag{35}$$

where

$$\omega_{jc}^* = \frac{\ell c T_j}{q_j n_{cj}}\frac{D n_{cj}}{D\alpha},$$

and

$$N_{\rm pass}^{(j)} = \frac{8\pi}{m_j^2}\int \frac{d\epsilon d\mu}{|v_{\|a}|}F_j L_a' = 2\int_{\rm anchor} \frac{ds}{B} n_{pj}$$

is equivalent to the number of passing particles in the two anchor regions per flux tube, with n_{pj} the passing particle density in the plugs.

The actual value of $N_{\rm pass}^{(j)}$ is fairly sensitive to the equilibrium potential and magnetic field axial profiles. For definiteness we use the idealized profiles shown in Fig. 1. If we assume $|q_j\Delta\Phi/T_j| >> 1$, we find that $N_{\rm pass}^{(j)}$ is given by

$$N_{\rm pass}^{(e)} = \frac{2n_c}{B_M}\frac{L_a}{\pi^{1/2}}\left(\frac{T}{|q(\Phi_a - \Phi_c)|}\right)^{1/2}$$

$$N^{(i)}_{\rm pass} = \frac{2n_c}{B_a} L_\Phi \left[1 - \left(1 - \frac{B_a}{B_M} \right)^{1/2} \right].$$

The other simple case is the long central cell limit, where $\Omega\big(t_b, t(L_c)\big) << \Omega\big(t(L_c), 0\big)$. In this case

$$\begin{aligned} W^{(b)}_{\rm kin} &\doteq - \sum_j \frac{8\pi q_j^2 \Delta\phi^2}{T_j m_j^2 c^2 \ell^2} \int d\epsilon d\mu F_j (\omega - \omega_{Ec} - \hat{\omega}^*_{jc}) \Omega\big(t_b, t(L_c)\big) \\ &= - \sum_j \frac{8\pi q_j^2 \Delta\phi^2 L'_a}{T_j m_j^2 c^2 \ell^2} \int \frac{d\epsilon d\mu}{|v_{\|a}|} F_j (\omega - \omega_{Ea})(\omega - \omega_{Ec} - \hat{\omega}^*_{jc}) \\ &= - \sum_j \frac{q_j^2 \Delta\phi^2}{T_j c^2 \ell^2} N^{(j)}_{\rm pass} (\omega - \omega_{Ec} - \omega^*_{jc})(\omega - \omega_{Ea}) \left[1 + O\left(\frac{T_j}{|q_j(\Phi_a - \Phi_c)|} \right) \right]. \end{aligned} \quad (36)$$

We see from Eq. (34) that an imaginary response for real ω is possible which was the resonance $\Omega(t_b, 0) \approx 0$. Additional contributions in the imaginary part comes from nearly zero parallel velocity particles by taking into account higher order bounce resonances. For definiteness we just exhibit the response of ions. The detailed calculation is very similar to what has been presented in previous work (see Appendix C of Ref. 13) where it was assumed $L_\Phi / L_c << 1$ and zero transverse electric field. The reader is referred to that calculation, and one finds when one accounts for modifications of the transverse electric fields and temperature gradients, the following result,

$$\begin{aligned} {\rm Im}\, W^{(b)}_{\rm kin,i} = \frac{-.05 \Delta\phi^2 q_i^2}{T_i c^2 \ell^2} N^{(i)}_{\rm pass} (\omega - \omega_{Ea})^2 \left[\omega - \omega_{Ec} - \omega^*_{ic} \left(1 - \frac{3}{2}\eta_i \right) \right] \\ \cdot \frac{(\omega - \omega_{Ec})^2 L_c^2 L_\Phi ln\left(\frac{B_M - B_c}{B_M - B_a} \right)}{(T_i/m_i)^{3/2} (1 - B_c/B_a) \left[1 - (1 - B_a/B_M)^{1/2} \right]} \end{aligned} \quad (37)$$

where $W^{(b)}_{\rm kin,i}$ is the contribution of ions to $W^{(b)}_{\rm kin}$.

In region (b) $\Omega(t_b, 0) >> 1$, and $\cot\Omega(t_b, 0)$ is most easily treated by the substitution $\cot\Omega(t_b, 0) \approx -i$, as justified in Ref. 14. We still have, $\cos\Omega\big(t_b, t(L_c)\big) \approx 1$, $\sin\Omega\big(t_b, t(L_c)\big) \approx \Omega\big(t_b, t(L_c)\big)$. Thus, in region (b) we may write $W_{\rm kin,i}$ as

$$\begin{aligned} W^{(b)}_{\rm kin,i} = - \frac{8\pi \Delta\phi^2 q_i^2}{\ell^2 T_i m_i^2 c^2} \int_{L_c}^{\infty} ds \int d\epsilon d\mu \frac{F_i}{|v_\||} (\omega - \omega_{Ec} - \hat{\omega}^*_{ic})(\omega - \omega_{Ea}) \\ \cdot \left[1 + i\pi\Omega\big(t_b, t(L_c)\big) \right]. \end{aligned} \quad (38)$$

Then using Eqs. (30) and (31) in a manner parallel to the corresponding calculations in Ref. 13, we find

$$W^{(b)}_{\text{kin,i}} = -\frac{\Delta\phi^2 q_i^2 N^{(i)}_{\text{pass}}(\omega-\omega_{Ea})\left[\omega-\omega_{Ea}-\omega^*_{ic}\left(1-\eta_i/2\right)\right]}{\ell^2 T_i c^2} \cdot \left[1+i\frac{(\omega-\omega_{Ea})L_\Phi \ell n\left(\frac{B_M}{B_M-B_a}\right)}{(2\pi)^{1/2}[1-(1-B_a/B_M)^{1/2}](T_i/m_i)^{1/2}}\right]. \tag{39}$$

Finally in region (c), where $\Omega\left(t_b, t(L_c)\right) >> 1$, we estimate $\sin^2[\Omega\left(t_b, t(L_c)\right)] \approx 1/2$, $\sin[\Omega\left(t_b, t(L_c)\right)]\cos[\Omega\left(t_b, t(L_c)\right)] \approx 0$ and again $\cot[\Omega(t_b, 0)] \simeq -i$. Now substituting these expressions into Eq. (29), we readily find,

$$W^{(b)}_{\text{kin,i}} - \frac{i\Delta\phi^2 q_i^2 N^{(i)}_{\text{pass}}[\omega-\omega_{Ec}-\omega^*_{ic}(1+\eta_i/2)](T_i/m_i)^{1/2}}{\ell^2 T_i c^2 (2\pi)^{1/2} L_\Phi\left[1-(1-B_a/B_M)^{1/2}\right]}. \tag{40}$$

IV. Non-Eikonal Expressions for Symmetric Mirror

We have derived the quadratic form in the eikonal limit including electrostatic and FLR terms in an ordering where $k_\perp^2 a_i^2 << 1$. We can obtain the appropriate non-eikonal forms for our equations, using principles of asymptotic matching, by comparing our results with those of other theories. When we perform this matching, we shall assume that the equilibrium is azimuthally symmetric where the perturbations are finite. In this case $\kappa_\theta = 0$ and we define $\kappa = \kappa_\alpha B r$. This assumption enables us to make detailed calculations analytically tractable, justify the self-adjointness structure of the quadratic form,[14] [i.e., a perturbed component, $A(s,\alpha,\theta)$, is taken to vary as $A(\alpha,s)\exp(i\ell\theta)$ and the adjoint function $A^\dagger(\alpha,s,\theta)$ then varies as $A(\alpha,s)\exp(-\ell\theta)$], and neglect the contribution of the equilibrium parallel current.

Our expression for the variational form is

$$W = W_{\text{local}} + W_{\text{kin}} \tag{41}$$

where W_{local} is given in Eq. (18) and W_{kin} in Eq. (19). First we compare our expression with that of Antonsen and Lee,[8] which was derived in the limit $a_i \to 0$, but with ω_i^*/ω

finite. Their expression also assumed $\omega/\omega_b << 1$, where ω_b is the particle bounce frequency. In their theory the usual particle inertia was ordered to be small and was neglected. If in Eqs. (18) and (19) we neglect the inertia FLR terms and consider the limit $\omega/\omega_b \rightarrow 0$, then one can show that these equations differ from that of Ref. 8, by the factor

$$\sigma\left[\frac{B_\perp^2}{B} - \frac{k_\perp^2}{B}\left(\frac{\partial\chi}{\partial s}\right)^2\right]$$

where $B_\perp$ is the perturbed perpendicular magnetic field. $B_\perp^2$ can be written as

$$\begin{aligned}\sigma B_\perp^2 &= \sigma B^2 r^2\left[\frac{\partial}{\partial s}\left(\frac{\partial\chi}{\partial\alpha} - \frac{\phi}{B}\frac{\partial B}{\partial\alpha} + \frac{Q_L}{B}\right)\right]^2 + \frac{\ell^2\sigma}{r^2}\left(\frac{\partial\chi}{\partial s}\right)^2 \\ &= \sigma k_\perp^2\left(\frac{\partial\chi}{\partial s}\right)^2 + \mathcal{O}\left(\frac{1}{k_\perp r}\right).\end{aligned} \tag{42}$$

Thus, to lowest order in the eikonal limit, and with $k_\perp^2 a_i^2,\ \omega/\omega_b \rightarrow 0$, the two results agree. Therefore, to have an appropriate non-eikonal expression, in the small FLR limit, we need to replace $k_\perp^2(\partial\chi/\partial s)^2$ in Eq. (18) by the complete expression for $B_\perp^2$ given in Eq. (42). Thus, we can generalize one of the quadratic terms of the eikonal theory to the non-eikonal case, by recognizing the correspondence of the eikonal bending term to the following form

$$\int\frac{ds}{B}k_\perp^2\sigma\left(\frac{\partial\chi}{\partial s}\right)^2 \longrightarrow 2\pi\int\frac{dsd\alpha}{B}\left\{B^2r^2\sigma\left[\frac{\partial}{\partial s}\left(\frac{\partial\chi}{\partial\alpha} - \frac{\phi}{B}\frac{\partial B}{\partial\alpha} + \frac{Q_L}{B}\right)\right]^2 + \frac{\ell^2\sigma}{r^2}\left(\frac{\partial\chi}{\partial s}\right)^2\right\} \tag{43}$$

where we have also added the phase-space integration over α and θ.

We also note that if $a_i \rightarrow 0$, and inertia terms are neglected, the remaining terms of our quadratic form are independent of $\lambda \equiv \partial S/\partial\alpha$. Thus, these terms would be the same in eikonal and non-eikonal theory.

The extension of the FLR terms beyond the eikonal expression can be obtained by comparing our result with Pearlstein-Krall[16] and Newcomb.[5] Pearlstein and Krall obtained the non-eikonal expressions in a system where the equilibrium doesn't vary along the field line, and Newcomb showed that this form is correct even with a slow equilibrium variation along the field line. Both papers use $\partial\chi/\partial s = \partial\phi/\partial s$. The orderings of these theories are $1 >> \kappa r \sim a^2/r^2 \sim Q_L r^2/\phi$ and $|\partial P_\perp/\partial\alpha| >> |\kappa B/r|$. When we compare FLR and inertia terms of our theory with the quadratic forms of the non-eikonal theory, we find the following correspondence

$$\int \frac{ds}{B^3}\phi^2 \frac{k_\perp^2 m_j}{\ell^2}\Bigg[(\omega-\omega_E)^2 n_j - (\omega-\omega_E)\frac{\ell c}{q_j B}\frac{D}{D\alpha}(P_{\perp j}B) + \frac{\ell^2 c^2}{2q_j^2}\frac{1}{B}\frac{\partial}{\partial\alpha}B\frac{D}{D\alpha}L_j\Bigg]$$
$$\to 2\pi\int\frac{dsd\alpha n_j m_j}{B\ell^2}\left\{\omega^2\left\{\frac{\ell^2}{B^2r^2}\phi^2 + r^2B^2\left[\frac{\partial}{\partial\alpha}\left(\frac{\phi}{B}\right)\right]^2\right\} + A_j\left\{\left[\frac{\partial}{\partial\alpha}\left(\frac{\phi}{Br}\right)\right]^2 B^2r^4 + (\ell^2-1)\frac{\phi^2}{B^2r^2}\right\}\right\} \tag{44}$$

where

$$A_j = \omega_E^2 - 2\omega_E\omega - \frac{(\omega-\omega_E)}{n_j}\frac{\ell c}{q_j B}\frac{D}{D\alpha}(P_{\perp j}B) - \frac{\ell^2 c^2}{2n_j q_j^2}\frac{1}{B}\frac{\partial B}{\partial\alpha}\frac{D}{D\alpha}L_j.$$

We also note that even though the non-eikonal studies assumed $\partial\chi/\partial s = \partial\phi/\partial s$, we can assert that the FLR corrections only depend on ϕ, not on χ. This is because in the most primitive form of the Vlasov equation, all χ dependence was proportional to $\partial/\partial s$, while the FLR corrections does not have any explicit dependence on $\partial/\partial s$. The FLR terms must be proportional only to ϕ even when $\partial\phi/\partial s \neq \partial\chi/\partial s$. This is of course explicitly derived in our eikonal form given in Eq. (18).

In our theory we have neglected the FLR contribution proportional to $\partial(\phi-\chi)/\partial s$ as we assumed that this contribution is small compared to the lowest order response in $\partial(\phi-\chi)/\partial s$.

In a recent work by Berk and Wong, [16] a non-eikonal analysis is considered when $r^2 Q_L/\phi \approx \beta$. In this case the Q_L term needs to be included in the inertia term. Then assuming the FLR correction in Q_L is unimportant, we find on comparing the inertia terms of Eqs. (18) with the inertia term of Ref. 16, that this term needs to be further modified to,

$$\frac{2\pi}{\ell^2}\int\frac{ds}{B}d\alpha n_j m_j\omega^2\left\{\frac{\ell^2}{B^2r^2}\phi^2 + r^2\left[B\frac{\partial}{\partial\alpha}(\phi/B)\right]^2\right\}$$
$$\to \frac{2\pi}{\ell^2}\int\frac{dsd\alpha}{B}n_j m_j\omega^2\left\{r^2\left[B\frac{\partial}{\partial\alpha}(\phi/B) + \frac{Q_L}{B}\right]^2 + \frac{\ell^2\phi^2}{B^2r^2}\right\}. \tag{45}$$

Now, incorporating all the changes we have made, we find that the quadratic form governing the perturbed response of an axisymmetric mirror machine is given by,

$$
\begin{aligned}
W = \pi \int \frac{d\alpha ds}{B} \Bigg\{ & \frac{\sigma}{4\pi}\left[\frac{1}{r^2}\left(\frac{\partial \chi}{\partial s}\right)^2\right] + \frac{B^2 \sigma r^2}{4\pi \ell^2}\left[\frac{\partial}{\partial s}\left(\frac{\partial \chi}{\partial \alpha} + \frac{Q_L}{B} - \frac{\phi}{B}\frac{\partial B}{\partial \alpha}\right)\right]^2 \\
& + \frac{\tau}{4\pi}\left(Q_L - \frac{\sigma\kappa}{\tau}\frac{\phi}{r}\right)^2 - \frac{\kappa\phi^2}{Br}\left(\frac{\sigma}{\tau}\frac{\partial}{\partial \alpha}P_\perp + \frac{\sigma}{\tau}\frac{\partial \Phi}{\partial \alpha}\frac{\partial P_\perp}{\partial \Phi} + \frac{\partial}{\partial \alpha}P_\parallel\right) \\
& - \sum_j n_j m_j \omega^2 \left\{\frac{r^2}{\ell^2}\left[B\frac{\partial}{\partial \alpha}(\phi/B) + \frac{Q_L}{B}\right]^2 + \frac{\phi^2}{B^2 r^2}\right\} \\
& - \sum_j n_j m_j A_j \left\{ r^4 B^2 \left[\frac{\partial}{\partial \alpha}\left(\frac{\phi}{\ell r B}\right)\right]^2 + \left(1 - \frac{1}{\ell^2}\right)\frac{\phi^2}{B^2 r^2}\right\}\Bigg\} + W_{\text{kin}}
\end{aligned}
\tag{46}
$$

with

$$
\begin{aligned}
W_{\text{kin}} = i \sum_j \left(\frac{2\pi}{\ell c m_j}\right)^2 \int d\alpha d\epsilon d\mu \left(\omega \frac{\partial F_j}{\partial \epsilon} + \frac{\ell c}{q_j}\frac{\partial F_j}{\partial \alpha}\right) \int_0^T dt \int_{-\infty}^{\infty} dt' \exp\left[i\Omega(t,t')\mathrm{sg}(t-t')\right] \\
\cdot \left[-i\ \mathrm{sg}(t-t') q_j \frac{d}{dt'}\left(\phi(t') - \chi(t')\right) + \ell c \mu Q_L(t') + q_j \phi(t')\omega_\kappa\right] \\
\left[i\ \mathrm{sg}(t-t') q_j \frac{d}{dt}\left(\phi(t) - \chi(t)\right) + \ell c \mu Q_L(t) + q_j \phi(t)\omega_\kappa\right].
\end{aligned}
\tag{47}
$$

We also note that

$$
\frac{\sigma}{\tau}\frac{\partial}{\partial \alpha}P_\perp + \frac{\sigma}{\tau}\frac{\partial \Phi}{\partial \alpha}\frac{\partial P_\perp}{\partial \alpha} + \frac{\partial P_\parallel}{\partial \alpha} = \frac{D}{D\alpha}(P_\perp + P_\parallel)[1 + \mathcal{O}(\kappa r)].
$$

Appendix A

In the reduction of Eq. (15) to Eq. (17) the kinetic integrals which appear can be expressed as moments and derivatives of moments of the equilibrium distribution function. We list below some of the identities used in obtaining this reduction,

$$n_j(\alpha, \Phi, B) \equiv \int d^3v F_j = \frac{4\pi B}{m_j^{3/2}} \int_{q\Phi}^{\infty} d\epsilon \int_0^{(\epsilon - q\Phi)/B} d\mu \frac{F_j(\epsilon, \mu, \alpha)}{2^{1/2}(\epsilon - \mu B - q_j\Phi)^{1/2}}$$

$$\frac{\partial n_j}{\partial \alpha} = \int d^3v \frac{\partial F_j}{\partial \alpha}$$

$$\frac{\partial n_j}{\partial \Phi} = q_j \int d^3v \frac{\partial F_j}{\partial \epsilon}$$

$$\frac{\partial n_j}{\partial B} = -\int d^3v \frac{\mu}{B}\frac{\partial F_j}{\partial \mu} = \frac{n_j}{B} + \int d^3v \mu \frac{\partial F_j}{\partial \epsilon}$$

$$P_\perp(\alpha, \Phi, B) \equiv \sum_j P_{\perp_j} \equiv \sum_j \int d^3v F_j \mu B$$

$$\frac{\partial P_\perp}{\partial \Phi} = \sum_j q_j \int d^3v \mu B \frac{\partial F_j}{\partial \epsilon}$$

$$\frac{\partial P_\perp}{\partial B} = -\sum_j \int d^3v \mu^2 \frac{\partial F_j}{\partial \mu}$$

$$\frac{\partial P_\perp}{\partial \alpha} = \sum_j \int d^3v \mu B \frac{\partial F_j}{\partial \alpha}$$

$$P_{||}(\alpha, \Phi, B) \equiv \sum_j P_{||_j} \equiv \sum_j \int d^3v 2(\epsilon - \mu B - q_j\Phi) F_j$$

$$\frac{\partial P_{||}}{\partial B} = \frac{P_{||} - P_\perp}{B} = -\sum_j m_j \int \frac{d^3v}{B} \mu \frac{\partial F}{\partial \mu} 2(\epsilon - \mu B - q_j\Phi)$$

$$\frac{\partial P_{||}}{\partial \Phi} = -\sum_j q_j n_j = 0$$

$$\frac{\partial P_{||}}{\partial \alpha} = \sum_j \int d^3v 2(\epsilon - \mu B - q_j\Phi) \frac{\partial F_j}{\partial \alpha}$$

$$L_j(\alpha, \Phi, B) \equiv \int d^3v (\mu B)^2 F_j$$

$$\frac{\partial L_j}{\partial \Phi} = q_j B^3 \frac{\partial}{\partial B}\left(\frac{P_{\perp_j}}{B^2}\right)$$

$$B\frac{\partial}{\partial B}\left(\frac{L_j}{B^3}\right) = \int d^3v \mu^3 \frac{\partial F_j}{\partial \epsilon}$$

$$\frac{1}{B}\frac{\partial L_j}{\partial B} = \int d^3v \mu^3 \frac{\partial F}{\partial \mu}$$

$$\frac{\partial L_j}{\partial \alpha} = \int d^3v (\mu B)^2 \frac{\partial F_j}{\partial \alpha}$$

$$\frac{D}{D\alpha} \equiv \frac{\partial}{\partial \alpha} + \frac{\partial \Phi}{\partial \alpha}\frac{\partial}{\partial \Phi} + \frac{\partial B}{\partial \alpha}\frac{\partial}{\partial B}. \qquad (A-1)$$

By way of illustration we derive the FLR term

$$T \equiv -\frac{m_j k_\perp^2}{2B^2}\phi^2 \frac{c^2}{q_j^2}\frac{1}{B}\frac{\partial B}{\partial \alpha}\frac{DL_j}{D\alpha}. \qquad (A-2)$$

After expanding the Bessel functions $J_0 = 1 - x^2/4$, $J_1 = x(1 - x^2/8)/2$, a number of terms contribute to T:

$$\begin{aligned} T_1 &= \int d^3v \frac{k_\perp^2 v_\perp^2}{2\omega_{c_j}^2}\left(\frac{\omega_E}{\omega}\frac{\partial F_j}{\partial \epsilon} + \frac{\ell c}{q_j \omega}\frac{\partial F_j}{\partial \alpha}\right)\frac{\omega_B}{\omega} q^2 \psi^2 \\ &= \frac{k_\perp^2 m_j c^2}{B^2}\psi^2 \left(\frac{\ell c}{q_j \omega}\right)^2 \left[\frac{\partial \Phi}{\partial \alpha}\frac{1}{B}\frac{\partial L_j}{\partial \Phi}\frac{\partial B}{\partial \alpha} + \frac{1}{B}\frac{\partial B}{\partial \alpha}\frac{\partial L_j}{\partial \alpha}\right] \end{aligned} \qquad (A-3)$$

$$\begin{aligned} T_2 &= \frac{3}{4}\int d^3v \frac{k_\perp^2 v_\perp^2}{\omega_{c_j}^2}\mu B_\parallel q_j \psi \left[\frac{\ell c}{q_j \omega}\frac{\partial F_j}{\partial \alpha} - \frac{\omega_E}{\omega}\frac{1}{B}\frac{\partial F_j}{\partial \mu}\right] \\ &= \frac{3}{2}\frac{k_\perp^2 m_j c^2}{B^2} B_\parallel \psi \left[\frac{\ell c}{q_j^2 \omega}\frac{1}{B}\frac{\partial L_j}{\partial \alpha} + \frac{\ell c}{q_j \omega}\frac{\partial \Phi}{\partial \alpha}\frac{\partial P_{\perp_j}}{\partial B}\right] \end{aligned} \qquad (A-4)$$

and

$$T_3 = \frac{1}{4}\int d^3v \mu^2 B_\parallel^2 \frac{k_\perp^2 v_\perp^2}{\omega_{c_j}^2}\frac{1}{B}\frac{\partial F_j}{\partial \mu} = \frac{k_\perp^2 m_j c^2}{q_j^2 B^2} B_\parallel^2 \frac{1}{2B^2}\frac{\partial L}{\partial B}. \qquad (A-5)$$

Changing variables to

$$Q_L = B_\parallel + \phi \frac{\partial B}{\partial \alpha}$$

$$\phi = \frac{\ell c \psi}{\omega} \qquad (A-6)$$

and examining only the ϕ^2 terms, we obtain

$$T_1 + T_2 + T_3 = \frac{k_\perp^2 m_j c^2}{q_j^2 B^2}\phi^2 \left\{ \left[\frac{1}{B}\frac{\partial B}{\partial \alpha}\left(\frac{\partial \Phi}{\partial \alpha}\frac{\partial L_j}{\partial \Phi} + \frac{\partial L_j}{\partial \alpha} \right) - \frac{3}{2}\frac{1}{B}\frac{\partial B}{\partial \alpha}\left(\frac{\partial L}{\partial \alpha} + \frac{\partial \Phi}{\partial \alpha}\frac{\partial L_j}{\partial \Phi} \right) \right.\right.$$

$$\left.\left. - \frac{1}{2}\left(\frac{\partial B}{\partial \alpha} \right)^2 \frac{1}{B}\frac{\partial L_j}{\partial B} \right] \right\} - 3\frac{k_\perp^2 m_j c \phi^2}{B^2 q_j \ell}\omega_E \frac{P_\perp}{B}\frac{\partial B}{\partial \alpha}. \qquad (A-7)$$

The final term in Eq. (A-7) contributes to part of the term proportional to $(\omega - \omega_E)(\ell c/q_j)(P_\perp/B)(\partial B/\partial \alpha)$ in the full FLR expression. Summing the remaining terms in Eq. (A-7) and using the chain rule for $DL_j/D\alpha$ yields Eq. (A-2).

Appendix B

We present below an alternate form for the reduction of Eq. (1). In this form, terms proportional to $Q_L k_\perp^2 a_i^2$ are retained. Previously, in reducing the non-local term, D_0, in Eq. (1), we separated the term, $A_j(t)$ of Eq. (3) into two parts. If we do not make this separation, but instead integrate the total $A_j(t)$ term by parts, the following quadratic expression is obtained,

$$W = W_{\text{local}} + \tilde{W}_{\text{local}} + \tilde{W}_{\text{kin}}$$

where W_{local} is given by Eq. (15),

$$\tilde{W}_{\text{local}} = -\int \frac{ds}{B} \sum_j \int d^3v \left(\omega \frac{\partial F_j}{\partial \epsilon} + \frac{\ell c}{q_j} \frac{\partial F_j}{\partial \alpha} \right) \frac{1}{(\omega - \omega_{d_j})} \cdot \left[\frac{q_j B_\|}{k_\perp c} \left(\frac{2\mu B}{m_j} \right)^{1/2} J_1 + \frac{(\omega_{B_j} + \omega_{\kappa_j})}{\omega} q_j \psi J_0 \right]^2 \qquad (B-1)$$

$$\tilde{W}_{\text{kin}} = i \sum_j \frac{2\pi}{m_j^2} \int d\epsilon d\mu \left(\omega \frac{\partial F_j}{\partial \epsilon} + \frac{\ell c}{q_j} \frac{\partial F_j}{\partial \alpha} \right) \cdot \int_{t^-}^{t^+} dt \frac{d}{dt} \left(\frac{A_j(t)}{\omega - \omega_{d_j}(t)} \right) \int_{-\infty}^{\infty} dt' \exp[i\Omega(t,t')\text{sg}(t-t')] \frac{d}{dt'} \left(\frac{A_j(t')}{(\omega - \omega_{d_j}(t'))} \right).$$

(B - 2)

We now evaluate the velocity integrals in W_{local} assuming that $k_\perp a_j < 1$ with a_j the Larmor radius. In the finite Larmor radius terms, all terms proportional to $(\kappa r) k_\perp^2 a_j^2$ are neglected although terms proportional to $Q_L k_\perp^2 a_j^2$ are retained. Curvature terms are are retained to all orders in the zero Larmor radius terms. In addition we assume that $r^{-2} B^{-1} (\partial Q/\partial\theta)/(\partial Q/\partial\alpha) \propto \mathcal{O}(\kappa r)$ where $Q(\alpha, \theta)$ is an equilibrium quantity. Thus to the order of this calculation we may neglect terms proportional to $(\partial Q/\partial\theta)$ in the finite Larmor radius corrections. With these approximations we find $W = W_{\text{local}} + W_{FLR} + \tilde{W}_{\text{local}} + \tilde{W}_{\text{kin}}$, with

$$W_{\rm local} = \int_{-\infty}^{+\infty} \frac{ds}{B}\left\{ \frac{1}{4\pi}\frac{\sigma k_\perp^2}{\ell^2}\left(\frac{\partial \chi}{\partial s}\right)^2 + \frac{\tau}{4\pi}\left(Q_L - \frac{\sigma}{\tau}B\phi\left(\kappa_\alpha - \frac{\lambda}{\ell}\kappa_\theta\right)\right)^2 \right.$$

$$\left. - \phi^2\frac{\sigma}{\tau}\left[\frac{\partial P_\perp}{\partial \alpha} + \frac{\partial P_\perp}{\partial \Phi}\frac{\partial \Phi}{\partial \alpha}\right]\left(\kappa_\alpha - \frac{\lambda}{\ell}\kappa_\theta\right) - \phi^2\frac{\partial P_\parallel}{\partial \alpha}\left(\kappa_\alpha - \frac{\lambda}{\ell}\kappa_\theta\right)\right\}$$

$$W_{FLR} = -\int_{-\infty}^{+\infty} \frac{ds}{B}\sum_j \frac{m_j k_\perp^2}{B^2\ell^2}\left\{\phi^2\left[n_j(\omega-\omega_E)^2 - (\omega-\omega_E)\frac{\ell c}{q_j B}\frac{D(P_{\perp_j}B)}{D\alpha}\right.\right.$$

$$\left. + \frac{1}{2}\frac{\ell^2 c^2}{q_j^2}\frac{1}{B}\frac{\partial B}{\partial \alpha}\frac{DL_j}{D\alpha}\right]$$

$$- \phi Q_L\left[(\omega-\omega_E)\frac{\ell c}{q_j}3\frac{P_{\perp_j}}{B} + \frac{3}{2}\frac{\ell^2 c^2}{q_j^2}\frac{1}{B}\frac{DL_j}{D\alpha} + \frac{1}{2}\frac{\ell^2 c^2}{q_j^2}\frac{1}{B}\frac{\partial B}{\partial \alpha}\frac{\partial L_j}{\partial B}\right]$$

$$\left. + \frac{1}{2}Q_L^2\frac{\ell^2 c^2}{q_j^2}\frac{1}{B}\frac{\partial L_j}{\partial B}\right\}$$

$$\tilde{W}_{\rm local} = -\int \frac{ds}{B}\sum_j \int d^3v\left(\omega\frac{\partial F_j}{\partial \epsilon} + \frac{\ell c}{q_j}\frac{\partial F_j}{\partial \alpha}\right)\frac{1}{\omega - \omega_{dj}}$$

$$\left[\left(\mu Q_L + m_j v_\parallel^2\phi\left(\kappa_\alpha - \frac{\lambda}{\ell}\kappa_\theta\right)\right)^2\right.$$

$$\left. - \frac{k_\perp^2(2\mu B)}{4\omega_{cj}^2 m_j}\mu Q_L\left(\mu Q_L + \mu\phi\frac{\partial B}{\partial \alpha}\right)\right]$$

$$\tilde{W}_{\rm kin} = i\sum_j \frac{2\pi}{(\ell c m_j)^2}\int d\epsilon d\mu\left(\omega\frac{\partial F_j}{\partial \epsilon} + \frac{\ell c}{q_j}\frac{\partial F_j}{\partial \alpha}\right)$$

$$\times \int_{t^-}^{t^+} dt\frac{d}{dt}\left[q_j\left(\phi(t) - \chi(t)\right) + \frac{\mu\ell c Q_L(t)}{\omega - \omega_{d_j}(t)}\right.$$

$$\left. + \phi(t)\frac{m v_\parallel^2(t)\ell c\left(\kappa_\alpha(t) - \lambda\kappa_\theta(t)/\ell\right)}{\omega - \omega_{d_j}(t)}\right]$$

$$\times \int_{-\infty}^{+\infty} dt' \exp\left[i\Omega(t,t')\mathrm{sg}(t-t')\right]\frac{d}{dt'}\left[q_j\left(\phi(t') - \chi(t')\right) + \frac{\mu\ell c Q_L(t')}{\omega - \omega_{d_j}(t')}\right.$$

$$\left. + \phi(t')\frac{m v_\parallel^2(t')\ell c\left(\kappa_\alpha(t') - \lambda\kappa_\theta(t')/\ell\right)}{\omega - \omega_{d_j}(t')}\right]. \qquad (B-3)$$

We note that $\tilde{W}_{\rm kin}$ vanishes for a flute except for terms proportional to $(\partial\omega_{d_j}(t)/\partial t)$.

The quantity $\tilde{W}_{\rm local}$ cannot generally be evaluated in closed form and we thus consider two limits. In the absence of a hot, fast drifting species the mode frequency will be larger than the particle drift frequencies and thus we take $\omega/\omega_d > 1$. In this limit $\tilde{W}_{\rm local}$ contributes order β corrections to the coefficient of Q_L^2 as well as a stabilizing "compressional" term proportional to $(\kappa r)^2$. Explicitly, neglecting FLR terms,

$$\tilde{W}_{\rm local} = -\int \frac{ds}{B}\left[Q_L^2 B\frac{\partial}{\partial B}\left(\frac{P_\perp}{B^2}\right) - \frac{2Q_L\phi P_\perp}{B}\left(\kappa_\alpha - \frac{\lambda}{\ell}\kappa_\theta\right)\right.$$
$$\left. - \phi^2 3p_\parallel\left(\kappa_\alpha - \frac{\lambda}{\ell}\kappa_\theta\right)^2\right]. \qquad (B-4)$$

Combining this expression with $W_{\rm local}$ gives

$$W_{\rm local} + \tilde{W}_{\rm local} = \int \frac{ds}{B}\left\{\frac{\sigma k_\perp^2}{4\pi\ell^2}\left(\frac{\partial\chi}{\partial s}\right)^2\right.$$
$$+ \frac{(1+\beta_\perp)}{4\pi}\left(Q_L - \phi B\left(\kappa_\alpha - \frac{\lambda}{\ell}\kappa_\theta\right)\frac{(1-\beta_\parallel/2)}{(1+\beta_\perp)}\right)^2 + \phi^2\left[-\left(\kappa_\alpha - \frac{\lambda}{\ell}\kappa_\theta\right)\frac{D}{D\alpha}(P_\perp + P_\parallel)\right.$$
$$\left.\left. + B^2\left(\kappa_\alpha - \frac{\lambda}{\ell}\kappa_\theta\right)^2\left(\sigma + \frac{3}{2}\beta_\parallel - \frac{(1-\beta_\parallel/2)^2}{(1+\beta_\perp)}\right)\right]\right\} \qquad (B-5)$$

where $\beta_{\perp,\parallel} \equiv 8\pi P_{\perp,\parallel}/B^2$. This result reproduces the zero Larmor radius result for flutes in a z-pinch.[17]

In the presence of a hot species for which $\omega < \omega_{B_j}$, it is convenient to introduce the auxilliary function $C(\alpha, s)$ defined by

$$Q_L \equiv C(\alpha, s)\frac{4\pi}{B}\frac{DP_\perp}{D\alpha}. \qquad (B-6)$$

In the case of hot particles the terms proportional to Q_L^2 in $W_{\rm local}$ and $\tilde{W}_{\rm local}$ cancel to order (κr) if the core β of the background plasma is $O(\kappa r)$. Thus while in the case considered above where $\omega > \omega_B$, minimization with respect to Q_L gives the result that $Q_L \sim \phi\kappa/r$ and is thus small, in present case $Q_L \sim \phi 4\pi B^{-1}(DP_\perp^{\rm hot}/D\alpha)$. Explicitly for $\omega_{Bh} > \omega$ and with finite Larmor radius corrections, $W_{\rm local}$ and $\tilde{W}_{\rm local}$ combine to give

$$W_{\rm local} + \tilde{W}_{\rm local} = \int \frac{ds}{B}\left\{\frac{1}{4\pi}\frac{\sigma k_\perp^2}{\ell^2}\left(\frac{\partial\chi}{\partial s}\right)^2 - (C+\phi)^2\frac{D}{D\alpha}(P_\perp + P_\parallel)\left(\kappa_\alpha - \frac{\lambda}{\ell}\kappa_\theta\right)\right.$$

$$+\sum_j \frac{k_\perp^2 m_j c^2}{2q_j^2 B^2}\frac{1}{B}\frac{DL_j}{D\alpha}(C+\phi)^2\frac{4\pi}{B}\frac{DP_\perp}{D\alpha}$$

$$+C^2\left[\frac{D}{D\alpha}\left(\frac{P_{\perp c}}{B^2}\right)\frac{DP_{\perp h}}{D\alpha}+\frac{c\omega q_h B}{\ell}\frac{D}{D\alpha}\left(\frac{n_h}{B}\right)\right]-\frac{k_\perp^2 n_j m_j}{\ell^2 B^2}\phi^2\omega^2\Bigg\}. \qquad (B-7)$$

where the subscript h refers to the hot species and the subscript c refers to the background species for which $\omega > \omega_{Bc}$. This result has been obtained in Ref. 19 when $\partial(\phi-\chi)/\partial s = 0$.

Acknowledgments

We are indebted to H. Vernon Wong for fruitful discussions. This work was supported by the U.S. Department of Energy, Contract no. DE-FG05-80ET-53088.

References

1. H.V. Wong, Phys. Fluids **25**, 1811 (1982).
2. M.D. Kruskal, and C.R. Oberman, Phys. Fluids **1**, 275 (1958).
3. M.N. Rosenbluth and N. Rostoker, Phys. Fluids **2**, 23 (1959).
4. M. Rosenbluth and A. Simon, Phys. Fluids **8**, 1300 (1965).
5. W. Newcomb, J. Plasma Physics **26**, 529 (1981).
6. T.M. Antonsen, Jr., B.G. Lane, and J.J. Ramos, Phys. Fluids **24**, 1465 (1981).
7. J.W. Van Dam, M.N. Rosenbluth, and Y.C. Lee, Phys. Fluids **25**, 1349 (1982).
8. T.M. Antonsen, Jr. and Y.C. Lee, Phys. Fluids **25**, 132 (1982).
9. J. Kesner and B. Lane, Phys. Fluids **28**, 634 (1985).
10. M. Kotschenreuther, Institute for Fusion Studies Report #172, The University of Texas at Austin.
11. P.H. Rutherford and E.A. Frieman, Phys. Fluids **11**, 569 (1968).
12. J.A. Byers and R.H. Cohen, Phys. Fluids **28**, 1591 (1985).
13. H.L. Berk, M.N. Rosenbluth, R.C. Cohen and W.M. Nevins, Institute for Fusion Studies Report no. 160, submitted to Phys. Fluids.
14. H.L. Berk and L.D. Pearlstein, Phys. Fluids **14**, 1810 (1971).
15. R.R. Dominguez and H.L. Berk, Phys. Fluids **27**, 1142 (1984).
16. L.D. Pearlstein and N.A. Krall, Phys. Fluids, **9**, 2231 (1966).
17. H.L. Berk and H.V. Wong, Institute for Fusion Studies Report no. 131, to be published in Phys. Fluids.
18. H.L. Berk, J.W. Van Dam, M.N. Rosenbluth and D.A. Spong **26**, 201 (1983).
19. H.L. Berk, H.V. Wong, et al., in Proceedings of the Conference on Plasma on Plasma Physics and Controlled Fusion, 1984 (in press).

Figure Caption

Axial variation of magnetic field, ambipolar potential and electric field drift frequency about midplane of a model tandem mirror configuration.

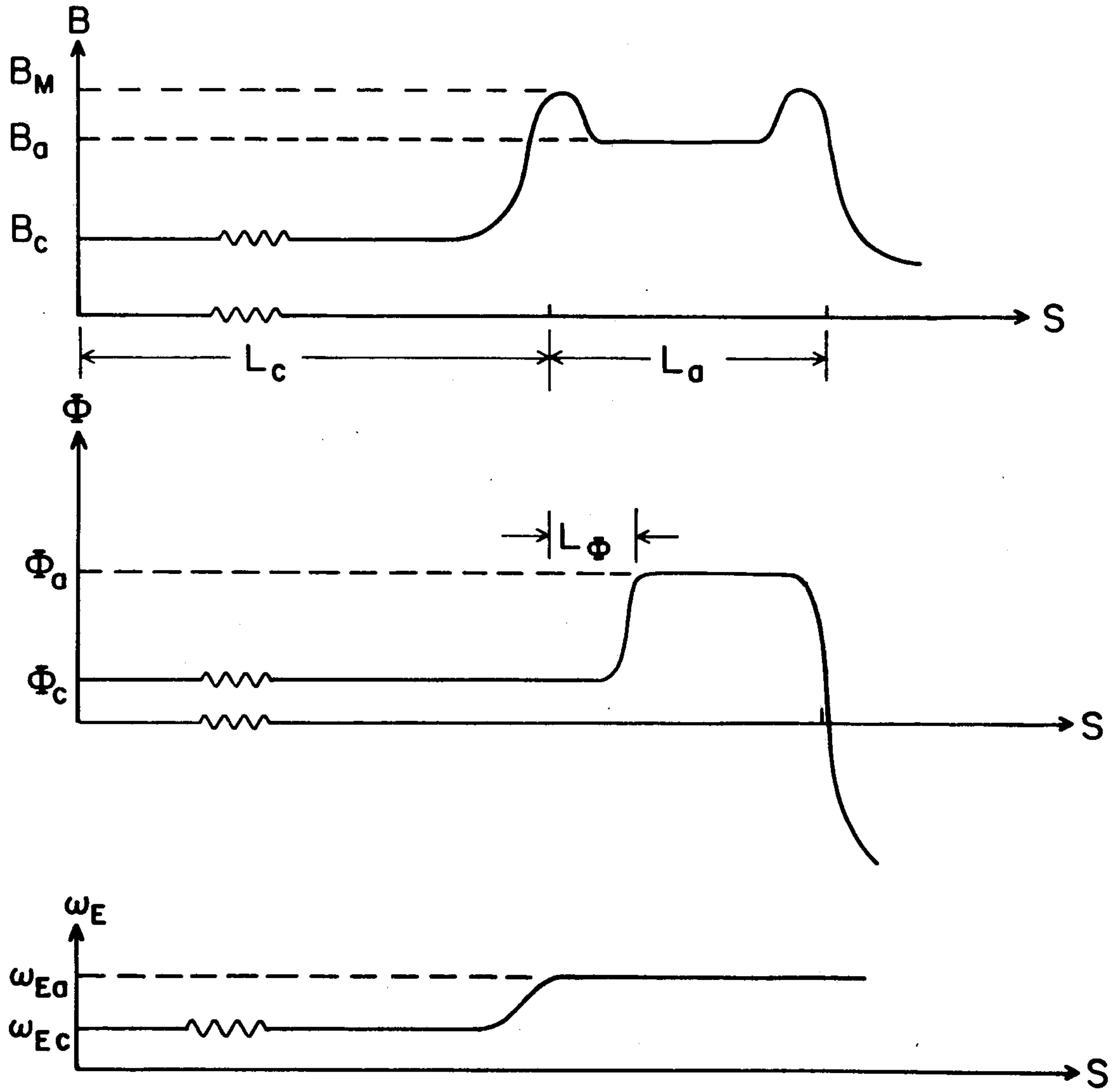
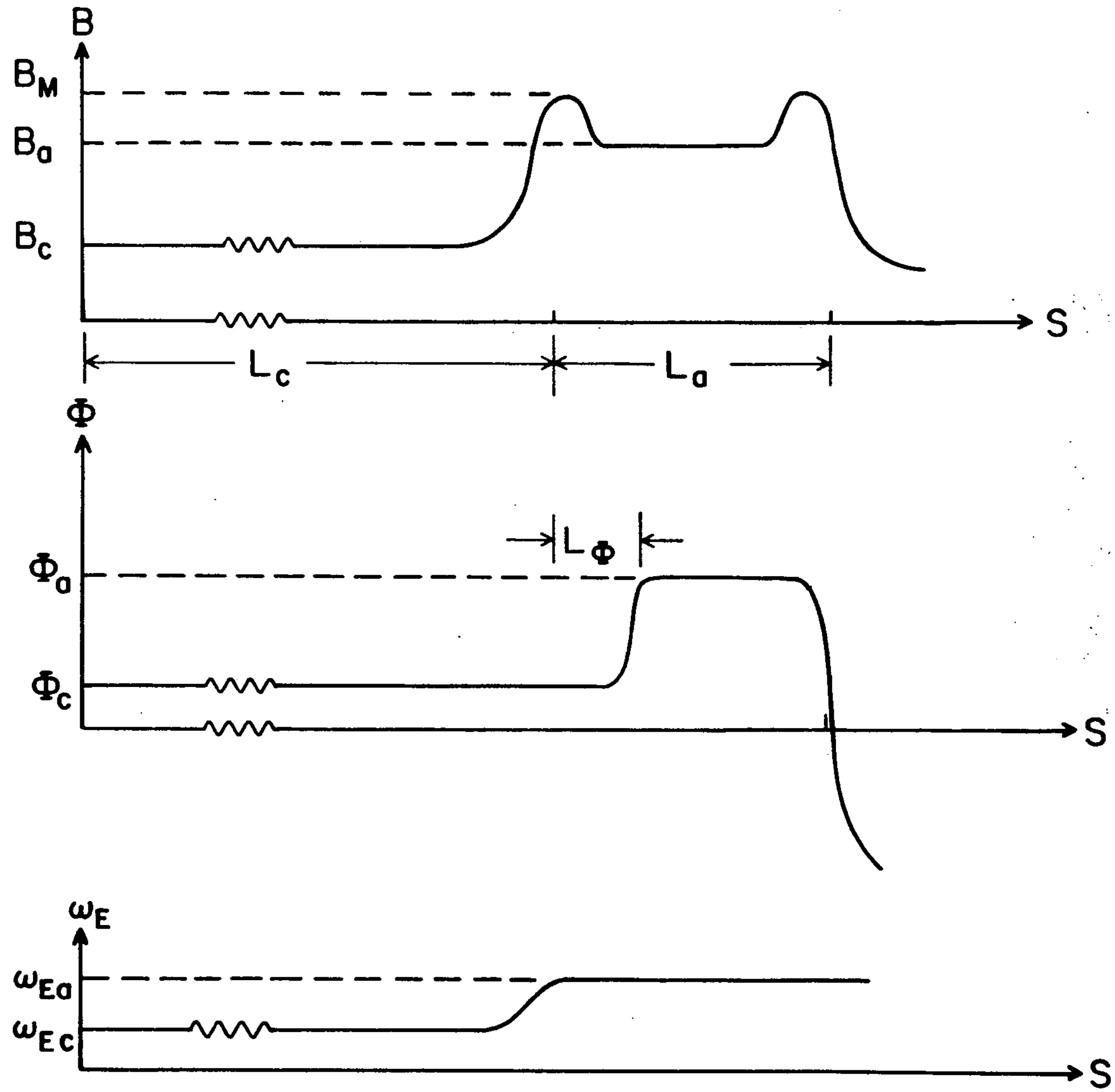
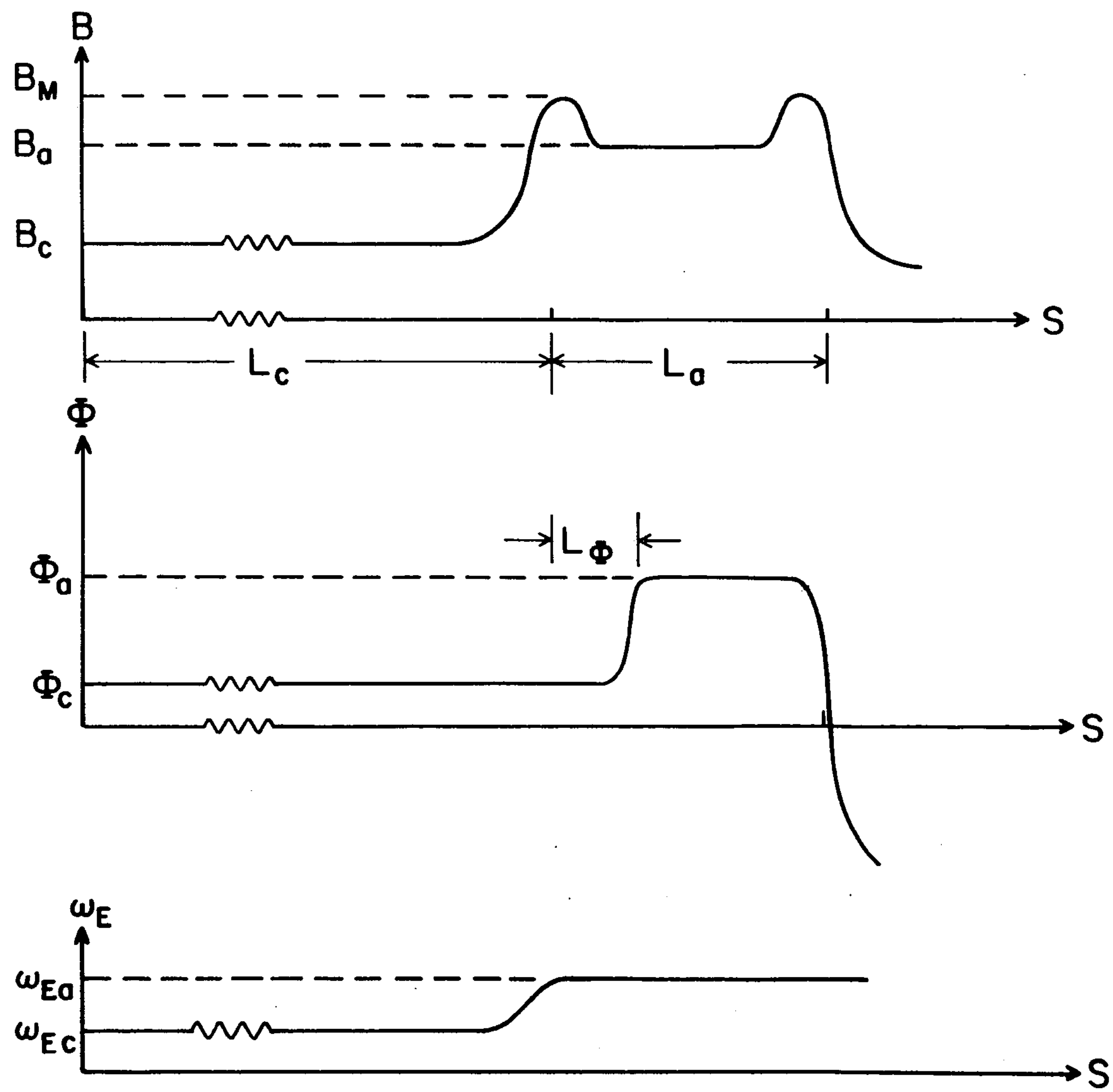


Fig. 1



B
B_M
B_a
B_c
S
L_c
L_a
Φ
L_Φ
Φ_a
Φ_c
S
ω_E
ω_{Ea}
ω_{Ec}
S



B
B_M
B_a
B_c
S
L_c
L_a
Φ
L_Φ
Φ_a
Φ_c
S
ω_E
ω_Ea
ω_Ec
S