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COVARIANT POISSON BRACKETS FOR CLASSICAL FIELDS

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ABSTRACT

Poisson brackets that are covariant under spacetime coordinate changes are presented for relativistic field theories. The formalism described here is an alternative to the symplectic formulation of field theories and has several advantages. It applies to relativistic fluids and plasmas written in Eulerian variables, while the symplectic formulation does not. It is expected to simplify and clarify the transition to the dynamical (or 3+1) Hamiltonian formalism as well.

1. Introduction

Recall that the canonical Hamiltonian equations

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i} \tag{1.1}$$

can be written as $\delta S = 0$, where the action

$$S[\gamma] = \int (p_i \dot{q}^i - H(q,p))\, dt \tag{1.2}$$

is regarded as a functional on Γ, the space of paths $\gamma(t) = (q(t), p(t))$ in phase space with appropriate boundary conditions (see e.g. Arnold [1978], p. 243). Let us rewrite this variational principle in terms of a Poisson bracket on Γ. For functionals F and G of paths γ, set

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$$\{F,G\}(\gamma) = \int\left(\frac{\delta F}{\delta q^i}\frac{\delta G}{\delta p_i} - \frac{\delta G}{\delta q^i}\frac{\delta F}{\delta p_i}\right)dt \tag{1.3}$$

where the functional derivatives are defined by

$$\frac{d}{ds}F(\gamma + s\delta\gamma)\Big|_{s=0} = \int\left(\frac{\delta F}{\delta\gamma}\,\delta\gamma\right)dt = \int\left(\frac{\delta F}{\delta q^i}\,\delta q^i + \frac{\delta F}{\delta p_i}\,\delta p_i\right)dt \tag{1.4}$$

for variations $\delta\gamma$ vanishing at the endpoints of γ. It is straightforward to check that $\delta S[\gamma] = 0$; i.e. γ solves Hamilton's equation if and only if

$$\{F,S\}(\gamma) = 0 \tag{1.5}$$

for all functionals F. It is this variational principle for Hamilton's equations that we shall generalize and apply to field theory. The covariant theory does not, of course, single out a time direction; rather space and time occur on equal footing.

We shall show that the Euler-Lagrange equations for a classical Lagrangian field theory can be recast in the form

$$\{F,S\}_\mu(z) = 0 \tag{1.6}$$

where $\mu = 0,1,2,3,$ is a spacetime index, $z = (\phi^A, \pi^\nu_A)$ is a field and its covariant conjugate momentum, and the Lagrangian S is a covariant version of (1.2) The operation $F \mapsto \{F,S\}_\mu$ may be viewed as a variation of S along a direction in function space which is determined by F and the spacetime direction $\partial/\partial x^\mu$.

Field theories of the traditional Euler-Lagrange form can be analyzed by symplectic methods in a fairly well-developed way. See, for example, Dedecker [1977], Kijowski and Tulczyjew [1979] and Gimmsy [1985] and references therein. However, a number of important field theories do not fit this mold, just as the rigid body equations in body representation admit a simple Poisson description, but not so easily a symplectic one (since, for example, there are

3 equations); see for example, Sudarshan and Mukunda [1974] and Holmes and Marsden [1983] for an account. The same thing is true for fluids and plasmas in Eulerian description, as is now well-known. For reviews, see Morrison [1982], Holm and Kupershmidt [1983], Marsden, Weinstein, Ratiu, Schmid and Spencer [1983], Marsden and Morrison [1984], and Marsden [1984]. We shall show, however, that the relativistic version of these theories (either interacting with gravity or with a fixed background) do admit a simple covariant Poisson bracket description. For these theories written in Lagrangian (material) description or in a 3+1 dynamical formulation a canonical or Poisson bracket formulation is known; see Bialynicki-Birula and Iwinski [1973], Iwinski and Turski [1976], Bialynicki-Birula and Hubbard [1982], Kunzle and Nester [1984], Tulczyjew [1983], Holm and Kupershmidt [1984], Bao, Marsden and Walton [1984], and Holm [1985].

In Gimmsy [1985], it is shown, amongst other things, how to obtain the 3+1 adjoint Hamiltonian form of Fischer and Marsden [1976,1979] starting with a symplectic formulation of classical field theory. We expect that the results here give an alternative setting for the same procedures and a formulation that will allow for a direct passage to the brackets of Bao, Marsden and Walton [1984]. In addition, the incorporation of covariant momentum maps should be possible for these covariant Poisson structures, as well as a covariant version of the reduction procedure (Marsden and Weinstein [1974]). The latter would enable one, for example, to pass directly from a covariant Hamiltonian description of a relativistic fluid or plasma in material representation to one in space-time representation (see Holm [1985]).

The plan of the paper is as follows. We will first present Maxwell's equations, the relativistic Maxwell-Vlasov system, general relativity and general relativistic fluids as examples. The covariant Poisson bracket form is exhibited explicitly in each case. We note with these examples at hand additional cases, such as the Einstein-Maxwell, Yang-Mills or relativistic Liouville equations, are immediate. (For the non-relativistic Liouville equation, see Marsden, Morrison and Weinstein [1984]). We conclude with some remarks on how these results suggest a general formulation of classical field theory.

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2. Electromagnetism

To begin, we deal with Maxwell's equations on Minkowski spacetime. (The results are generalizable to arbitrary background spacetimes and to general gauge fields). Let A denote the four vector potential, thought of as a one-form on Minkowski space. Let

$$F = dA, \quad \text{i.e.} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.1}$$

be the electromagnetic field tensor, where $\partial_\mu = \partial/\partial x^\mu$, $\mu = 0,1,2,3$ and x^0, x^1, x^2, x^3 are the usual Minkowski coordinates.

The standard Lagrangian for the theory with an external current density J^μ is

$$L[A] = \int \mathcal{L} := \int (-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu J^\mu) \, d^4x \tag{2.2}$$

where indices are raised and lowered using the Minkowski metric. In order to define a Legendre transformation, we introduce the covariant momentum variables, $\pi^{\mu\nu}$ as follows:

$$\pi^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = F^{\mu\nu} . \tag{2.3}$$

The primary constraint manifold is defined to be the image of the map defined by (2.3), where $\mathcal{L}$ is regarded as defined on the space of A_μ and $\partial_\nu A_\mu$'s. This image space is the space of pairs of fields $(A_\mu, \pi^{\mu\nu})$ with $\pi^{\mu\nu}$ skew symmetric, and is our basic covariant phase space.

If F is a functional of A and π, the functional derivatives are defined as usual, being cautious about the constraint on $\pi^{\mu\nu}$ (just like one must be cautious about the div $B = 0$ constraint in the MHD and Maxwell-Vlasov equations). Namely, $\frac{\delta F}{\delta \pi^{\mu\nu}}$ is a <u>skew</u> tensor satisfying

$$\frac{d}{ds} F(\pi^{\mu\nu} + s\delta\pi^{\mu\nu})\Big|_{s=0} = \int \frac{\delta F}{\delta \pi^{\mu\nu}} \delta\pi^{\mu\nu} d^4x \tag{2.4}$$

for $\delta\pi^{\mu\nu}$ a skew symmetric perturbation.

The covariant Poisson bracket of two functions F and G of A_μ and $\pi^{\mu\nu}$ is defined by

$$\{F,G\}_V(A,\pi) = \int \left(\frac{\delta F}{\delta A_\mu} \frac{\delta G}{\delta \pi^{\mu\nu}} - \frac{\delta G}{\delta A_\mu} \frac{\delta F}{\delta \pi^{\mu\nu}} \right) V^\nu d^4x , \tag{2.3}$$

where V^ν is an arbitrary vector field on spacetime, and functional derivatives are defined as usual. (The vector field V is related to the passage to dynamical equations — see remark 3 in §6). The bracket (2.3) can be written as

$$\{F,G\}_V\ (A,\pi) = \int \{F,G\}_\nu V^\nu d^4x \qquad (2.4)$$

where

$$\{F,G\}_\nu = \frac{\delta F}{\delta A_\mu}\frac{\delta G}{\delta \pi^{\mu\nu}} - \frac{\delta G}{\delta A_\mu}\frac{\delta F}{\delta \pi^{\mu\nu}} \qquad (2.5)$$

is the associated density.

Let S be defined by the covariant analogue of (1.2), namely

$$S[A,\pi] = \int [\pi^{\mu\nu} A_{\mu,\nu} - H(A,\pi)]\, d^4x \qquad (2.6)$$

where

$$H(A,\pi) = \frac{1}{4}\pi_{\mu\nu}\pi^{\mu\nu} + A_\mu J^\mu \qquad (2.7)$$

We claim that Maxwell's equations are equivalent to

$$\{F,S\}_V(A,\pi) = 0 \qquad (2.8)$$

for all V and F. The statement (2.8) is clearly equivalent to

$$\frac{\delta S}{\delta \pi^{\mu\nu}} = 0 \quad \text{and} \quad \frac{\delta S}{\delta A_\mu} = 0 \qquad (2.9)$$

i.e. to

$$\pi^{\mu\nu} = -(\partial^\mu A^\nu - \partial^\nu A^\mu) \quad \text{and} \quad \pi^{\mu\nu}{}_{,\nu} = -J^\mu \qquad (2.10)$$

which, together with $F = dA$, are the Maxwell equations. [We remark that the choice $\hat{S}(A,\pi) = \int [\frac{1}{2}\pi^{\mu\nu}F_{\mu\nu} - H(A,\pi)]d^4x$ would have yielded skew symmetry of π as one of the consequences of (2.8), but (2.6) seems to be a more useful version for the general theory; in fact one has, in general, a fair amount of freedom in the choice of S. We have followed the analogue of the form (1.2) in all cases to remove any ambiguity.]

3. The Relativistic Maxwell-Vlasov Equations

A special relativistic particle moves in an external electromagnetic field $F = dA$ according to the Lorentz force law:

$$\frac{dx^\mu}{d\tau} = u^\mu; \quad \frac{du^\mu}{d\tau} = \frac{e}{m} F^{\mu\nu} u_\nu \tag{3.1}$$

where τ is the particle's proper time, e is its charge and m its rest mass. Declare

$$p_\mu = mu_\mu + \frac{e}{c} A_\mu \tag{3.2}$$

to be canonically conjugate to x^μ and set

$$H = \frac{m}{2} u^\mu u_\mu = \frac{1}{2m} (p^\mu - \frac{e}{c} A^\mu)(p_\mu - \frac{e}{c} A_\mu) . \tag{3.3}$$

Thus (3.1) are equivalent to Hamilton's equations

$$\frac{dx^\mu}{d\tau} = \frac{\partial H}{\partial p_\mu} ; \quad \frac{dp_\mu}{d\tau} = \frac{\partial H}{\partial x^\mu} = \frac{e}{c} u^\nu \frac{\partial A_\nu}{\partial x^\mu} . \tag{3.4}$$

A relativistic plasma density $f(x,p)\ d^4x d^4p$ is constant along its particles' world lines:

$$\frac{df}{d\tau} = \frac{\partial f}{\partial x^\mu} u^\mu + \frac{e}{c} \frac{\partial f}{\partial p_\mu} u^\nu \frac{\partial A_\nu}{\partial x^\mu} = 0 . \tag{3.5}$$

We may rewrite this as

$$\{f,H\}_{xp} = 0 \quad \text{where} \quad \{f,g\}_{xp} = \frac{\partial f}{\partial x^\mu} \frac{\partial g}{\partial p_\mu} - \frac{\partial g}{\partial x^\mu} \frac{\partial f}{\partial p_\mu} . \tag{3.6}$$

The basic field for the Vlasov theory is the plasma phase space density function. As in Iwinski and Turski [1976] and in the nonrelativistic case (Morrison [1980] and Marsden and Weinstein [1982]) we define the bracket of two functionals F, G of f to be of Lie-Poisson form:

$$\{F,G\}(f) = \int f\left\{\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right\}_{xp} d^4x\, d^4p\,. \tag{3.7}$$

Let

$$S[f] = \int f(x,p)\; H(x,p)\; d^4x d^4p \tag{3.8}$$

so $\delta S/\delta f = H$. An integration by parts shows that the covariant bracket equation

$$\{F,S\}(f) = 0 \tag{3.9}$$

is equivalent to the relativistic Vlasov equation (3.5) [or (3.6)].

The basic fields for the relativistic Maxwell-Vlasov equations are triples $(A_\mu, \pi^{\mu\nu}, f)$. The bracket of two functions of (A,π,f) is just the sum of (2.3) and (3.7):

$$\{F,G\}_V(A,\pi,f) = \int\left(\frac{\delta F}{\delta A_\mu}\frac{\delta G}{\delta \pi^{\mu\nu}} - \frac{\delta G}{\delta A_\mu}\frac{\delta F}{\delta \pi^{\mu\nu}}\right) V^\nu d^4x$$

$$+ \int f\left\{\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right\}_{xp} d^4x\, d^4p\,. \tag{3.10}$$

Let

$$S[A,\pi,f] = \int(\pi^{\mu\nu}A_{\mu,\nu} - \frac{1}{4}\pi_{\mu\nu}\pi^{\mu\nu})\, d^4x$$

$$+ \int f(x,p)\; \frac{1}{2m}\,(p_\mu - \frac{e}{c}A_\mu)(p^\mu - \frac{e}{c}A^\mu)\, d^4x d^4p. \tag{3.11}$$

The field equations are

$$\{F,S\}_V\,(A,\pi,f) = 0 \tag{3.12}$$

for all F and all V. These are obviously equivalent to

and

$$\left.\begin{aligned} &\frac{\delta S}{\delta \pi^{\mu\nu}} = 0 \quad \text{and} \quad \frac{\delta S}{\delta A_\mu} = 0 \\ &\int f\left\{\frac{\delta F}{\delta f}, \frac{\delta S}{\delta f}\right\}_{xp} d^4x\,d^4p = 0. \end{aligned}\right\} \tag{3.13}$$

These are, in turn, equivalent to the relativistic Maxwell-Vlasov equations:

$$\left.\begin{aligned} &\frac{\partial f}{\partial x^\mu} u^\mu + \frac{e}{c}\frac{\partial f}{\partial p_\mu} u^\nu \frac{\partial A_\nu}{\partial x^\mu} = 0 \\ &\partial_\mu F^{\mu\nu} = \frac{e}{c}\int u^\nu f(x,p)\, d^4p \\ &F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \end{aligned}\right\} \tag{3.14}$$

<u>Remarks</u>. 1. Here we have not mentioned the obvious physical constraint that f vanishes unless $u^\alpha u_\alpha = -1$. This can be treated a posteriori since it can be shown that if $f(x,p)$ is a solution of the relativistic Vlasov equation (3.5) defined on all of xp space, then $g(x,p) = f(x,p)\,\delta(u^\alpha u_\alpha + 1)$, where $u_\alpha = \frac{1}{m}\left(p_\alpha - \frac{eA_\alpha}{c}\right)$, also is a solution. Alternatively, this constraint can be treated by restricting to density functions concentrated on the mass shell at the outset.

2. The bracket (3.7) is literally the Lie-Poisson bracket for the group of canonical transformations on xp space, the cotangent

bundle of spacetime. Thus, this part of the bracket can be regarded as the reduction from canonical coordinates in Lagrangian representation by the particle relabeling group. In Lagrangian representation, the bracket has a form similar to (2.3); the vector field V^ν should dissappear during reduction because one relabels by world lines, not by points (x,p). This is part of a general covariant reduction process which is planned for future development.

3. Another reduction process that we plan to pursue is the elimination of the gauge freedom for electromagnetism via reduction. This should re-express the bracket in terms of $F^{\mu\nu}$ and f alone and build in the div E constraint. When expressed dynamically, this should reproduce the known bracket for relativistic plasmas (Iwinski and Turski [1976] and Bialynicki-Birula and Hubbard [1982]), and should coincide with the non-relativistic bracket (Morrison [1980], Marsden and Weinstein [1982]).

4. <u>General Relativity</u>

The basic field variables we use for general relativity are the contravariant symmetric two-tensor $g^{\alpha\beta}$ representing the dual metric and the "conjugate momenta" $\pi^\mu_{\alpha\beta}$, which are symmetric in α and β. We shall identify $\pi^\mu_{\alpha\beta}$ with the affine connection; this is standard, although somewhat awkward from the point of view of the Legendre transformation because of second derivatives of $g_{\alpha\beta}$ in the Lagrangian density (see Misner, Thorne and Wheeler [1973], Ch. 21 and Szczyrba [1976]). Because of this, we shall need to depart from the $p\dot{q}-L$ prescription somewhat.

The Poisson brackets are of the same form as (2.3), namely

$$\{F,G\}_V(g,\pi) = \int\left(\frac{\delta F}{\delta g^{\alpha\beta}}\frac{\delta G}{\delta\pi^{\mu}_{\alpha\beta}} - \frac{\delta G}{\delta g^{\alpha\beta}}\frac{\delta F}{\delta\pi^{\mu}_{\alpha\beta}}\right)V^{\mu}d^4x \tag{4.1}$$

Here functional derivatives are defined so that those with respect to $g^{\alpha\beta}$ are tensors:

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} F(g+\lambda\delta g) = \int \frac{\partial F}{\partial g^{\alpha\beta}}\ \delta g^{\alpha\beta}\ \sqrt{-g}\ d^4x$$

whereas those with respect to $\pi^{\mu}_{\alpha\beta}$ are tensor densities:

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} F(\pi+\lambda\delta\pi) = \int \frac{\delta F}{\delta\pi^{\gamma}_{\alpha\beta}}\ \delta\pi^{\gamma}_{\alpha\beta}\ d^4x\ .$$

The action is the usual one written in terms of $g^{\alpha\beta}$ and $\pi^{\mu}_{\alpha\beta}$:

$$S[g,\pi] = \int g^{\alpha\beta}R_{\alpha\beta}(\pi)\sqrt{-g}\ d^4x - 8\pi\int L^*\ \sqrt{-g}\ d^4x\ , \tag{4.2}$$

where $\sqrt{-g}\ d^4x$ is the volume element on spacetime,

$$R_{\alpha\beta} = \partial_\lambda\pi^{\lambda}_{\alpha\beta} - \partial_\beta\pi^{\lambda}_{\alpha\lambda} - \pi^{\lambda}_{\nu\beta}\pi^{\nu}_{\alpha\lambda} + \pi^{\lambda}_{\nu\lambda}\pi^{\nu}_{\alpha\beta} \tag{4.3}$$

and where

$$\frac{\delta}{\delta g^{\alpha\beta}}\int L^*\ \sqrt{-g}\ d^4x = T_{\alpha\beta}$$

is an (externally imposed) stress-energy tensor.

The covariant bracket equations are

$$\{F,S\}_V = 0 \quad \text{for all} \quad F,V: \quad \text{i.e.}$$

$$\frac{\delta S}{\delta g^{\alpha\beta}} = 0 \quad \text{and} \quad \frac{\delta S}{\delta \pi^{\mu}_{\alpha\beta}} = 0. \tag{4.4}$$

The first equation yields the field equations

$$G_{\alpha\beta} = 8\pi \, T_{\alpha\beta} \tag{4.5}$$

where

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R^{\lambda}{}_{\lambda} \, ,$$

while the second equation can be shown to imply

$$(g^{\alpha\beta}\sqrt{-g})_{;\mu} := \partial_\mu (g^{\alpha\beta}\sqrt{-g}) + \pi^{\alpha}_{\mu\nu} g^{\nu\beta}\sqrt{-g} + \pi^{\beta}_{\mu\nu} g^{\nu\alpha}\sqrt{-g} - \pi^{\nu}_{\mu\nu} g^{\alpha\beta}\sqrt{-g} = 0 \tag{4.6}$$

which implies that π is the Levi-Civita connection:

$$\pi^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}) \tag{4.7}$$

(See Misner, Thorne, and Wheeler [1973], Ch. 21, sec. 2.)

We may think of (4.2) as depending parametrically on a set of matter and radiation fields ϕ^A through an additional Lagrangian L^*. To couple these fields to the gravitational fields $(g^{\alpha\beta}, \pi^{\gamma}_{\alpha\beta})$ we need a covariant bracket for the ϕ. Then the equations

$$\{F(\phi), S\}_V = 0$$

ought to be equivalent to

$$\nabla \cdot T = 0$$

and the field equations for the ϕ. This procedure is followed in the next section.

5. General Relativistic Fluids

We consider a perfect adiabatic fluid coupled to gravity; see Misner, Thorne, and Wheeler [1973], Ch. 22 for background. One can similarly treat, we presume, plasmas coupled to general relativity (the Maxwell-Einstein-Vlasov system) or charged general relativistic fluids or general relativistic MHD.

The basic fluid quantities are the following scalar fields:

ρ = fluid mass-energy per unit rest three volume

n = baryon number density per unit rest three volume

σ = entropy per unit rest three volume

p = pressure in a rest frame

s = entropy per baryon

μ = relativistic inertial mass per unit rest three volume.

We have the relations

$$\sigma = ns \quad \text{and} \quad \mu = p + \rho. \tag{5.1}$$

The equation of state has the form

$$\rho = \rho(n,\sigma) \tag{5.2}$$

and the pressure is determined by the Legendre transform

$$p = n\frac{\partial\rho}{\partial n} + \sigma\frac{\partial\rho}{\partial\sigma} - \rho \tag{5.3}$$

The basic fluid variables are taken to be

$$n, \sigma, \text{ and } M_\alpha = \mu u_\alpha ,$$

Here u^α is the four velocity of the fluid, which satisfies $u^\alpha u_\alpha = -1$, i.e. $M^\alpha M_\alpha = \mu^2$. This constraint is to be imposed <u>after</u> functional derivatives, i.e. variations are taken. Here indices are raised and lowered using the Lorentz metric $g^{\alpha\beta}$. The constraint $u^\alpha u_\alpha = -1$ can either be imposed directly, as we do, or can be viewed as a constraint in the sense of Dirac associated to the gauge symmetry of curve reparametrizations. [The latter requires some work on covariant momentum maps - see §6 below.]

The fluid brackets are taken to be Lie-Poisson with a similar structure as in the non-relativistic case (Morrison and Greene [1980], Dzyaloshinskii an Volovick [1980]):

$$\{F,G\}(M,n,\sigma) = \int d^4x \sqrt{-g} \left[M_\alpha \left(\frac{\delta G}{\delta M_\beta} \partial_\beta \frac{\delta F}{\delta M_\alpha} - \frac{\delta F}{\delta M_\beta} \partial_\beta \frac{\delta G}{\delta M_\alpha} \right) \right.$$

$$\left. - n \left(\frac{\delta G}{\delta M_\alpha} \partial_\alpha \frac{\delta F}{\delta n} - \frac{\delta F}{\delta M_\alpha} \partial_\alpha \frac{\delta G}{\delta n} \right) - \sigma \left(\frac{\delta G}{\delta M_\alpha} \partial_\alpha \frac{\delta F}{\delta \sigma} - \frac{\delta F}{\delta M_\alpha} \partial_\alpha \frac{\delta G}{\delta \sigma} \right) \right] . \tag{5.4}$$

The Lie algebra underlying this Lie-Poisson bracket is a semi direct product of vector fields and (densities × densities), similar to the nonrelativistic case (see Marsden [1982], Holm and Kupershmidt [1983] and Marsden et. al. [1983]). Here, functional derivatives are defined to be vectors or scalars, <u>not</u> densities:

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} F(M + \lambda \delta M) = \int \frac{\delta F}{\delta M_\alpha} \delta M_\alpha \sqrt{-g}\, d^4x$$

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} F(n + \lambda\delta n) = \int \frac{\delta F}{\delta n}\, \delta n \sqrt{-g}\, d^4x$$

and

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} F(\sigma + \lambda\delta\sigma) = \int \frac{\delta F}{\delta \sigma}\, \delta \sigma \sqrt{-g}\, d^4x.$$

We note that the two minus signs in (5.4) are in apparent disagreement with the nonrelativistic and 3+1 version of the theory (see the above references and Bao, Marsden and Walton [1985], equation (1C.13)). However, when the covariant theory is decomposed into its 3+1 parts, this discrepancy will disappear (for example, when a bracket of vector fields on space time is decomposed, the result looks like a semi direct product bracket, but with a relative sign switch due to the signature (+++-) of the spacetime metric; cf. Fischer and Marsden [1979], Appendix II.)

For the coupled system we use the variables

$$(g^{\alpha\beta},\ \pi^{\mu}_{\alpha\beta},\ M_{\alpha},\ n,\ \sigma)$$

and use the bracket (5.4) plus (4.1). For the action we take

$$S[g,\pi,M,n,\sigma) = \int g^{\alpha\beta} R_{\alpha\beta}(\pi)\sqrt{-g}\, d^4x - 8\pi \int\left\{ \frac{-1}{2\mu} g^{\alpha\beta} M_\alpha M_\beta + V(n,\sigma)\right\}\sqrt{-g}\, d^4x \qquad (5.5)$$

where $R_{\alpha\beta}$ is given by (4.3) and

$$V(n,\sigma) = \frac{1}{2}[p(n,\sigma) - \rho(n,\sigma)]. \qquad (5.6)$$

We note that the fluid term in (5.5), when evaluated on the constraint set $M^\alpha M_\alpha = \mu^2$ is proportional to the integral of the pressure. The covariant bracket equations are

$$\{F,S\}_V = 0 \qquad (5.7)$$

for all F and V. Choosing: $F = F(g,\pi)$ gives

$$\pi^\mu_{\alpha\beta} = \text{Levi-Civita connection of } g$$

and

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta},$$

$$\text{where} \quad T_{\alpha\beta} = \mu u_\alpha u_\beta + p g_{\alpha\beta}, \qquad (5.8)$$

as in the previous section. In getting $\frac{\delta S_{fluid}}{\delta g^{\alpha\beta}} = 8\pi T_{\alpha\beta}$, we used the constraint $g^{\alpha\beta} M_\alpha M_\beta = -\mu^2$ after taking the variation. Choosing $F = F(n)$ and $F = F(\sigma)$ gives

$$(nu^\alpha)_{;\alpha} = 0 \quad \text{and} \quad (\sigma u^\alpha)_{;\alpha} = 0 \qquad (5.9)$$

i.e. conservation of baryon number and entropy. (The apparent discrepancy in (5.8) by a factor of 2 is discussed in Bao, Marsden and Walton [1984].) Finally, choosing $F = F(M_\alpha)$ gives $\nabla \cdot T = 0$ which, of course, also follows from (5.8) and the Bianchi identity.

6. General Canonical Field Theories

We sketch here a framework in which the canonical brackets (2.5) and (4.1) can be constructed and in which the Euler-Lagrange equations are equivalent to the covariant bracket equations. As we have remarked this covers, in principle, fluids and plasmas by reduction of this structure from Lagrangian (material) representation to Eulerian (spatial) representation. (In 3+1 form, the connection between these is discussed in Holm [1985]).

Our fields are assumed to be sections of a vector bundle $\pi : Y \to X$ over a base manifold X (we take X to be spacetime - but for plasmas it is T^*(Spacetime) or the mass hyperboloid therein). We suspect that most of what we describe also works for a general fiber bundle, but we have restricted to the vector bundle case for simplicity. The fields are described in local coordinates by $\phi^A(x^\mu)$, where A is a multi-index for field components and x^μ are spacetime coordinates. Let $\mathcal{L}$ be a given Lagrangian density defined on $J^1(Y)$, the first jet bundle of Y. Recall that the fiber $J^1_x(Y)$ of $J^1(Y)$ over $y \in Y_x$ is

$$J^1_y(Y) = Y_x \otimes T^*_x X = \mathcal{L}(T_x X, Y_x) \ . \tag{6.1}$$

The Lagrangian of a field ϕ is locally given by $\mathcal{L}(\phi^A, \partial_\mu \phi^A)$. The field equations are the usual Euler-Lagrange equations for $\mathcal{L}$:

$$\frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)} = \frac{\partial \mathcal{L}}{\partial \phi^A} \ , \tag{6.2}$$

and we set

$$\pi^\mu_A = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)} \ . \tag{6.3}$$

We now describe (6.3) intrinsically (cf. Kijowski and Tulczyjew [1979]). Let $\Lambda^4 X$ be the bundle of 4-forms (densities) over X so

$$\mathcal{L}: J^1(Y) \to \Lambda^4(X) . \tag{6.4}$$

Let P be the bundle over X whose fiber at x is

$$P_x = (Y_x \otimes T_x^* X)^* \otimes \Lambda_x^4 X \approx T_x X \otimes Y_x^* \otimes \Lambda_x^4 X . \tag{6.5}$$

Describe P by local coordinates (ϕ^A, π_A^μ). The Legendre transformation is the fiber derivative of $\mathcal{L}$:

$$F\mathcal{L}: J^1(Y) \to P$$

given locally by

$$(x^\mu, \phi^A, \partial_\mu \phi^A) \mapsto (x^\mu, \phi^A, \pi_A^\mu),$$

where π_A^μ is given by (6.3).

Let F and G be functionals of sections of P. Then we have an intrinsic bracket

$$\{F,G\}_V(\phi,\pi) = \int_X \left(\frac{\delta F}{\delta \phi^A} \frac{\delta G}{\delta \pi_A^\mu} - \frac{\delta G}{\delta \phi^A} \frac{\delta F}{\delta \pi_A^\mu} \right) V^\mu d^4 x \tag{6.6}$$

where $\frac{\delta F}{\delta \phi^A}$ is a section of $Y^* \otimes \Lambda^4 X \to X$ and $\frac{\delta G}{\delta \pi_A^\mu}$ is a section of $Y \otimes T^* X \to X$. They pair together to give a section of $T^* X \otimes \Lambda^4 X \to X$ which can be contracted with the vector field V and the resulting four form integrated over X. For each fixed V, the bracket makes the sections of P into a Poisson manifold.

The primary constraint set C is the image of the Legendre transformation. We will assume that it is a vector sub-bundle of P. This will be the case, for example, if $\mathcal{L}$ is quadratic in $\partial_\mu\phi^A$ and if its 'kinetic matrix' $\dfrac{\partial^2\mathcal{L}}{\partial(\partial_\mu\phi^A)\partial(\partial_\nu\phi^B)}$ has constant rank.

Let $\ell:P \to P$ be a smooth vector bundle projection with $\mathrm{im}\,\ell = C$. For example, in electromagnetism, ℓ would project any tensor density onto its skew-symmetric part. Then $\ell^*:Y \otimes T^*X \to Y \otimes T^*X$. Set $C^* = \mathrm{im}\,\ell^*$, a subbundle of $Y \otimes T^*X$. This is a bundle dual to C, so that functional derivatives with respect to the constrained covariant momenta naturally take values in C^*:

$$D_\pi F\cdot\delta\pi = \int_X \frac{\delta F}{\delta\pi^\mu_A}\cdot\delta\pi^\mu_A$$

where F is a functional of sections of C, $\delta\pi$ is a variation in C and $\delta F/\delta\pi^\mu_A$ takes values in C^*; thus $\dfrac{\delta F}{\delta\pi^\mu_A}\cdot\delta\pi^\mu_A$ is a density on X.

One may now define brackets on functionals on $\Gamma(C)$ (sections of C) by the same formula as before (6.6), however where $\dfrac{\delta}{\delta\pi^\mu_A}$ is interpreted as sections of C^*. These brackets satisfy all the conditions for Poisson brackets. The only non-obvious condition is the Jacobi identity. To check this, we extend functionals F on $\Gamma(C)$ to functionals $\tilde F = \ell^*F$ on $\Gamma(P)$ as follows:

$$\tilde F(\phi,\pi) = F(\phi,\ell\pi).$$

For a general extension $\hat F$, we have

$$\ell^*\frac{\delta\hat F}{\delta\pi} = \frac{\delta F}{\delta\pi}$$

by the fiber linearity of ℓ. However, for our extension,

$$\ell^* \frac{\delta\tilde{F}}{\delta\pi} = \frac{\delta\tilde{F}}{\delta\pi} = \frac{\delta F}{\delta\pi},$$

since $\ell_\perp^* \frac{\delta\tilde{F}}{\delta\pi} = 0$, where $\ell_\perp = \mathrm{id} - \ell$. Note that the bracket of two such extensions is again such an extension and hence

$$\{\tilde{F},\tilde{G}\} = \{F,G\}^\sim . \tag{6.7}$$

Now Jacobi's identity follows since it holds for the extended functions.

Remarks. 1. (6.7) is the statement that $(\phi,\pi) \mapsto (\phi,\ell\pi)$ is a Poisson map. Thus the constrainted brackets are the pull back of the full brackets by the projection $\Gamma(P) \to \Gamma(C)$. Note that the injection $\Gamma(C) \to \Gamma(P)$ is not a Poisson map for $C \neq P$. In fact,

$$\{F,G\} - \{\hat{F},\hat{G}\} = \int \frac{\delta\hat{F}}{\delta\phi} \; \ell_\perp^* \; \frac{\delta\hat{G}}{\delta\pi} - \frac{\delta\hat{G}}{\delta\phi} \; \ell_\perp^* \frac{\delta\hat{F}}{\delta\pi},$$

for arbitrary extentions.

2. For general fiber bundles, or if C is not a vector bundle, the results just described require extension. This development should be done in conjunction with examples such as general relativistic fluids written in the Lagrangian (material) picture.

The "Hamiltonian" is uniquely defined on C by

$$H(\phi,\pi) = \pi^\mu_A \partial_\mu \phi^A - \mathcal{L}(\phi^A, \partial_\mu\phi^A) . \tag{6.7}$$

At first, the right hand side is defined on $J^1(Y) \times C$. However, the partial derivative with respect to $\partial_\mu\phi^A$ is zero, so we get a well defined density on C. Set

$$S[\phi,\pi] = \int_X [\pi^\mu_A \partial_\mu \phi^A - H(\phi,\pi)] d^4x \tag{6.8}$$

and note that

$$\frac{\delta S}{\delta \phi^A} = 0 \quad \text{and} \quad \frac{\delta S}{\delta \pi^\mu_A} = 0\ , \tag{6.9}$$

i.e.,

$$\{F,S\}_V = 0 \quad \text{for all} \quad F,V \tag{6.10}$$

reproduce respectively

$$\frac{\partial}{\partial x^\mu} \pi^\mu_A = -\frac{\delta H}{\delta \phi^A} \quad \text{and} \quad \partial_\mu \phi^A = \frac{\delta H}{\delta \pi^\mu_A}\ , \tag{6.11}$$

which are equivalent to the Euler-Lagrange equations, (6.2).

Remarks. 1. In the above setting, only canonical brackets are described. Noncanonical brackets, such as those for fluids and plasmas are expected to come from canonical brackets in Lagrangian representation as in the nonrelativistic case (Marsden, Ratiu and Weinstein [1984a,b]) by a covariant version of the reduction process.

2. We conjecture that covariant momentum maps associated with a group action G on P should be defined to be maps $J : P \to \mathfrak{g}^* \otimes TX \otimes \Lambda^4 P$. These should be consistent with the covariant momentum maps defined in Gimmsy [1985] and should include standard Noether identities.

As in Gimmsy [1985], one can presumably show that for an appropriately covariant localized theory, J vanishes on solutions of the field equations and that these conditions $J = 0$ correspond to first class constraints in the sense of Dirac [1964]. The momentum maps should play a key role in the reduction process, as in the nonrelativistic case (Marsden et.al. [1983]).

3. This paper has not undertaken a systematic 3+1 analysis of the theory. This requires further development of the theory along the lines of remarks 1 and 2. Once this is done, the 3+1 analysis should proceed as in Gimmsy [1985]. In particular, the 3+1 procedure applied to the covariant brackets and field equations should directly yield the dynamical Poisson brackets and the evolution equations in bracket form (which is equivalent to the adjoint form of Fischer and Marsden [1976, 1979]). The vector V in the bracket (6.6) plays an important role in the 3+1 process. It corresponds to the arbitrariness in the choice of the direction of time and to the lapse and shift which appears in the dynamical formulation. Forming the variables ϕ^A and $\pi_A = \pi^\mu_A V_\mu$ is a first step in constructing conjugate variables for the 3+1 formalism. Subsequently, one must also eliminate the so called 'atlas fields' (such as the temporal component of A in electromagnetism), as in Gimmsy [1985].

4. The results of this paper also need to be studied with a view towards understanding limits and averaging (see for example, Weinstein [1983]). For example, we presume that the fluid bracket (5.4) can be derived from the plasma bracket (3.10) in the cold plasma limit and that, as in Marsden et.al. [1983], taking moments via reduction gives a Poisson map between these structures.

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